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TEXTBOOK

MATHEMATICS

GRADE 11

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Contents

Chapter 1

The Remainder Theorem and the Factor Theorem

In this chapter, the students will study the remainder theorem and how it applies to the factor theorem concerning the polynomials. A polynomial of degree n in one variable x can be written as $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ where the coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$ are real numbers with $a_0 \neq 0$ and n is non-negative integer. The degree of a polynomial is the greatest exponent of its variables. We will use the functional notation $f(x)$ to denote a polynomial in x, i.e., $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$.

1.1 Dividing Polynomials

When a polynomial $f(x)$ is divided by a divisor $d(x)$, we get a quotient $q(x)$ and the remainder $r(x)$. This can be written as $f(x) = d(x) \cdot q(x) + r(x)$, where $r(x) = 0$ or the degree of the remainder $r(x)$ is less than the degree of the divisor $d(x)$.

Example 1.

Divide $2x^4 + 3x^3 + 5x - 1$ by $x^2 - 2x + 2$ using the long division.

Solution

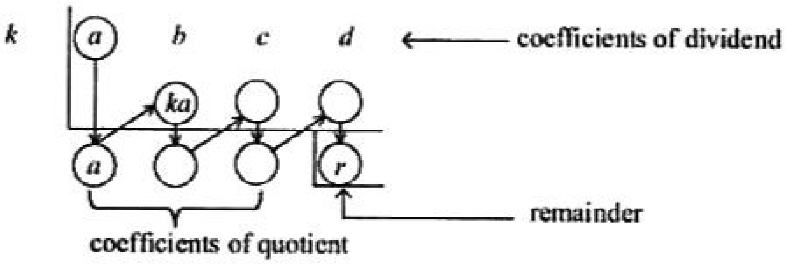
$$
\begin{array}{r|rrrrr}
 & 2x^2 & +\,7x & +\,10 & & \\
\hline
x^2 - 2x + 2 & 2x^4 + 3x^3 & +\,0x^2 & +\,5x & -\,1 & \\
 & 2x^4 - 4x^3 & +\,4x^2 & & & \\
\hline
 & 7x^3 & -\,4x^2 & +\,5x & & \\
 & 7x^3 & -\,14x^2 & +\,14x & & \\
\hline
 & & 10x^2 & -\,9x & -\,1 & \\
 & & 10x^2 & -\,20x & +\,20 & \\
\hline
 & & & 11x & -\,21 &
\end{array}
$$

This tells us that when the dividend $2x^4 + 3x^3 + 5x - 1$ is divided by the divisor $x^2 - 2x + 2$, the quotient is $2x^2 + 7x + 10$ and the remainder is $11x - 21$. This can be written as

$$\underbrace{2x^4 + 3x^3 + 5x - 1}_{f(x)} = \underbrace{(x^2 - 2x + 2)}_{d(x)}\underbrace{(2x^2 + 7x + 10)}_{q(x)} + \underbrace{11x - 21}_{r(x)}.$$

1.2 Synthetic Division

Synthetic division is another way to divide a polynomial by the divisors of the form $x-k$, where k is a constant. This method normally reduces the writing and calculations that involved in the long division. The pattern of dividing a cubic polynomial $ax^3 + bx^2 + cx + d$ by $x - k$ is illustrated below.



Vertical pattern: Add terms in columns.
Digonal pattern: Multiply results by k.
The degree of quotient polynomial is one less than that of dividend.
For higher degree polynomial, the pattern is similar.

Example 2.

Use synthetic division and long division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$.

Solution

For this division, we rewrite $x + 3$ as $x - (-3)$.

-3	1	0	-10	-2	4
		-3	9	3	-3
	1	-3	-1	1	1

$$
\begin{array}{r|l}
 & x^3 - 3x^2 - x + 1 \\ \hline
x+3 & x^4 \qquad -10x^2 - 2x + 4 \\
 & \underline{x^4 + 3x^3} \\
 & -3x^3 - 10x^2 \\
 & \underline{-3x^3 - 9x^2} \\
 & -x^2 - 2x \\
 & \underline{-x^2 - 3x} \\
 & x + 4 \\
 & \underline{x + 3} \\
 & 1
\end{array}
$$

Quotient $= x^3 - 3x^2 - x + 1$
Remainder $= 1$.

1.3 The Remainder Theorem

We know how to divide a polynomial expression $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ by the polynomial of the first degree, such as $x - 1, x + 2, 2x + 3, x - \frac{1}{2}$ and so on. In this case, the remainder is always a constant. That is, the remainder does not depend on x. Our aim is to find the remainder without doing the actual division.

Theorem 1.1 (Remainder Theorem). If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $f(k)$.

Proof. Let $q(x)$ be the quotient and R be the remainder when $f(x)$ is divided by $x - k$. Since R does not depend on x, we can then write $f(x)$ as

$$f(x) = (x - k) \cdot q(x) + R.$$

Then

$$\begin{aligned} f(k) &= (k - k) \cdot q(k) + R \\ &= 0 \cdot q(k) + R \\ &= R. \end{aligned}$$

That is $R = f(k)$.

Example 3.

Find the remainder when $4x^2+3x-1$ is divided by $x-3$, by using remainder theorem.

Solution

Let $f(x) = 4x^2 + 3x - 1$.
When $f(x)$ is divided by $x - 3$,
the remainder $= f(3) = 4(3)^2 + 3(3) - 1 = 44$.

Example 4.

Find the remainder when $x^8 + 4x^6 - 3x$ is divided by $x + 1$, by using remainder theorem.

Solution

Let $f(x) = x^8 + 4x^6 - 3x$.

When $f(x)$ is divided by $x + 1$,

the remainder $= f(-1) = (-1)^8 + 4(-1)^6 - 3(-1) = 8$.

Example 5.

When the polynomial $px^3 + 11x^2 + 2px - 5$ is divided by $x + 2$, the remainder is 15. Find the value of p.

Solution

Let $f(x) = px^3 + 11x^2 + 2px - 5$.

When$f(x)$ is divided by $x + 2$,

$$\begin{aligned} \text{the remainder} &= f(-2) \\ &= p(-2)^3 + 11(-2)^2 + 2p(-2) - 5 \\ &= -8p + 44 - 4p - 5 \\ &= -12p + 39. \end{aligned}$$

$$\begin{aligned} \text{By the given condition, } -12p + 39 &= 15 \\ 12p &= 24 \\ p &= 2. \end{aligned}$$

Theorem 1.2. If a polynomial $f(x)$ is divided by $ax - b, a \neq 0$, then the remainder is $f(\frac{b}{a})$.

Proof. If $f(x)$ is divided by $ax - b$, let $q(x)$ be the quotient and R the remainder. Then

$$f(x) = (ax - b) \cdot q(x) + R.$$

Now putting $x = \frac{b}{a}$ on both sides, we have

$$f(\frac{b}{a}) = (a(\frac{b}{a}) - b) \cdot q(\frac{b}{a}) + R = R.$$

Hence $R = f(\frac{b}{a})$.

Example 6.

Find the remainder when $2x^2 + 5x - 3$ is divided by $2x - 3$.

Solution

Let $f(x) = 2x^2 + 5x - 3$.

When $f(x)$ is divided by $2x - 3$,

$$\begin{aligned}\text{the remainder} &= f(\frac{3}{2}) \\ &= 2\left(\frac{3}{2}\right)^2 + 5\left(\frac{3}{2}\right) - 3 \\ &= \frac{9}{2} + \frac{15}{2} - 3 \\ &= 9.\end{aligned}$$

Example 7.

When $f(x) = 2x^3 - 7x^2 + px + 5$ is divided by $2x - 1$, the remainder is 1. Find the value of p.

Solution

Let $f(x) = 2x^3 - 7x^2 + px + 5$.

When $f(x)$ is divided by $2x - 1$, the remainder $= f(\frac{1}{2})$.

So we have

$$\begin{aligned}f(\frac{1}{2}) &= 1 \\ 2\left(\frac{1}{2}\right)^3 - 7\left(\frac{1}{2}\right)^2 + p\left(\frac{1}{2}\right) + 5 &= 1 \\ \frac{1}{4} - \frac{7}{4} + \frac{p}{2} &= -4 \\ 1 - 7 + 2p &= -16 \\ 2p &= -10 \\ p &= -5.\end{aligned}$$

Exercise 1.1

1. Using synthetic division, find the remainder and quotient when
   (a) $3x^2 - 2x + 1$ is divided by $x - 1$
   (b) $3 - 4x - 2x^2$ is divided by $x + 1$
   (c) $4x^3 + 2x - 3$ is divided by $x - 3$
   (d) $2x^3 + x^2 + 2x + 1$ is divided by $x + \frac{1}{2}$
   (e) $x^4 - 6x^2 + 9$ is divided by $x - \sqrt{3}$
2. Using the remainder theorem, find the remainder when
   (a) $5x^2 + 7x + 9$ is divided by $x + 1$.
   (b) $-x^3 + 3x^2 - 7x$ is divided by $x - 2$.
   (c) $2x^3 + 3x^2 + 5$ is divided by $x - \frac{1}{2}$.
   (d) $x^6 - 6x^4 + 12x^2 - 15$ is divided by $x + \sqrt{2}$.
   (e) $x^4 - 2x^2 + 6$ is divided by x.
3. Find the remainder when
   (a) $6x^2 + x - 7$ is divided by $2x + 3$.
   (b) $10x^4 + 5x^3 + 4x^2 - 9$ is divided by $2x - 1$.
4. When $x^3 + 3x^2 - mx + 4$ is divided by $x - 2$, the remainder is $m + 3$. Find the value of m.
5. The polynomial $x^3 + ax^2 + bx - 3$ leaves a remainder of 27 when divided by $x - 2$ and a remainder of 3 when divided by $x + 1$. Calculate the remainder when the polynomial is divided by $x - 1$.
6. The expressions $x^3 - 7x + 6$ and $x^3 - x^2 - 4x + 24$ have the same remainder when divided by $x + p$. Find the possible values of p.
7. Given that the expression $x^3 - ax^2 + bx + c$ leaves the same remainder when divided by $x + 1$ or $x - 2$, find 'a' in terms of 'b'.
8. Given that the remainder when $f(x) = x^3 - x^2 + ax$ is divided by $x + a$ where $a > 0$, is twice the remainder when $f(x)$ is divided by $x - 2a$, find the value of a. Find also the remainder when $f(x)$ is divided by $x - 2$.
9. When the expression $x^3 + ax^2 + 4$ is divided by $x + 1$ the remainder is 6 greater than the remainder when it is divided by $x - 2$. Find the value of a.
10. What number should be subtracted from $3x^3 - 5x^2 + 6x$ so that on dividing it by $x - 3$, the remainder is 8?
11. The remainders when $f(x) = ax^2 + bx + c$ is divided by $x - 1, x + 1, x - 2$ are 1, 25, 1 respectively. Show that $f(x)$ is a perfect square.

12. When the polynomials ax^3+5x^2+3x+4 and $3x^3+9x^2+ax-6$ are divided by $x+3$, the remainders are A and B respectively. Find the values of a if $A+B=4$.
13. The remainder when ax^3+bx^2+2x+3 is divided by $x-1$ is twice that when it is divided by $x+1$, show that $b=3a+3$.
14. The remainder when $2x^3+kx^2+7$ is divided by $x-2$ is half the remainder when the same expression is divided by $2x-1$. Find the value of k.
15. The remainder when x^4+3x^2-2x+2 is divided by $x+a$ is the square of the remainder when x^2-3 is divided by $x+a$. Calculate the possible values of a.
16. The expression ax^3-x^2+bx-1 leaves the remainders of -33 and 77 when divided by $x+2$ and $x-3$ respectively. Find the values of a and b and the remainder when divided by $x-2$.

1.4 The Factor Theorem

The factor theorem is an application of the remainder theorem. If the remainder is zero when the polynomial $f(x)$ is divided by $x-k$, then the divisor $x-k$ is a factor of $f(x)$. Repeated application of the factor theorem may be used to factorize the polynomial.

Theorem 1.3 (The Factor Theorem). Let $f(x)$ be a polynomial. Then $x-k$ is a factor of $f(x)$ if and only if $f(k)=0$.

Proof. Suppose that $x-k$ is a factor of $f(x)$. Then

$$f(x)=(x-k)\cdot g(x)$$

for some polynomial $g(x)$. It follows that

$$f(k)=(k-k)\cdot g(k)=0.$$

Conversely, suppose that $f(k)=0$. We have known that

$$f(x)=(x-k)\cdot q(x)+R,$$

for some polynomial $q(x)$. Then $f(k)=(k-k)\cdot q(x)+R=R$, we have $R=0$. Thus

$$f(x)=(x-k)\cdot q(x).$$

Therefore $x-k$ is a factor of $f(x)$.

Example 8.

Show that $x-2$ is a factor of $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$. Using (i) the factor theorem, (ii) synthetic division.

Solution

$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$

(i) $f(2) = 2(2)^4 + 7(2)^3 - 4(2)^2 - 27(2) - 18$

$= 32 + 56 - 16 - 54 - 18 = 0$

By the factor theorem, $x-2$ is a factor of $f(x)$.

(ii)

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & \boxed{0} \end{array}$$

The remainder $= 0$.

Therefore $x-2$ is a factor of $f(x)$.

Example 9.

Find what value c must have in order that $x-5$ may be a factor of $x^3 + 4x^2 - 3x + c$.

Solution

Let $f(x) = x^3 + 4x^2 - 3x + c$.
$x-5$ is a factor of $f(x)$ when $f(5) = 0$.

$$\begin{aligned} 5^3 + 4(5)^2 - 3(5) + c &= 0 \\ 125 + 100 - 15 + c &= 0 \\ c &= -210. \end{aligned}$$

Example 10.

Find the factors of $x^3 + 3x^2 - 13x - 15$.

Solution

Let $f(x) = x^3 + 3x^2 - 13x - 15$.
We shall try the divisors of 15, namely $\pm1, \pm3, \pm5, \pm15$.

$$\begin{aligned} f(1) &= 1 + 3 - 13 - 15 \neq 0 \\ f(-1) &= (-1)^3 + 3(-1)^2 - 13(-1) - 15 \\ &= -1 + 3 + 13 - 15 = 16 - 16 = 0. \end{aligned}$$

By the factor theorem, $x+1$ is a factor of $f(x)$.
The other factors can be formed by actual division as follows:

$$
\begin{array}{r|l}
 & x^2 + 2x - 15 \\
\hline
x+1 & x^3 + 3x^2 - 13x - 15 \\
 & x^3 + x^2 \\
\hline
 & 2x^2 - 13x \\
 & 2x^2 + 2x \\
\hline
 & -15x - 15 \\
 & -15x - 15 \\
\hline
 & 0
\end{array}
$$

$$
\begin{aligned}
f(x) &= (x+1)(x^2+2x-15) \\
&= (x+1)(x-3)(x+5).
\end{aligned}
$$

Therefore the factors are $(x+1), (x-3)$ and $(x+5)$.

A root of a polynomial equation $f(x) = 0$ is a number r such that $f(r) = 0$. With regard to the rational roots of a polynomial equation with integral coefficients, we have the following theorem:

Theorem 1.4 (Rational Root Theorem). If the rational number $\frac{p}{q}$, a fraction in the lowest terms, is a root of the equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$, where $a_i (i = 0, 1, 2, \ldots n)$ are integral coefficients and $a_0 \neq 0, a_n \neq 0$, then p is an exact divisor of a_n and q is an exact divisor of a_0.

Proof. Since $\frac{p}{q}$ is a root of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0,$$

we have

$$a_0\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + a_2\left(\frac{p}{q}\right)^{n-2} + \ldots + a_{n-1}\left(\frac{p}{q}\right) + a_n = 0.$$

Multiply each term of this equation by q^n, we get

$$a_0p^n + a_1p^{n-1}q + a_2p^{n-2}q^2 + \ldots + a_{n-1}pq^{n-1} + a_nq^n = 0.$$

If we add $-a_nq^n$ to the both sides and divide both members by p.

$$a_0p^{n-1} + a_1p^{n-2}q + a_2p^{n-3}q^2 + \ldots + a_{n-1}q^{n-1} = \frac{-a_nq^n}{p}.$$

In the left hand side of the above equation, since each a_i, p and q is an integer, and therefore, $\frac{-a_n q^n}{p}$ is an integer. Also, p and q have no common factor, so that p does not divide q^n. Thus p is an exact divisor of a_n.

Similarly, $a_1 p^{n-1} + a_2 p^{n-2} q + \ldots + a_{n-1} p q^{n-2} + a_n q^{n-1} = \frac{-a_0 p^n}{q}$.

By the same type of argument, we have the fact that q is an exact divisor of a_0.

To use this theorem, we should first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational roots} = \frac{\text{Factors of constant term}}{\text{Factors of leading coefficient}}$$

The following theorem follows directly from the above theorem.

Theorem 1.5. Any rational root of the equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$$

where each a_i is an integral coefficient, must be an integer which is the exact divisor of the constant term a_n.

Example 11.

Solve the equation $3x^3 - x^2 - 3x + 1 = 0$.

Solution

Let $f(x) = 3x^3 - x^2 - 3x + 1$.

The leading coefficient is 3 and the constant term is 1.

Possible rational roots are $\pm 1, \pm\frac{1}{3}$.

$f(1) = 3(1)^3 - (1)^2 - 3(1) + 1 = 0$.

Therefore $x - 1$ is a factor of $f(x)$.

$$\begin{array}{r|rrrr}
 & 3x^2 & +\,2x & -\,1 & \\
x-1 & 3x^3 & -\,x^2 & -\,3x & +\,1 \\
 & 3x^3 & -\,3x^2 & & \\ \hline
 & & 2x^2 & -\,3x & \\
 & & 2x^2 & -\,2x & \\ \hline
 & & & -x & +\,1 \\
 & & & -x & +\,1 \\ \hline
 & & & & 0
\end{array}$$

$f(x) = (x-1)(3x^2+2x-1) = (x-1)(3x-1)(x+1)$

$$\begin{aligned} f(x) &= 0 \\ (x-1)(3x-1)(x+1) &= 0 \end{aligned}$$

$x-1=0$ or $3x-1=0$ or $x+1=0$

Therefore $x=1$ or $x=\frac{1}{3}$ or $x=-1$.

Example 12.

Solve the equation $4x^4+8x^3-7x^2-11x+6=0$.

Solution

Let $f(x) = 4x^4+8x^3-7x^2-11x+6$.

The leading coefficient is 4 and the constant term is 6.

Possible rational roots are $\pm1, \pm2, \pm3, \pm6, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{3}{2}, \pm\frac{3}{4}$.

$$\begin{aligned} f(1) &= 4(1)^4+8(1)^3-7(1)^2-11(1)+6 \\ &= 4+8-7-11+6=0. \end{aligned}$$

Therefore $x-1$ is a factor of $f(x)$.

$$\begin{aligned} f(-2) &= 4(-2)^4+8(-2)^3-7(-2)^2-11(-2)+6 \\ &= 64-64-28+22+6=0. \end{aligned}$$

Therefore $x+2$ is a factor of $f(x)$.

$(x-1)(x+2) = x^2+x-2$ is also a factor of $f(x)$.

	$4x^2$	$+4x$	-3		
x^2+x-2	$4x^4$	$+8x^3$	$-7x^2$	$-11x$	$+6$
	$4x^4$	$+4x^3$	$-8x^2$		
		$4x^3$	$+x^2$	$-11x$	
		$4x^3$	$+4x^2$	$-8x$	
			$-3x^2$	$-3x$	$+6$
			$-3x^2$	$-3x$	$+6$
					0

$$\begin{aligned} f(x) &= (x^2+x-2)(4x^2+4x-3) \\ &= (x-1)(x+2)(2x-1)(2x+3) \end{aligned}$$

$$\begin{aligned} f(x) &= 0 \\ (x-1)(x+2)(2x-1)(2x+3) &= 0 \end{aligned}$$

$x-1=0$ or $x+2=0$ or $2x-1=0$ or $2x+3=0$.

Therefore $x=1$ or $x=-2$ or $x=\dfrac{1}{2}$ or $x=-\dfrac{3}{2}$.

Example 13.

If the equations $ax^3+4x^2-5x-10=0$ and $ax^3-9x-2=0$ have a common root, then show that $a=2$ or 11. Hence, solve the equation $2x^3-9x-2=0$.

Solution

Let c be a common root of $ax^3+4x^2-5x-10=0$ and $ax^3-9x-2=0$.

Then $ac^3+4c^2-5c-10=0$ and
$ac^3-9c-2=0$.

Subtracting the equations, we get

$4c^2+4c-8=0$

$c^2+c-2=0$

$(c+2)(c-1)=0$

$c+2=0$ or $c-1=0$

$c=-2$ or $c=1$.

If $c=-2$, then
$$\begin{aligned} a(-2)^3-9(-2)-2 &= 0 \\ -8a+18-2 &= 0 \\ a &= 2. \end{aligned}$$

If $c=1$, then
$$\begin{aligned} a(1)^3-9(1)-2 &= 0 \\ a-9-2 &= 0 \\ a &= 11. \end{aligned}$$

Therefore $a=2$ or 11.

Let $f(x)=2x^3-9x-2$.

Since $c=-2$ is a root of $f(x)=0$, $f(-2)=0$.
Therefore $(x+2)$ is a factor of $f(x)$.

$$
\begin{array}{r|rrrr}
 & 2x^2 & -\ 4x & -\ 1 & \\
\hline
x+2 & 2x^3 & & -\ 9x & -\ 2 \\
 & 2x^3 & +\ 4x^2 & & \\
\hline
 & & -4x^2 & -\ 9x & \\
 & & -4x^2 & -\ 8x & \\
\hline
 & & & -\ x & -\ 2 \\
 & & & -\ x & -\ 2 \\
\hline
 & & & & 0
\end{array}
$$

$f(x) = (x+2)(2x^2 - 4x - 1)$

$$
\begin{aligned}
f(x) &= 0 \\
(x+2)(2x^2 - 4x - 1) &= 0
\end{aligned}
$$

$x + 2 = 0$ or $2x^2 - 4x - 1 = 0$

$x = -2$ or $x = \dfrac{4 \pm \sqrt{16+8}}{4} = 1 \pm \dfrac{\sqrt{6}}{2}$.

Exercise 1.2

1. Show that $x - 2$ is a factor of $x^3 - 6x^2 + 12x - 8$.
2. Show that $2x + 1$ is a factor of $6x^4 + 5x^3 - 9x^2 - x + 2$.
3. By using synthetic division, show that $x - 3$ is a factor of $3x^3 - 19x^2 + 33x - 9$.
4. Find the factors of $2x^3 + 11x^2 + 18x + 9$.
5. Find the factors of $x^3 + 5x^2 + 2x - 8$.
6. Find the factors of $2x^4 - 3x^3 - 4x^2 + 3x + 2$.
7. Solve the following equations:
   (a) $6x^3 + x^2 - 19x + 6 = 0$
   (b) $9x^3 + 10x^2 - 17x = 2$
   (c) $x^4 - 9x^2 - 4x + 12 = 0$
8. Use synthetic division to find all the factors of $x^3 + 6x^2 - 9x - 54$.
9. If the expression $f(x) = 6x^3 + 13x^2 - 40x - 4p$ is divisible by $2x - 1$, find the value of p.

10. If $x^2 - 5x + 4$ is a factor of $f(x) = 2x^3 + ax^2 + bx - 12$, find the values of a and b, and the third factor.

11. The polynomial $ax^3 + bx^2 - 5x + 2a$ is exactly divisible by $x^2 - 3x - 4$. Calculate the values of a and b, and factorize the polynomial completely.

12. $x^3 + ax^2 - x + b$ and $x^3 + bx^2 - 5x + 3a$ have a common factor $x + 2$. Find a and b.

13. If $x - k$ is a common factor of expressions $f(x) = x^3 + ax + b$ and $g(x) = x^3 + px + q$, show that $k = \dfrac{q - b}{a - p}$.

14. Given that $f(x) = 2x^3 + px^2 + qx - 9$ has a factor $x - 3$ but leaves a remainder of 8 when it is divided by $x + 1$. Find the values of p and q. Show that $2x + 1$ is a factor of $f(x)$.

15. If $x - 1$ and $x + 2$ are factors of $x^3 + (5a + 1)x^2 + (b - 3)x - 10$, find the values of a and b. And then factorize the given expression completely.

16. Given $f(x) = x^3 + px^2 - 2x + 4\sqrt{3}$ has a factor $x + \sqrt{2}$, find the value of p. Show that $x - 2\sqrt{3}$ is also a factor, and solve the equation $f(x) = 0$.

17. Given that $kx^3 + 2x^2 + 2x + 3$ and $kx^3 - 2x + 9$ have a common factor, what are the possible values of k?

18. Given $f(x) = 2x^3 + ax^2 - 7a^2x - 6a^3$, determine whether or not $x - a$ and $x + a$ are factors of $f(x)$. Hence find, in terms of 'a', the roots of $f(x) = 0$.

19. If the equations $ax^4 + x^3 - x^2 + 3x + 2 = 0$ and $ax^4 - x^2 + 3x + 1 = 0$ have a common root, then find the value of a.

20. Given that $4x^4 - 9a^2x^2 + 2(a^2 - 7)x - 18$ is exactly divisible by $2x - 3a$, show that $a^3 - 7a - 6 = 0$ and hence find the possible values of a.

21. Given that $f(x) = x^{2n} - (p + 1)x^2 + p$, where n and p are positive integers. Show that $x - 1$ is a factor of $f(x)$, for all values of p. When $p = 4$, find the value of n for which $x - 2$ is a factor of $f(x)$ and hence factorize $f(x)$ completely.

Chapter 2

The Binomial Theorem

An algebraic expression containing two terms, for example $x + y$, $a - 2$, $2x + 1$, etc., is called a binomial expression. The expression of a binomial with a small positive power can be calculated by ordinary multiplication, but for large power, the actual multiplication is difficult and complicated. By means of the binomial theorem, it can be easily calculated. In this chapter, we will study the expansion of $(a + b)^n$ for a positive integer n.

2.1 Binomial Expansion

We can express powers of $(x + y)$ as follows:

$$
\begin{aligned}
(x+y)^1 &= x + y \\
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
(x+y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5
\end{aligned}
$$

We can notice the facts in the above expansions.

- If the power of the binomial is n, there are $n + 1$ terms.
- The sum of the powers of x and y in each term is equal to the power of the binomial.
- The power of x decreases by 1 in each consecutive term while the power of y increases by 1 in each consecutive term.
- From the above expansions of the powers of $(x+y)$, the coefficients form a pattern as below.

Binomials	Coefficients
$(x+y)$	1 1
$(x+y)^2$	1 2 1
$(x+y)^3$	1 3 3 1
$(x+y)^4$	1 4 6 4 1
$(x+y)^5$	1 5 10 10 5 1

- The coefficients for binomial expansions in the above table is called **Pascal's Triangle**, in honour of the great French mathematician, Blaise Pascal (1623-1662). In each row of this triangle, the outer terms are always 1 and each of the inner term is obtained by adding two values above it.

Example 1.

Write down and simplify the following expansions.

(i) $(1+2x)^4$ (ii) $(2-x)^5$ (iii) $(x+\frac{2}{x})^5$

Solution.

(i) $(1+2x)^4 = (1)^4 + 4(1)^3(2x) + 6(1)^2(2x)^2 + 4(1)(2x)^3 + (2x)^4$
$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$

(ii) $(2-x)^5 = (2+(-x))^5$
$= (2)^5 + 5(2)^4(-x) + 10(2)^3(-x)^2 + 10(2)^2(-x)^3 + 5(2)(-x)^4 + (-x)^5$
$= 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$

(iii) $(x+\frac{2}{x})^5 = (x)^5 + 5(x)^4(\frac{2}{x}) + 10(x)^3(\frac{2}{x})^2 + 10(x)^2(\frac{2}{x})^3 + 5(x)(\frac{2}{x})^4 + (\frac{2}{x})^5$
$= x^5 + 10x^3 + 40x + \frac{80}{x} + \frac{80}{x^3} + \frac{32}{x^5}$

Example 2.

Expand and simplify $(5-3x)(2+x)^4$.

Solution

$$\begin{aligned}(5-3x)(2+x)^4 &= (5-3x)[(2)^4 + 4(2)^3x + 6(2)^2x^2 + 4(2)x^3 + x^4] \\ &= (5-3x)(16 + 32x + 24x^2 + 8x^3 + x^4) \\ &= 80 + 160x + 120x^2 + 40x^3 + 5x^4 \\ &\quad -48x - 96x^2 - 72x^3 - 24x^4 - 3x^5 \\ &= 80 + 112x + 24x^2 - 32x^3 - 19x^4 - 3x^5\end{aligned}$$

Example 3.

Find the first three terms, in ascending power of x, of the expansion of $(3-x)^5$. Use the result to find the coefficient of x^2 in the expansion of $(1-2x)(3-x)^5$.

Solution

$$\begin{aligned}(3-x)^5 &= (3)^5 + 5(3)^4(-x) + 10(3)^3(-x)^2 + \dots \\ &= 243 - 405x + 270x^2 + \dots\end{aligned}$$

$(1-2x)(3-x)^5 = (1-2x)(243 - 405x + 270x^2 + \dots)$

The coefficient of $x^2 = 1(270) - 2(-405) = 270 + 810 = 1080$

Exercise 2.1

1. Expand the following.
   (a) $(1-x)^3$ (b) $(a+2b)^3$ (c) $(1+\frac{2}{3}x)^3$
   (d) $(1-\frac{1}{4}x)^4$ (e) $(x+\frac{1}{x})^4$ (f) $(\frac{1}{2}x+\frac{1}{3}y)^4$
   (g) $(x-2)^5$ (h) $(\frac{1}{2}-2x)^5$ (i) $(\frac{a}{2}-\frac{3}{b})^5$
2. Find, in ascending powers of x, the first three terms of the expansions of $(1-2x)^4$ and $(2+x^2)^5$. Hence find the coefficient of x^2 in the expansion of $(1-2x)^4(2+x^2)^5$.
3. Find, in ascending powers of x, the first three terms of the expansions of $(1+2x)^4$ and $(2-\frac{1}{2}x)^5$. Hence find the coefficient of x^2 in the expansion of $(1+2x)^4(2-\frac{1}{2}x)^5$.
4. Find, in ascending powers of x, the first three terms of the expansions of $(1+2x)^5$ and $(3-x)^5$. Hence find the coefficient of x^2 in the expansion of $(3+5x-2x^2)^5$.
5. Find the value of $(x+\sqrt{2})^4 + (x-\sqrt{2})^4$.

2.2 The Binomial Theorem

Pascal's triangle can be used to expand the power of binomial expressions, but it is really useful only for small powers. If we want to expand a binomial expression with a large power, finding the binomial coefficients by the use of Pascal's triangle is impractical. It would require too much work to extend the triangle. In such cases, using the binomial theorem with combinatorial coefficients is better.

The numbers ^nC_r count the number of r-element subsets of an n-element set. Consider the set $X = \{a, b, c, d\}$ and the number of subsets of X.

Subset	Number of subsets
ϕ	$^4C_0$
$\{a\}, \{b\}, \{c\}, \{d\}$	$^4C_1$
$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}$	$^4C_2$
$\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}$	$^4C_3$
$\{a,b,c,d\}$	$^4C_4$

For any integers n and r where $0 \leq r \leq n$,
We define:

$$^nC_0 = 1, \qquad ^nC_n = 1$$

$$^nC_1 = n,$$

$$^nC_2 = \frac{n(n-1)}{1 \cdot 2}$$

$$^nC_3 = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$$

$$^nC_r = \frac{n(n-1)(n-2)\cdots(n-(r-1))}{1 \cdot 2 \cdot 3 \ldots r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots r}.$$

for example, (i) $^4C_0 = 1$, $^5C_0 = 1$, $^7C_0 = 1$, $^{10}C_0 = 1$

(ii) $^3C_3 = 1$, $^4C_4 = 1$, $^8C_8 = 1$, $^{13}C_{13} = 1$

(iii) $^4C_1 = 4$, $^5C_1 = 5$, $^9C_1 = 9$, $^{12}C_1 = 12$

(iv) $^4C_2 = \frac{4 \cdot 3}{1 \cdot 2} = 6$, $^7C_3 = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35$, $^9C_3 = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$

Theorem 2.1 (The Binomial Theorem). For any x and y,

$$\boxed{(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y + \ldots + {}^nC_r x^{n-r} y^r + \ldots + {}^nC_n x^0 y^n}$$

where n is a positive integer.

Proof. The left-hand side represents the product of n copies of $(x + y)$.

$$(x+y)^n = \underbrace{(x+y)(x+y)\ldots(x+y)}_{n \text{ factors}}$$

We will expand this expression, multiplying all the terms out together. Since there are n linear factors, each term in the expansion would be of the form $x^{n-k}y^k$, for some $k = 0, 1, 2, \ldots, n$ which represents how many times of y was chosen. The coefficient of $x^{n-k}y^k$ is the number of ways of choosing y exactly k times. This is the same as choosing a subset of k of the n factors from which to choose y (with x being chosen from the rest).

- We get x^n by choosing x from every one of the n factors. It can be done in one way. The number of ways of choosing x^n from n factors is ${}^nC_0$. So the term with x^n is ${}^nC_0x^ny^0$.
- We get $x^{n-1}y$ terms by choosing y from one of the n factors and choose x from the remaining $n-1$ factors. The number of ways of choosing one y from n factors is ${}^nC_1$. So the term with y is ${}^nC_1x^{n-1}y$.
- In general, for any r where $0 \leq r \leq n$, we get $x^{n-r}y^r$ terms by choosing y from r of the n factors and choose x from the remaining $n-r$ factors. The number of ways of choosing y^r from n factors is ${}^nC_r$. So the term with y^r is ${}^nC_rx^{n-r}y^r$.
- We get y^n by choosing y from every one of the n factors. It can be done in one way. The number of ways of choosing y^n from n factors is ${}^nC_n$. So the term with y^n is ${}^nC_nx^0y^n$.

Hence
$$(x+y)^n = {}^nC_0x^ny^0 + {}^nC_1x^{n-1}y + \ldots + {}^nC_rx^{n-r}y^r + \ldots + {}^nC_nx^0y^n.$$

The coefficients of successive terms ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$ are called **binomial coefficients.**

Special Case

$$\boxed{(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots + {}^nC_{n-1}x^{n-1} + x^n}$$

Note.

Consider the expansion of $(x+y)^n$ where n is a positive integer,

1. The binomial coefficients are all integers.
2. The coefficients of terms equidistant from the beginning and end of the expansion are equal.
   That is

$$\begin{aligned} {}^nC_0 &= {}^nC_n = 1 \\ {}^nC_1 &= {}^nC_{n-1} = n \\ &\ldots \\ {}^nC_r &= {}^nC_{n-r} \end{aligned}$$

Corollary 2.2. For any integers n and r, $0 \leq r \leq n$, ${}^nC_r = {}^nC_{n-r}$.

Proof. ${}^nC_r = \dfrac{n(n-1)(n-2)\ldots(n-(r-1))}{1\cdot2\cdot3\ldots\cdot r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1\cdot2\cdot3\ldots\cdot r}$

${}^nC_{n-r} = \dfrac{n(n-1)(n-2)\ldots(n-(n-r-1))}{1\cdot2\cdot3\ldots(n-r)} = \dfrac{n(n-1)(n-2)\ldots(r+1)}{1\cdot2\cdot3\ldots(n-r)}$

$$\frac{{}^nC_r}{{}^nC_{n-r}} = \frac{\dfrac{n(n-1)(n-2)\dots(n-r+1)}{1\cdot 2\cdot 3\dots\cdot r}}{\dfrac{n(n-1)(n-2)\dots(r+1)}{1\cdot 2\cdot 3\dots(n-r)}}$$

$$= \frac{n(n-1)(n-2)\dots(n-r+1)}{1\cdot 2\cdot 3\dots\cdot r} \times \frac{1\cdot 2\cdot 3\dots(n-r)}{n(n-1)(n-2)\dots(r+1)}$$

$$= \frac{1\cdot 2\cdot 3\dots(n-r)(n-r+1)\dots(n-2)(n-1)n}{1\cdot 2\cdot 3\dots\cdot r(r+1)\dots(n-2)(n-1)n}$$

$$= 1$$

$$\therefore {}^nC_r = {}^nC_{n-r}$$

The number of r-element subsets of n-element set is equal to the number of $(n-r)$-element subsets of n-element set.

for example, ${}^7C_5 = {}^7C_2$

${}^{13}C_{10} = {}^{13}C_3$

General term

The general term is used to find out the specific term.

The general term of $(x+y)^n$ is

$$\boxed{\text{the } (r+1)^{th} \text{ term} = {}^nC_r x^{n-r} y^r}$$

Example 4.

Prove that ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$.

Solution

$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots + {}^nC_n x^0 y^n$

Letting $x = 1$ and $y = 1$,

$$(1+1)^n = {}^nC_0 1^n 1^0 + {}^nC_1 1^{n-1} 1^1 + {}^nC_2 1^{n-2} 1^2 + \dots + {}^nC_n 1^0 1^n$$

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$$

Example 5.

Find the first three terms, in ascending power of x, in the expansion of (i) $(1+2x)^5$ (ii) $(2-\frac{1}{2}x)^6$.

Hence find the coefficient of x^2 in the expansion of $(1+2x)^5(2-\frac{1}{2}x)^6$.

Solution

(i) $(1+2x)^5 = 1 + {}^5C_1(2x) + {}^5C_2(2x)^2 + \dots$

$= 1 + 5(2x) + \frac{5\cdot 4}{1\cdot 2}4x^2 + \dots$

$= 1 + 10x + 40x^2 + \dots$

(ii) $(2-\frac{1}{2}x)^6 = {}^6C_0 2^6 + {}^6C_1(2)^5(-\frac{1}{2}x) + {}^6C_2(2)^4(-\frac{1}{2}x)^2 + \dots$

$= 64 - 6(32)(\frac{1}{2}x) + \frac{6\cdot 5}{1\cdot 2}(16)(\frac{1}{4}x^2) + \dots$

$= 64 - 96x + 60x^2 + \dots$

$(1+2x)^5(2-\frac{1}{2}x)^6 = (1 + 10x + 40x^2 + \dots)(64 - 96x + 60x^2 + \dots)$

the coefficient of $x^2 = 1(60) + 10(-96) + 40(64) = 60 - 960 + 2560 = 1660$

Example 6.

Expand $(1-2x)^{16}$ in ascending powers of x as far as the term in x^3. Use this result to find 0.998^{16} correct to 6 decimal places.

Solution

(i) $(1-2x)^{16} = 1 + {}^{16}C_1(-2x) + {}^{16}C_2(-2x)^2 + {}^{16}C_3(-2x)^3 + \dots$

$= 1 + 16(-2x) + \frac{16\cdot 15}{1\cdot 2}4x^2 + \frac{16\cdot 15\cdot 14}{1\cdot 2\cdot 3}(-8x^3) + \dots$

$= 1 - 32x + 480x^2 - 4480x^3 + \dots$

Since $0.998 = 1 - 0.002$,

$0.998^{16} = (1-0.002)^{16}$

$= (1 - 2(0.001))^{16}$

$= 1 - 32(0.001) + 480(0.001)^2 - 4480(0.001)^3 + \dots$

$= 1 - 0.032 + 0.00048 - 0.00000448 + \dots$

$= 0.968476$ (6 decimal places)

Example 7.

Find the term independent of x and the coefficient of x^5 in the expansion of $(3-2x)^9$.

Solution

$(3-2x)^9$

$(r+1)^{th}$ term $= {}^9C_r(3)^{9-r}(-2x)^r$

$= {}^9C_r(3)^{9-r}(-2)^r(x)^r$

To get the term independent of x, put $r = 0$, the term independent of $x = {}^9C_0(3)^{9-0}(-2)^0 = 3^9 = 19683$

To get the coefficient of x^5, put $r = 5$,

$$\begin{aligned}\text{the coefficient of } x^5 &= {}^9C_5(3)^{9-5}(-2)^5\\ &= \frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5}(3)^4(-2)^5\\ &= 126(81)(-32)\\ &= -326592\end{aligned}$$

Example 8.

The binomial expansion of $(1 + ax)^n$, where $n > 0$, in ascending power of x, is $1 + 20x + 45a^2x^2 + bx^3 + \ldots$. Find the values of n, a and b.

Solution

$$\begin{aligned}(1 + ax)^n &= 1 + {}^nC_1(ax) + {}^nC_2(ax)^2 + {}^nC_3(ax)^3 + \ldots\\ &= 1 + nax + \frac{n(n-1)}{1\cdot 2}a^2x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^3x^3 + \ldots\end{aligned}$$

By the problem, $(1 + ax)^n = 1 + 20x + 45a^2x^2 + bx^3 + \ldots$

So,

$$na = 20 \qquad (1)$$

$$\frac{n(n-1)}{1\cdot 2} = 45 \qquad (2)$$

$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^3 = b \qquad (3)$$

From equation (2),

$$\begin{aligned}n(n-1) &= 90\\ n(n-1) &= 10\times 9\\ \therefore n &= 10\end{aligned}$$

substituting $n = 10$ in equation (1),

$$a = 2$$

substituting $n = 10$ and $a = 2$ in equation (3),

$$b = \frac{10(10-1)(10-2)}{1\cdot 2\cdot 3}(2)^3 = \frac{10\cdot 9\cdot 8}{6}\cdot 8 = 960$$

Middle term in the expansion

In the expansion of $(x + y)^n$, there are $(n + 1)$ terms.

- If n is even, then $(n + 1)$ will be odd. So, $\left(\frac{n+2}{2}\right)^{th}$ term will be the only one middle term in the expansion.

- If n is odd, then $(n+1)$ will be even. So, $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+3}{2}\right)^{th}$ term will be the two middle terms in the expansion.

Example 9.

Find the middle term in the expansion of $\left(1-\frac{x^2}{2}\right)^{14}$.

Solution

$(r+1)^{th}$ term of $\left(1-\frac{x^2}{2}\right)^{14} = {}^{14}C_r(1)^{14-r}\left(-\frac{x^2}{2}\right)^r = {}^{14}C_r\left(-\frac{x^2}{2}\right)^r$

In the expansion of $\left(1-\frac{x^2}{2}\right)^{14}$, there are 15 terms.

$$\begin{aligned}\text{the middle term} &= \left(\frac{14+2}{2}\right)^{th} \text{ term} = 8^{th} \text{ term}\\ &= (7+1)^{th} \text{ term}\\ &= {}^{14}C_7\left(-\frac{x^2}{2}\right)^7\\ &= \frac{14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\left(-\frac{x^{14}}{128}\right)\\ &= -\frac{429x^{14}}{16}\end{aligned}$$

Exercise 2.2

1. Find the 5^{th} term in the expansion of $(2x^2-\frac{3}{x})^{10}$.
2. Find the 8^{th} term in the expansion of $(\sqrt{x}+\frac{2}{\sqrt{x}})^{12}$.
3. Find and simplify the coefficient of x^5 in the expansion of $(2x^2-\frac{3}{x})^{10}$.
4. Find the term independent of x in the expansion of $(\frac{1}{2x^2}-x)^9$.
5. Find the middle term in the expansion of $(\frac{a}{2}-\frac{b}{3})^{10}$.
6. Find the middle terms in the expansion of $(2x+\frac{1}{x})^7$.
7. If the coefficient of x^4 in the expansion of $(3+2x)^6$ is equal to the coefficient of x^4 in the expansion of $(k+3x)^6$, find k.
8. In the expansion of $(x^2+\frac{a}{x})^8$, $a\neq 0$, the coefficient of x^7 is equal to the coefficient of x^{10}. Find the value of a.

9. Given that the coefficient of x^3 in the expansion of $(a+x)^5+(1-2x)^6$ is -120, calculate the possible values of a.
10. In the expansion of $(2+3x)^n$, the coefficients of x^3 and x^4 are in the ratio $8:15$. Find the value of n.
11. Given that the coefficient of x^2 in the expansion of $(4+kx)(2-x)^6$ is zero, find the value of k.
12. Given that the coefficient of x^2 in the expansion of $(1-ax)^6$ is 60 and that $a>0$, find the value of a.
13. Write down the third and fourth terms in the expansion of $(a+bx)^n$. If these terms are equal, show that $3a=(n-2)bx$.
14. If $(1+ax)^n=1+20x+150x^2+\cdots$, find the value of n and of a.
15. The first three terms in the binomial expansion of $(a+b)^n$, in ascending power of b, are denoted by p, q and r respectively. Show that $\frac{q^2}{pr}=\frac{2n}{n-1}$.
16. Write down the binomial expansion of $(1+2x)^n$ in ascending power of x as far as the term containing x^3. Given that the coefficient of x^3 is twice the coefficient of x^2 and that both are positive, find the value of n.
17. Write down the first three terms in the binomial expansion, in ascending power of x, of $(1+ax)^n$ where $a\neq 0$. Given that the coefficient of x in this expansion is twice the coefficient of x^2. Show that $n=\frac{a+1}{a}$. Find the value of the coefficient of x^2 when $a=\frac{1}{2}$.
18. If the expansion, in ascending powers of x, as far as the term in x^2, of $(2-x)(1+ax)^6$ is $1+bx^2+\cdots$, find the value of a and of b.
19. Find the first three terms, in ascending powers of a, in the expansion of $(1+a)^{10}$. By substituting $a=2x-5x^2$, find the coefficient of x^2 in the expansion of $(1+2x-5x^2)^{10}$.
20. Let $a=2-\sqrt{2}$. Using the binomial expansion, express a^5 in the form $m+n\sqrt{2}$.
21. Find the binomial expansion of $(x+2)^3$. Hence find the exact value of $(2.001)^3$.

Chapter 3

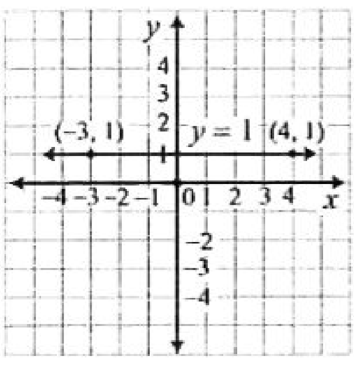
Elementary Functions and Transformations

In Grade 10, elementary functions: linear function, quadratic function, absolute value function, square root function, and rational function, were introduced. Students also studied graphs of these functions. This chapter contains, first, a collection of the simplest forms of these elementary functions, including cubic function and cube root function, and their graphs. After that, transformations of these functions appear as a continuous study of those of quadratic functions and absolute value functions from Grade 10.

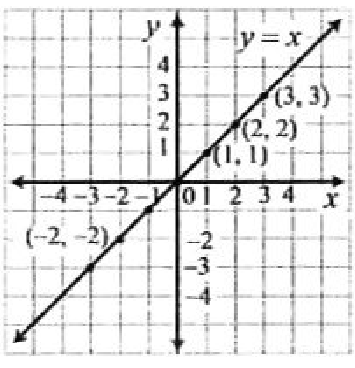
3.1 Elementary Functions

The following are graphs of the elementary functions:

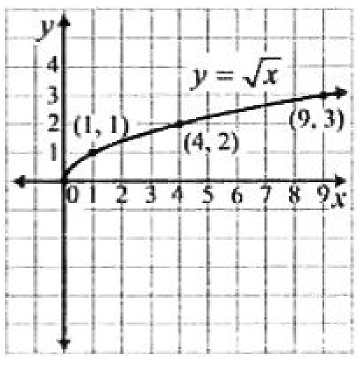
$$y = 1,\ y = x,\ y = \sqrt{x},\ y = x^2,\ y = |x|,\ y = \frac{1}{x},\ y = x^3 \text{ and } y = \sqrt[3]{x}.$$



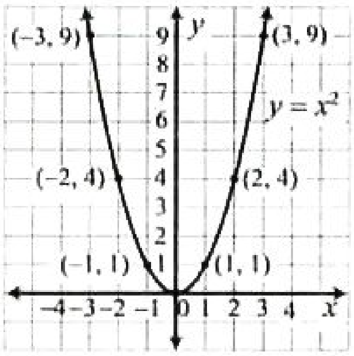
Graph of $y = 1$



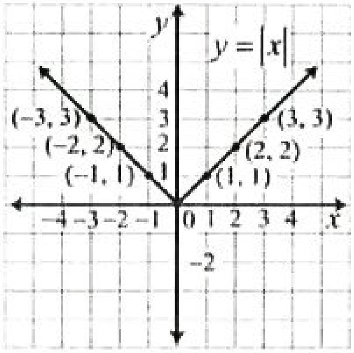
Grpah of $y = x$



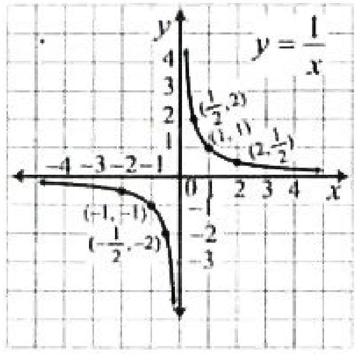
Graph of $y = \sqrt{x}$



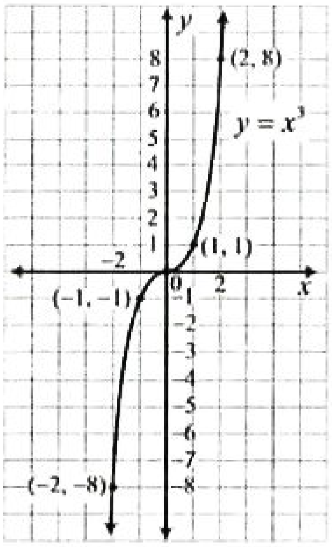
Graph of $y = x^2$



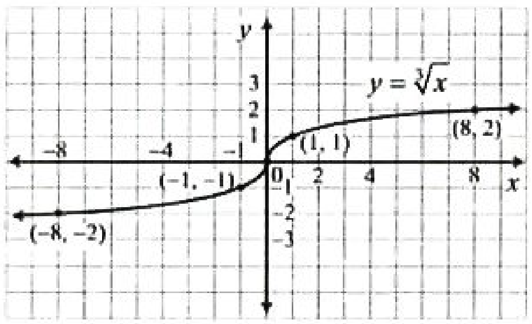
Grpah of $y = |x|$



Graph of $y = \frac{1}{x}$



Graph of $y = x^3$



Graph of $y = \sqrt[3]{x}$

3.2 Transformations

Let us consider three types of transformations: translation, reflection, and scaling.

Translations

Students already learned translations of quadratic functions and absolute value functions in Grade 10. Now we consider translations of arbitrary functions.

Let $y = f(x)$ be a given function.

Vertical translation: The graph of $\boxed{y = f(x) + k}$ is translation vertically, k units, up (when $k > 0$) or down (when $k < 0$) of the graph of $y = f(x)$ as

$$(x, y) \to (x, y + k)$$

since if (x, y) is on the graph of $y = f(x)$ then $(x, y+k)$ is on the graph of $y = f(x)+k$.

Horizontal translation: The graph of $\boxed{y = f(x-h)}$ is translation horizontally, h units, right (when $h > 0$) or left (when $h < 0$) of the graph of $y = f(x)$ as

$$(x, y) \to (x + h, y)$$

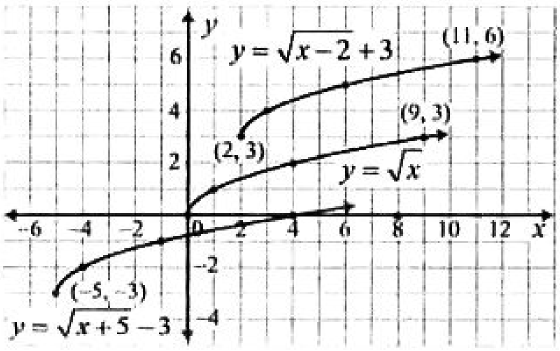
since if (x, y) is on the graph of $y = f(x)$ then $(x+h, y)$ is on the graph of $y = f(x-h)$.

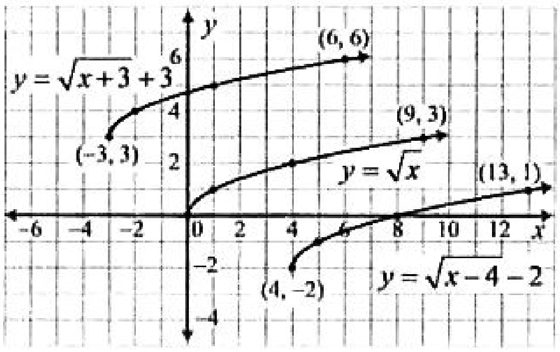
Translation: The graph of $\boxed{y = f(x-h)+k}$ is translation of h units horizontally and k units vertically of the graph of $y = f(x)$ as

$$(x, y) \to (x + h, y + k)$$

since if (x, y) is on the graph of $y = f(x)$ then $(x + h, y + k)$ is on the graph of $y = f(x - h) + k$.

The following are translations of the graph of $y = \sqrt{x}$.





Reflections

We will consider three types of reflections: reflection on the x-axis, reflection on the y-axis and reflection on the origin.

Reflection on the x-axis: The graph of $\boxed{y = -f(x)}$ is the reflection on the x-axis of the graph of $y = f(x)$ as

$$(x, y) \to (x, -y)$$

since if (x, y) is on the graph of $y = f(x)$ then $(x, -y)$ is on the graph of $y = -f(x)$.

Reflection on the y-axis: The graph of $\boxed{y = f(-x)}$ is the reflection on the y-axis of the graph of $y = f(x)$ as

$$(x, y) \to (-x, y)$$

since if (x, y) is on the graph of $y = f(x)$ then $(-x, y)$ is on the graph of $y = f(-x)$.

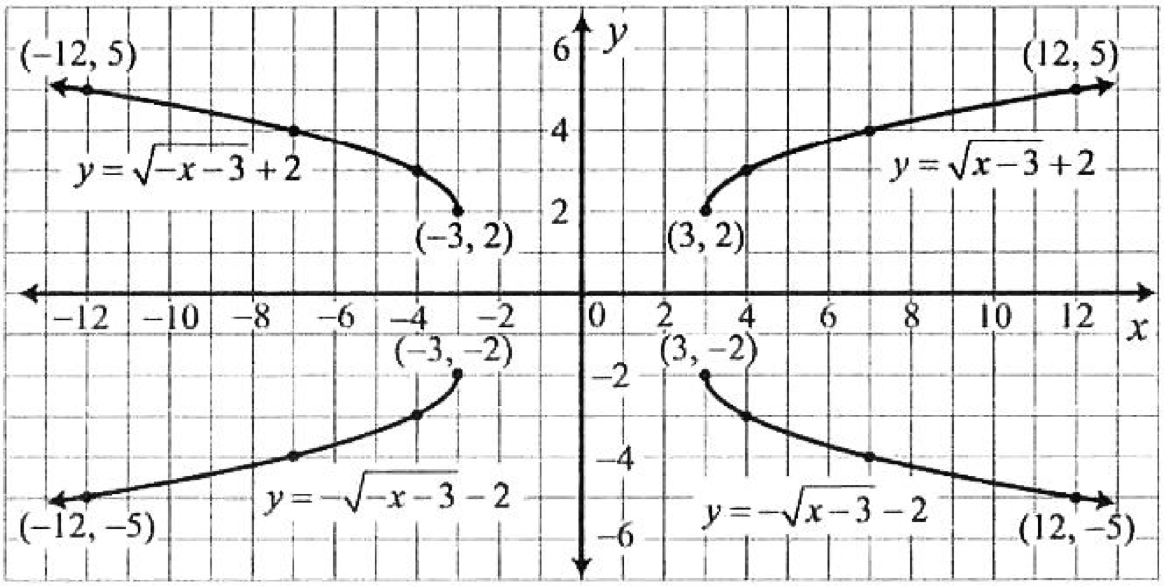
Reflection on the origin: The graph of $\boxed{y = -f(-x)}$ is the reflection on the origin of the graph of $y = f(x)$ as

$$(x, y) \to (-x, -y)$$

since if (x, y) is on the graph of $y = f(x)$ then $(-x, -y)$ is on the graph of $y = -f(-x)$.

For the graph of $y = \sqrt{x - 3} + 2$,

- reflection on the x-axis is the graph of $y = -\sqrt{x - 3} - 2$
- reflection on the y-axis is the graph of $y = \sqrt{-x - 3} + 2$
- reflection on the origin is the graph of $y = -\sqrt{-x - 3} - 2$



Note that for the graph of the function $y = f(x)$, the reflection on the origin can be seen as 2-step transformation as follows:

$$y = f(x) \xrightarrow[\text{on } x\text{-axis}]{\text{reflection}} y = -f(x) \xrightarrow[\text{on } y\text{-axis}]{\text{reflection}} y = -f(-x)$$

or

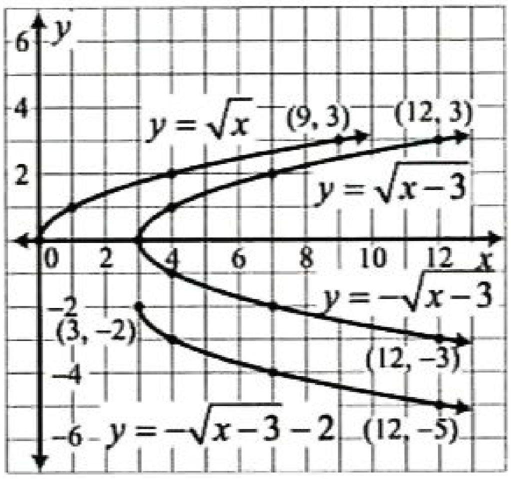
$$y = f(x) \xrightarrow[\text{on } y\text{-axis}]{\text{reflection}} y = f(-x) \xrightarrow[\text{on } x\text{-axis}]{\text{reflection}} y = -f(-x)$$

From the graph of $y = \sqrt{x}$, using translations and reflections,

- the graph of $y = -\sqrt{x-3} - 2$ can be obtained as follows:

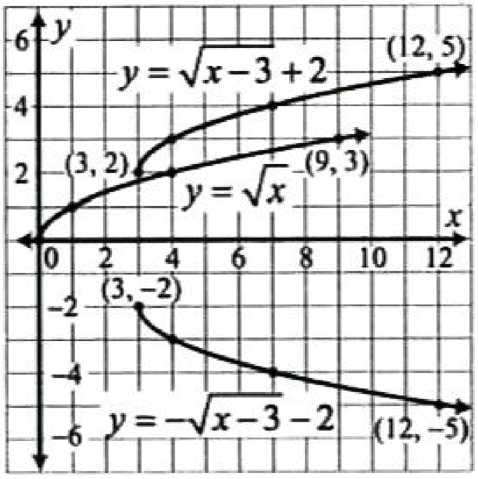
$$\sqrt{x} \xrightarrow[\text{translation } (h=3,k=0)]{f(x-3)} \sqrt{x-3} \xrightarrow[\text{reflection on } x\text{-axis}]{-f(x-3)} -\sqrt{x-3}$$

$$\xrightarrow[\text{translation } (h=0,k=-2)]{-f(x-3)-2} -\sqrt{x-3} - 2$$



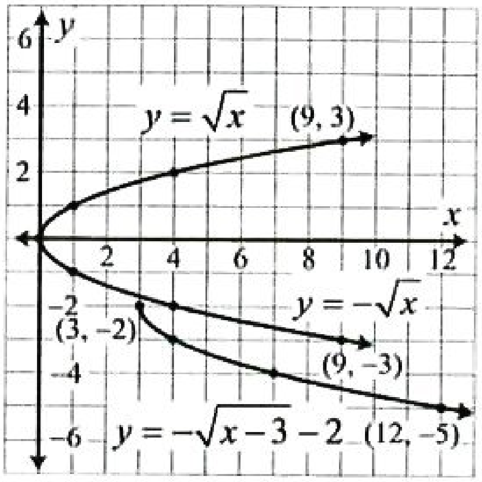
or

$$\sqrt{x} \xrightarrow[\text{translation } (h=3,k=2)]{f(x-3)+2} \sqrt{x-3} + 2 \xrightarrow[\text{reflection on } x\text{-axis}]{-f(x-3)-2} -\sqrt{x-3} - 2$$



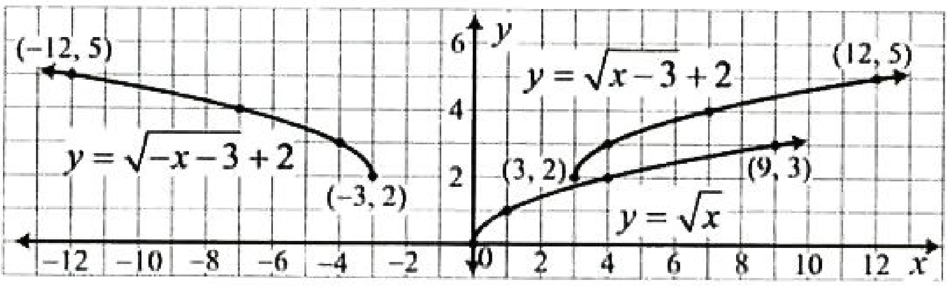
or

$$\sqrt{x} \xrightarrow[\text{reflection on } x\text{-axis}]{-f(x)} -\sqrt{x} \xrightarrow[\text{translation } (h=3,k=-2)]{-f(x-3)-2} -\sqrt{x-3} - 2$$



- the graph of $y=\sqrt{-x-3}+2$ can be obtained as

$$\sqrt{x} \xrightarrow[\text{translation } (h=3,k=2)]{f(x-3)+2} \sqrt{x-3}+2 \xrightarrow[\text{reflection on } y\text{-axis}]{f(-x-3)+2} \sqrt{-x-3}+2$$



Even functions: If $\boxed{f(x)=f(-x)}$ then the function f is called an **even function**. So the graph of an even function is symmetric with respect to the y-axis. $f(x)=x^2$ is an even function, because $f(-x)=(-x)^2=x^2=f(x)$.

Odd functions: If $\boxed{f(x)=-f(-x)}$ then the function f is called an **odd function**. So the graph of an odd function is symmetric with respect to the origin. $f(x)=x^3$ is an odd function, because $-f(-x)=-(-x)^3=-(-x^3)=x^3=f(x)$.

Scalings

Vertical scalings and horizontal scalings of the graphs of the function $y=f(x)$ are considered in this section.

Vertical scaling: For $p > 0$, the graph of $\boxed{y = pf(x)}$ is the vertical scaling of the graph of $y = f(x)$ as

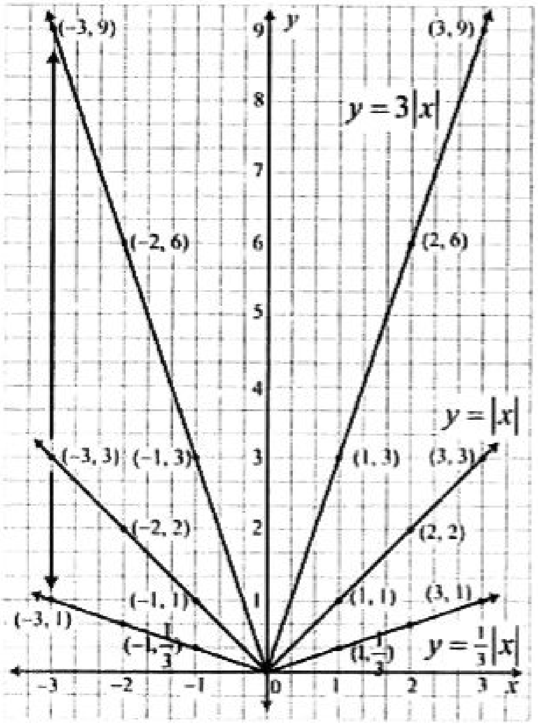
$$(x, y) \to (x, py)$$

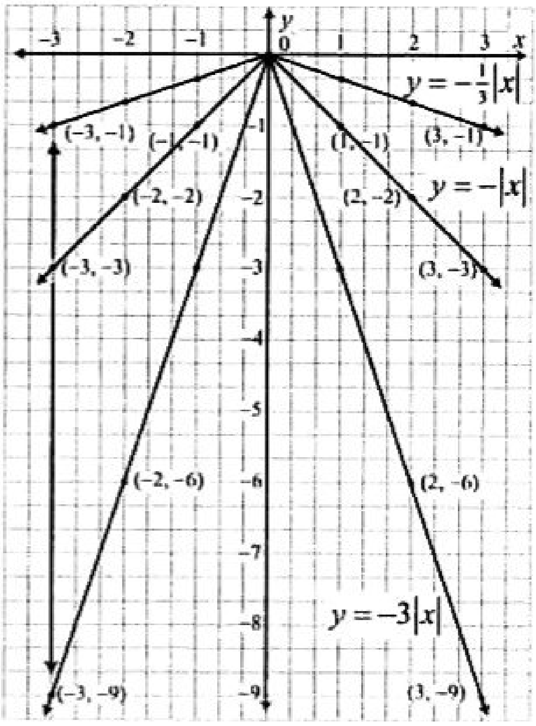
since if (x, y) is on the graph of $y = f(x)$ then (x, py) is on the graph of $y = pf(x)$.

When $0 < p < 1$, the graph of the function $y = pf(x)$ can be seen as points on the graph of $y = f(x)$ move nearer vertically by scale factor p to the x-axis.

When $p > 1$, the graph of the function $y = pf(x)$ can be seen as points on the graph of $y = f(x)$ move away vertically by scale factor p from the x-axis.

The following figures show vertically scaling graphs of the functions $y = |x|$ and $y = -|x|$.





Note that from the graph of $y = f(x)$, the graph of $y = -pf(x), p > 0$ can be obtained as

$$y = f(x) \xrightarrow[\text{on } x\text{-axis}]{\text{reflection}} y = -f(x) \xrightarrow[\text{scale factor } p]{\text{vertical scaling}} y = -pf(x)$$

or

$$y = f(x) \xrightarrow[\text{scale factor } p]{\text{vertical scaling}} y = pf(x) \xrightarrow[\text{on } x\text{-axis}]{\text{reflection}} y = -pf(x)$$

Horizontal scaling: For $q > 0$, the graph of $\boxed{y = f(qx)}$ is the horizontal scaling of the graph of $y = f(x)$ as

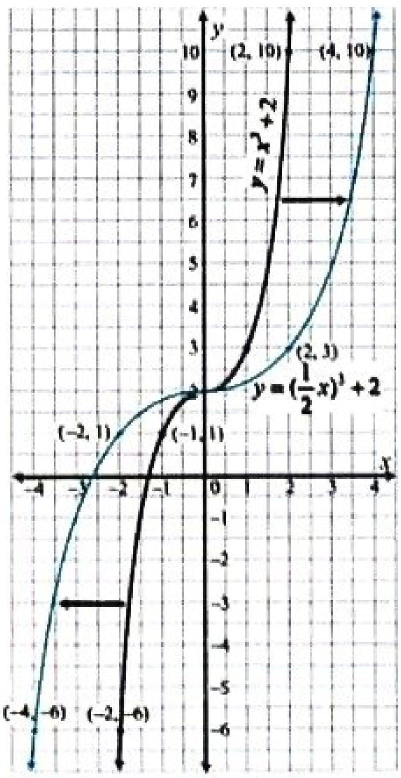
$$(x, y) \to (\frac{1}{q}x, y)$$

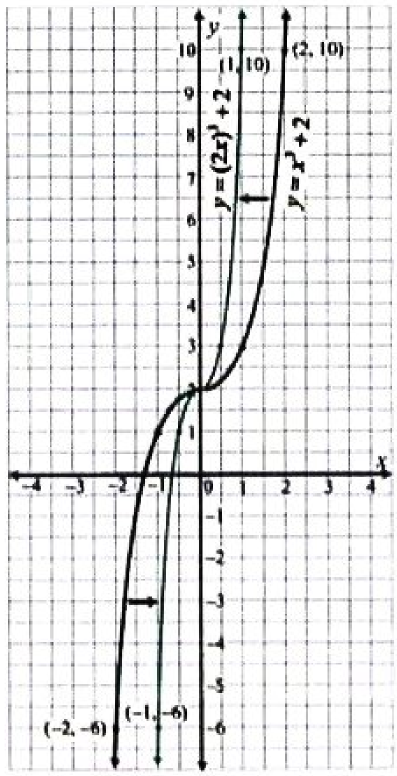
since if (x, y) is on the graph of $y = f(x)$ then $(\frac{1}{q}x, y)$ is on the graph of $y = f(qx)$.

> When $0 < q < 1$, the graph of the function $y = f(qx)$ can be seen as points on the graph of $y = f(x)$ move away horizontally by scale factor $\frac{1}{q}$ from the y-axis.

> When $q > 1$, the graph of the function $y = f(qx)$ can be seen as points on the graph of $y = f(x)$ move nearer horizontally by scale factor $\frac{1}{q}$ to the y-axis.

The following figures show horizontally scaling graphs of the function $y = x^3 + 2$.





Note that from the graph of $y = f(x)$, the graph of $y = f(-qx), q > 0$ can be obtained as

$$y = f(x) \xrightarrow[\text{on } y\text{-axis}]{\text{reflection}} y = f(-x) \xrightarrow[\text{scale factor } \frac{1}{q}]{\text{horizontal scaling}} y = f(-qx)$$

or

$$y = f(x) \xrightarrow[\text{scale factor } \frac{1}{q}]{\text{horizontal scaling}} y = f(qx) \xrightarrow[\text{on } y\text{-axis}]{\text{reflection}} y = f(-qx)$$

The graph of $y = af(bx + c) + d$ is combining all transformations—translation, reflection, and scaling, of the graph of $y = f(x)$ as

$$(x, y) \to (\frac{1}{b}(x - c), ay + d)$$

since if (x, y) is on the graph of $y = f(x)$ then $(\frac{1}{b}(x - c), ay + d)$ is on the graph of $y = af(bx + c) + d$.

Example 1.

From the function $y = f(x)$,

(a) find the step-by-step transformation of $y = f(3x - 3) + 2$.

(b) If the point (x, y) is on the graph of $y = f(x)$, then find the respective point on the graph of $y = f(3x - 3) + 2$.

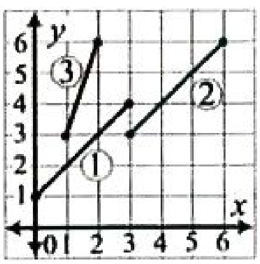
(c) Hence if $f(x) = x + 1,\ 0 \le x \le 3$, then draw step-by-step transformation graphs to get the graph of $y = f(3x - 3) + 2$.

Solution

(a) $y = f(x) \xrightarrow[h = 3, k = 2]{\text{translation}} y = f(x - 3) + 2 \xrightarrow[\text{scale factor} \frac{1}{3}]{\text{horizontal scaling}} y = f(3x - 3) + 2$

(b) $(x, y) \xrightarrow{f(x-3)+2} (x + 3, y + 2) \xrightarrow{f(3x-3)+2} (\frac{x + 3}{3}, y + 2)$

(c) $f(x) = x + 1,\ 0 \le x \le 3$



Example 2.

From the function $y = f(x)$,

(a) find the step-by-step transformation of $y = f(-\frac{1}{2}x + 3) - 2$.

(b) If the point (x, y) is on the graph of $y = f(x)$, then find the respective point on the graph of $y = f(-\frac{1}{2}x + 3) - 2$.

(c) Hence if $f(x) = x^2$, then draw step-by-step transformation graphs to get the graph of $y = (-\frac{1}{2}x + 3)^2 - 2$.

Solution

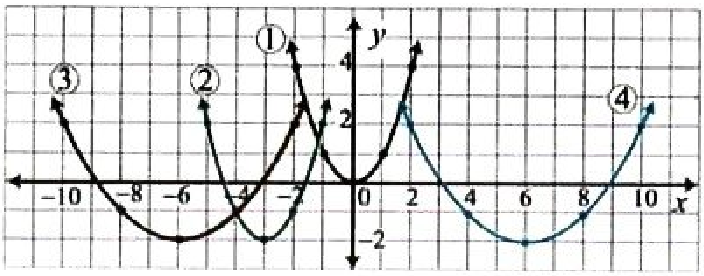
(a) $y = f(x) \xrightarrow[h=-3,\, k=-2]{\text{translation}} y = f(x+3) - 2 \xrightarrow[\text{scale factor 2}]{\text{horizontal scaling}} y = f(\frac{1}{2}x + 3) - 2$

$\xrightarrow[\text{on } y\text{-axis}]{\text{reflection}} y = f(-\frac{1}{2}x + 3) - 2$

(b) $(x, y) \xrightarrow{f(x+3)-2} (x-3, y-2) \xrightarrow{f(\frac{1}{2}x+3)-2} (2(x-3), y-2)$

$\xrightarrow{f(-\frac{1}{2}x+3)-2} (-2(x-3), y-2)$

(c)



Example 3.

Starting from the graph of $f(x) = \sqrt[3]{x}$, $-8 \le x \le 8$,

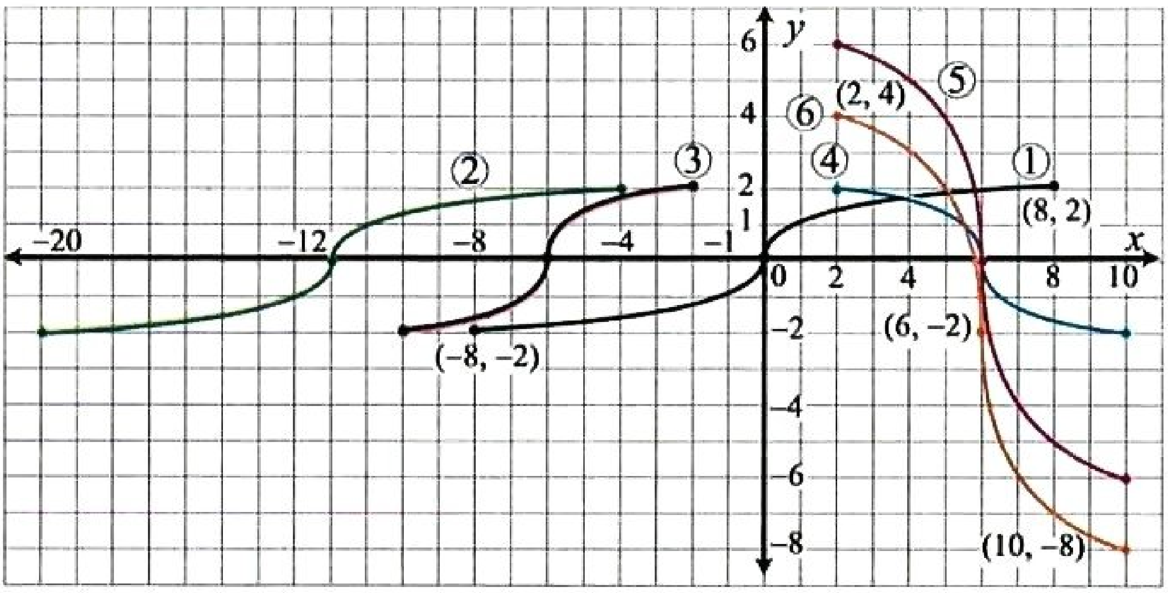
(a) draw step-by-step transformation graphs to get the graph of

$$f(x) = 3\sqrt[3]{-2x + 12} - 2.$$

(b) Check your answer by finding the respective points on the graph of $f(x) = 3\sqrt[3]{-2x + 12} - 2$ for the points $(0, 0), (8, 2), (-8, -2)$ on the graph of $f(x) = \sqrt[3]{x}$.

Solution

(a) $\sqrt[3]{x} \xrightarrow{f(x+12)} \sqrt[3]{x+12} \xrightarrow{f(2x+12)} \sqrt[3]{2x+12} \xrightarrow{f(-2x+12)} \sqrt[3]{-2x+12}$
$\xrightarrow{3f(-2x+12)} 3\sqrt[3]{-2x+12} \xrightarrow{3f(-2x+12)-2} 3\sqrt[3]{-2x+12}-2$



(b) $(0,0) \xrightarrow{f(x+12)} (-12,0) \xrightarrow{f(2x+12)} (-6,0) \xrightarrow{f(-2x+12)} (6,0) \xrightarrow{3f(-2x+12)} (6,0)$
$\xrightarrow{3f(-2x+12)-2} (6,-2)$

$(8,2) \xrightarrow{f(x+12)} (-4,2) \xrightarrow{f(2x+12)} (-2,2) \xrightarrow{f(-2x+12)} (2,2) \xrightarrow{3f(-2x+12)} (2,6)$
$\xrightarrow{3f(-2x+12)-2} (2,4)$

$(-8,-2) \xrightarrow{f(x+12)} (-20,-2) \xrightarrow{f(2x+12)} (-10,-2) \xrightarrow{f(-2x+12)} (10,-2)$
$\xrightarrow{3f(-2x+12)} (10,-6) \xrightarrow{3f(-2x+12)-2} (10,-8)$

Exercise 3.1

1. From the function $y = f(x)$, find step-by-step transformation of
   (a) $y = 2f(x-3)+2$ (b) $y = f(-2x+3)-2$ (c) $y = -3f(\frac{1}{3}x-2)+1$

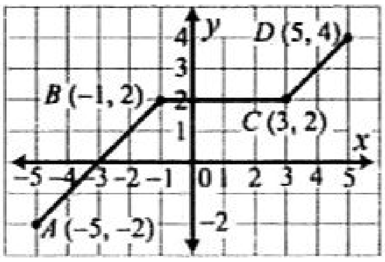
2. From the function $y = \frac{1}{x}$, find step-by-step transformation of
   (a) $y = \frac{1}{x+1}+3$ (b) $y = \frac{2}{x+1}-1$ (c) $y = -\frac{2}{3x-2}+5$

3. Let $y = f(x)$ be a given function. Match the following transformations to corresponding expressions.

(a) $y = f(x+3)$	i. horizontal scaling with scale factor 3
(b) $y = f(x)+3$	ii. 3 units vertical translation
(c) $y = f(-x)$	iii. vertical scaling with scale factor 3
(d) $y = -f(x)$	iv. -3 units horizontal translation
(e) $y = f(3x)$	v. reflection on y-axis
(f) $y = 3f(x)$	vi. reflection on x-axis
(g) $y = \frac{1}{3}f(x)$	vii. vertical scaling with scale factor $\frac{1}{3}$
(h) $y = f(\frac{1}{3}x)$	viii. horizontal scaling with scale factor $\frac{1}{3}$

4. Starting from the graph of $f(x) = x^3, -2 \leq x \leq 2$, draw step-by-step transformation graphs to get the graph of $f(x) = -2(\frac{1}{3}x - 1)^3 + 2$. Check your answer by finding the respective points on the graph of $f(x) = -2(\frac{1}{3}x - 1)^3 + 2$ for the points $(0,0), (2,8), (-2,-8)$ on the graph of $f(x) = x^3$.

5. The following figure shows the graph of a function $y = f(x)$.



Draw step-by-step transformation graphs to get the graph of

(a) $y = \frac{1}{2}f(x-3)+1$ (b) $y = f(-2x+1)-2$

(c) $y = 2f(\frac{1}{3}x - 2)+2$ (d) $y = -2f(2x-1)-1$

Check the answers by finding the respective points on the graphs of above transformations for the points $A(-5,-2), B(-1,2), C(3,2)$ and $D(5,4)$ on the given graph of $y = f(x)$.

6. Determine the following functions are even or odd or neither or both.

(a) $y = x^3 - 2x$ (b) $y = 3x^2 + 2$ (c) $y = 0$ (d) $y = 1$

(e) $y = 2x - 1$ (f) $y = 3|x| - 2$ (g) $y = \sqrt[3]{-x^3 - x}$

Chapter 4

Sequences and Series

In this chapter we introduce to the notion of a sequence and its related series. Then you will learn a particular types of sequence known as arithmetic progression, geometric progression and corresponding series.

4.1 Introduction to Sequences and Series

Basically, a sequence is a list of numbers placed in a specified order. The numbers in the sequence are called the terms. For example, $3, 6, 9, 12, 15, \ldots$ is a sequence in which the first term is 3, the second term is 6, and so on.

Sequence

A **sequence** is a function whose domain is a set of natural numbers or some subsets of the type $\{1, 2, 3, \ldots, k\}$ and the numbers in the range of function are called the **terms** of a sequence.

Sequences are written in subscript notation instead of function notation and the subscripts denote the position of the term. We use the notation u_n instead of $u(n)$ for the value of the function corresponding to the number n of the domain. For example, let domain A $= \{1, 2, 3, 4, 5, 6\}$ and if we define $u_n = 2n$, then the range of the function is $\{2, 4, 6, 8, 10, 12\}$.

Domain	n:	1	2	3	4	5	6
Range	u_n:	2 ↑ 1st term	4 ↑ 2nd term	6	8 ...	10	12 ↑ last term

Here $u_1 = 2, u_2 = 4, u_3 = 6, u_4 = 8, u_5 = 10, u_6 = 12$.
Then we get the sequence 2, 4, 6, 8, 10, 12.

u_n denotes the nth term or **general term** of a given sequence.

A sequence containing finite number of terms is called a **finite sequence**, otherwise it is called an **infinite sequence**. Here are some examples of finite sequences:

(i) $4, 8, 12, 16, 20, 24$ (ii) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}$ (iii) $1, 3, 5, 7, \ldots, 29$

The followings are some examples of infinite sequences:

(i) $1, 4, 9, 16, 25, \ldots$ (ii) $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$ (iii) $1, -2, 4, -8, 16, \ldots$

Example 1.

Find the first four terms of the sequence whose general term is $u_n = 2n + 1$.

Solution

$$\begin{aligned} u_n &= 2n + 1 \\ u_1 &= 2 \times 1 + 1 = 3 \\ u_2 &= 2 \times 2 + 1 = 5 \\ u_3 &= 2 \times 3 + 1 = 7 \\ u_4 &= 2 \times 4 + 1 = 9 \end{aligned}$$

Hence the first four terms are $3, 5, 7, 9$.

If we know the general term of a sequence, then we can write down the sequence. Moreover, we will be able to find any term in the sequence simply by replacing n with the number that represents the term we want. From the above example, the 10th term of given sequence is 21, since $u_{10} = 2 \times 10 + 1 = 21$.

If we know a sufficient number of terms and the terms of sequence have a certain pattern, we may be able to write a rule for that nth term of the sequence. For example, consider the sequence $7, 11, 15, 19, 23, \ldots$.

The sequence may be rewritten as $(4 \times 1) + 3, (4 \times 2) + 3, (4 \times 3) + 3, (4 \times 4) + 3, \ldots$ Therefore the nth term rule is $u_n = 4n + 3$.

We may also be able to describe a recursive rule for a sequence by computing term by term. From the above example, a sequence starts with 7 and each term add 4 to get the next term. Then $u_1 = 7$,

$$\begin{aligned} u_2 &= u_1 + 4 = 11 \\ u_3 &= u_2 + 4 = 15 \\ u_4 &= u_3 + 4 = 19 \\ u_5 &= u_4 + 4 = 23 \end{aligned}$$

and so on. Hence we get the recursive rule $u_n = u_{n-1} + 4$.

A sequence may also be described by the rule of formation being defined recursively using one or more of the previous term of the sequence.

Example 2.

Find the sequence whose first term is 2 and $u_n = 2u_{n-1} + 1$.

Solution

$$\begin{aligned} u_1 &= 2 \\ u_n &= 2u_{n-1} + 1 \\ u_2 &= 2u_1 + 1 = 5 \\ u_3 &= 2u_2 + 1 = 11 \\ u_4 &= 2u_3 + 1 = 23 \end{aligned}$$

and so on. Therefore the required sequence is $2, 5, 11, 23, \ldots$.

Example 3.

For each sequence describe the pattern and write a rule for the nth term.

(i) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ (ii) $2, 6, 12, 20, \ldots$

Solution

(i) The sequence may be rewritten as $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$\therefore u_n = \frac{1}{n}$

(ii) The sequence may be rewritten as $1 \times (1+1), 2 \times (2+1), 3 \times (3+1), 4 \times (4+1), \ldots$

$\therefore u_n = n(n+1)$.

Notice that if the first few terms of a sequence are given, then it cannot be sure that the sequence will continue on forever with the same pattern. For example, consider the sequence 2, 4, 8, ...

There may be $u_n = 2^n$ or $u_n = n^2 - n + 2$.

Hence if the first few terms only are given, then there may be many rules for the nth term of the sequence.

Series

The indicated sum $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ of the terms in a sequence

$$u_1, u_2, u_3, \ldots, u_n, \ldots$$

is called a **series**. A series can be finite or infinite. For example, given a finite sequence:

5, 7, 9, 11, ..., $2n + 3$, we may form a series $5 + 7 + 9 + 11 + \ldots + (2n + 3)$.

Exercise 4.1

1. Find the first four terms of the sequences given by the rule of formation

(a) $u_n = n + 1$
(b) $u_n = n^2$
(c) $u_n = n^3 - 1$
(d) $u_n = n(n + 1)$
(e) $u_n = (-\frac{1}{3})^n$
(f) $u_n = \frac{n}{n+1}$

2. Write down the next two terms of each of the following sequences and determine the nth term of each sequence.

(a) $1, 5, 9, 13, \ldots$
(b) $4, 7, 12, 19, \ldots$
(c) $-1, -8, -27, -64, \ldots$
(d) $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$
(e) $\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{3}{2}, \frac{3\sqrt{3}}{2}, \ldots$
(f) $-6, 12, -18, 24, \ldots$

3. In each case below an initial term and a recursion formula are given. Find u_n.

(a) $u_1 = 2, \quad u_n = u_{n-1} - n$
(b) $u_1 = 5, \quad u_n = 4u_{n-1}$
(c) $u_1 = 1, \quad u_n = u_{n-1} + 9(n + 1)$
(d) $u_1 = -3, \quad u_n = 2u_{n-1} + 5$

4. (a) Write down the first four terms of the sequence defined by $u_n = 5n + 2$.
(b) Which term of the sequence is 247?

5. (a) Write down the first four terms of the sequence defined by $u_n = 2^n - 1$.
(b) Which term of the sequence is 1023?

6. A culture of bacteria doubles in number every hour. If there were originally ten bacteria in the culture, how many will there be after two hours? Four hours? n hours?

7. A boy begins training by running 5m on first day. The second day he runs a total of 8m. The third day he runs a total of 11m and so on in the same increasing pattern. How far will he run in the nth day?

4.2 Arithmetic Progression (A.P.)

An **Arithmetic progression** or **Arithmetic sequence** is a sequence in which the difference between any two consecutive terms is constant. This constant is called the **common difference** of the progression and it is denoted by d. For example, the following sequences are arithmetic progressions:

(i) $3, 6, 9, 12, \ldots$ (ii) $1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \ldots$

The common difference for (i) is 3 and for (ii) is $-\frac{1}{6}$.

Since $u_2 - u_1 = u_3 - u_2 = u_4 - u_3 = \ldots = u_{n+1} - u_n = \ldots = d$, if we denote $u_1 = a$, then

$$u_2 = u_1 + d = a + d$$
$$u_3 = u_2 + d = a + d + d = a + 2d$$
$$u_4 = u_3 + d = a + 2d + d = a + 3d$$
$$u_5 = u_4 + d = a + 3d + d = a + 4d \text{ and so on.}$$

In general, the n^{th} **term** of an A.P. is given by

$$\boxed{u_n = a + (n-1)d}$$

where a is the first term and d is the common difference.

Example 4.

Find the 25^{th} term of the following arithmetic progressions:

(i) $4, 7, 10, 13, \ldots$ (ii) $1, \frac{1}{2}, 0, -\frac{1}{2}, -1, \ldots$

Solution

(i) In the A.P. $4, 7, 10, 13, \ldots$

$$a = 4, d = 3, n = 25$$
$$u_{25} = a + 24d = 4 + 72 = 76$$

Therefore the 25^{th} term is 76.

(ii) In the A.P. $1, \frac{1}{2}, 0, -\frac{1}{2}, -1, \ldots$

$$a = 1, d = -\frac{1}{2}, n = 25$$
$$u_{25} = a + 24d = 1 - 12 = -11$$

Therefore the 25^{th} term is -11.

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Example 5.

The first three terms of an arithmetic progression are $x+5, 3x-5$, and $2x-3$ respectively. Find the numerical value of x and the 10th term of the progression.

Solution

In the A.P. $x+5, 3x-5, 2x-3$,

$$\begin{aligned}(3x-5)-(x+5) &= (2x-3)-(3x-5)\\ 2x-10 &= -x+2\\ x &= 4.\end{aligned}$$

Hence the sequence is 9, 7, 5, ...

$a=9, d=-2$

$u_{10}=a+9d=9-18=-9$

Therefore the 10th term is -9.

Example 6.

The eighth term of an arithmetic progression is three times the third term and the twentieth term is 78. Find the first five terms of this progression.

Solution

$$u_8 = 3u_3$$

$$a+7d = 3(a+2d)$$

$$\therefore d = 2a. \qquad (1)$$

Since $u_{20} = 78$,

$$a+19d = 78$$

$$a+38a = 78$$

$$a = 2.$$

Substitute $a=2$ in equation (1), we get $d=4$.

Therefore the first five terms of the A.P. are $2, 6, 10, 14, 18$.

Example 7.

How many terms are there in the arithmetic progression whose first term is 8, common difference 9 and the last term is 890.

Solution

$$a=8,\ d=9.$$

Let n be the number of terms.

$$\begin{aligned} u_n &= 890 \\ a+(n-1)d &= 890 \\ 8+(n-1)9 &= 890 \\ 9n &= 891 \\ n &= 99 \end{aligned}$$

Therefore there are 99 terms in the given A.P.

Example 8.

Find the first 5 terms of A.P. given that the first term is 11 and the last term is -1.

Solution

$11, x_1, x_2, x_3, -1$ is an A.P. with $a = 11$, $u_5 = -1$.

$$\begin{aligned} \therefore a+4d &= -1 \\ 11+4d &= -1 \\ 4d &= -12 \\ d &= -3 \end{aligned}$$

$\therefore x_1 = 8, x_2 = 5, x_3 = 2$

Hence the arithmetic progression is $11, 8, 5, 2, -1$.

Arithmetic Mean (A.M.)

In a finite arithmetic progression,

> the **arithmetic mean** of two numbers x and $y = \dfrac{x+y}{2}$.

Then $x, \dfrac{x+y}{2}, y$ is an arithmetic progression.

Exercise 4.2

1. In each of the following A.P. find
   (a) the common difference (b) the 10th term (c) the nth term.

   (i) $1, 3, 5, 7, \ldots$ (ii) $1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \ldots$

   (iii) $1, 2\frac{1}{2}, 4, 5\frac{1}{2}, \ldots$ (iv) $11, 8, 5, 2, \ldots$

2. How many terms are there in each of the following A.P.

   (a) $2, 4, 6, 8, \ldots, 64$ (b) $18, 13, 8, \ldots, -102$

3. For a given sequence in which $u_1 = -2$ and $u_n = u_{n-1} + 3$, for $n \geq 2$. Find u_n in terms of n and hence calculate the 21^{st} term.

4. The first four terms of an A.P. are $3, x - y, 2x + y + 3, x - 3y$. Find

   (a) the values of x and y.

   (b) the numerical value of the 10th term of the A.P.

5. If the first, eighth and last terms of an A.P. are 4, 32 and 400 respectively, find the number of terms of the A.P.

6. The first row in a movie theater has 15 seats. Each subsequent row has 2 seats more than the row in front of it. If the last row has 55 seats, how many rows are there?

7. The eighth term of an A.P. is 38 and the sixteenth term is 118. Which term has the value 458?

8. The sixth term of an A.P. is 32 while the 10^{th} term is 48. Find the common difference and the 21^{st} term.

9. The four angles of a quadrilateral are in an A.P. Given that the value of the largest angle is three times the value of the smallest angle, find the values of all four angles.

10. Find the n^{th} term of the A.P. whose third term is 15 and the 8^{th} term exceeds the 4^{th} term by 12.

11. An A.P. contains 25 terms. If the first term is 15 and the last term is 111, find the middle term.

12. If the n^{th} term of an A.P. $2, 3\frac{7}{8}, 5\frac{3}{4}, \ldots$ is equal to the nth term of an A.P. $187, 184\frac{1}{4}, 181\frac{1}{2}, \ldots$, find n.

4.3 Arithmetic Series

The sum of the terms in an arithmetic progression is called an **arithmetic series**. Here are some arithmetic progression and their corresponding series.

arithmetic progression	arithmetic series
4, 7, 10, 13, 16	$4 + 7 + 10 + 13 + 16$
$1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \ldots$	$1 + \frac{5}{6} + \frac{2}{3} + \frac{1}{2} + \cdots$

To develop a formula for the sum of the first n terms of an arithmetic progression, consider the series below.

Let $u_1, u_2, u_3, \ldots, u_n, \ldots$ be given A.P.

Let S_n denote the sum of the first n terms of the A.P. Then

$$S_n = u_1 + u_2 + u_3 + \ldots + u_{n-1} + u_n$$

$$S_n = u_1 + (u_1 + d) + (u_1 + 2d) + \ldots + (u_1 + (n-2)d) + (u_1 + (n-1)d) \quad (1)$$

Writing the A.P. in the reversed order of terms, we have

$$S_n = u_n + (u_n - d) + (u_n - 2d) + \ldots + (u_n - (n-2)d) + (u_n - (n-1)d) \quad (2)$$

Adding equations(1) and (2), we have

$$2S_n = \underbrace{(u_1 + u_n) + (u_1 + u_n) + (u_1 + u_n) + \ldots + (u_1 + u_n) + (u_1 + u_n)}_{n \text{ times}}$$

$$= n(u_1 + u_n)$$

$$S_n = \frac{n}{2}(u_1 + u_n) \quad (3)$$

Substitute $u_1 = a$ and $u_n = a + (n-1)d$ in equation (3), we have

$$S_n = \frac{n}{2}\{a + a + (n-1)d\}$$

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

If we know the first term a and the common difference d of an A.P., then

$$S_n = \frac{n}{2}\{2a + (n-1)d\}.$$

If we know the first term a and last term l of an A.P., then

$$S_n = \frac{n}{2}(a + l).$$

Since $S_n = u_1 + u_2 + u_3 + \ldots + u_{n-1} + u_n$,
if $n = 1$, $S_1 = u_1$
if $n = 2$, $S_2 = u_1 + u_2$
if $n = 3$, $S_3 = u_1 + u_2 + u_3$
if $n = 4$, $S_4 = u_1 + u_2 + u_3 + u_4$
and so on. From the above equations we may get the following relations:

$$u_1 = S_1$$
$$u_2 = S_2 - S_1$$
$$u_3 = S_3 - S_2$$
$$u_4 = S_4 - S_3, \text{ and so on.}$$

From these relations we get the general term as follows:

$$\boxed{u_n = S_n - S_{n-1}, \text{ for } n \geq 2}$$

Example 9.

Find the sum of the first 16 positive odd integers.

Solution

1, 3, 5, ... are positive odd integers.

The sum $= 1 + 3 + 5 + 7 + 9 + 11 + \ldots$ to 16 terms. It is an arithmetic series with $a = 1$, $d = 2$, $n = 16$.

$$\begin{aligned} S_n &= \frac{n}{2}\{2a + (n-1)d\} \\ S_{16} &= \frac{16}{2}\{2 + (16-1)2\} \\ &= 8 \times 32 \\ &= 256. \end{aligned}$$

Example 10.

Find the sum of all two-digit numbers which are divisible by 3.

Solution

Two-digit numbers which are divisible by 3 are 12, 15, 18, ..., 99.

$a = 12$, $d = 3$

Let n be the number of terms.

$$\begin{aligned} \therefore l = u_n &= 99 \\ 12 + (n-1)3 &= 99 \\ (n-1)3 &= 87 \\ n - 1 &= 29 \\ n &= 30. \end{aligned}$$

The required sum is $S_{30} = \frac{30}{2}(a + l) = 15(12 + 99) = 1665.$

Example 11.

If the third term of an A.P. is 5, find the sum of the first five terms of the A.P. Given that the tenth term is 33. Find the first term and the common difference of the A.P.

Solution

$$\begin{aligned} u_3 &= 5 \\ \therefore a + 2d &= 5 \qquad (1) \\ S_5 &= \frac{5}{2}\{2a + 4d\} = 5(a + 2d) = 5 \times 5 = 25 \end{aligned}$$

Since $u_{10} = 33$,

$a + 9d = 33$ (2)

Solving equations (1) and (2), we get $d = 4, \quad a = -3$.

Example 12.

In an arithmetic progression 44, 40, 36, ...

(a) find the sum to first 12 terms.

(b) find the sum from 13^{th} term to 25^{th} term.

Solution

$a = 44, \quad d = -4$

(a) $S_{12} = \frac{12}{2}\{2(44) + (11)(-4)\} = 264.$

(b) Let S be the required sum. Then

$S = S_{25} - S_{12}.$

$$S_{25} = \frac{25}{2}\{2(44) + (25-1)(-4)\} = -100.$$

$$\therefore S = -100 - 264 = -364.$$

Example 13.

An arithmetic series has the sum of the first ten terms is half of the sum of the next ten terms and the 12th term is -15, find the first four terms of the series.

Solution

$$\begin{aligned} S_{10} &= \frac{1}{2}(S_{20} - S_{10}) \\ 2S_{10} &= S_{20} - S_{10} \\ 3S_{10} &= S_{20} \\ 3 \times \frac{10}{2}\{2a + 9d\} &= \frac{20}{2}\{2a + 19d\} \\ 6a + 27d &= 4a + 38d \\ \therefore 2a &= 11d. \qquad (1) \end{aligned}$$

Since $u_{12} = -15,$

$$a + 11d = -15 \qquad (2)$$

By solving equations (1) and (2), we get

$$a = -5,\ d = -\frac{10}{11}.$$

Therefore the first four terms of the A.P. are $-5, -5\frac{10}{11}, -6\frac{9}{11}, -7\frac{8}{11}$.

Example 14.

A circular disc is divided into n sectors such that the angles of the sectors form an arithmetic progression. If the smallest angle is 20° and the largest angle is 40°, calculate n.

Solution

$a = 20^\circ,\ \ l = 40^\circ,\ \ S_n = 360^\circ$

$$\begin{aligned} S_n &= \frac{n}{2}(a + l) \\ 360^\circ &= \frac{n}{2}(20^\circ + 40^\circ) \\ 30^\circ n &= 360^\circ \\ \therefore n &= 12. \end{aligned}$$

Example 15.

If m is a positive integer, show that the sum of the A.P. $2m+1, 2m+3, 2m+5, ..., 4m-1$ is divisible by 3.

Solution

$a = 2m + 1, d = (2m + 3) - (2m + 1) = 2$

Let n be the number of terms.

$$\therefore l = u_n = 4m - 1$$

$$2m + 1 + (n - 1)2 = 4m - 1$$

$$2m + 1 + 2n - 2 = 4m - 1$$

$$\therefore n = m$$

$$S_n = \frac{n}{2}(a + l)$$

$$S_m = \frac{m}{2}\{(2m + 1) + (4m - 1)\}$$

$$= \frac{m}{2}(6m)$$

$$= 3m^2.$$

Since m is positive integers, the sum is divisible by 3.

Exercise 4.3

1. Find the sum of each of the following arithmetic progressions.

   (a) 5, 3, 1, $-$ 1, ... to 18 terms

   (b) 2, 6, 10, 14, ... to 98 terms

   (c) $a - b, a - 2b, a - 3b, \ldots$ to 20 terms

   (d) $-3, -4, -5, -6, \ldots$ to 21 terms

2. The third and sixth terms of an A.P. are 13 and 22 respectively. Find the sum of the first n terms in terms of n.

3. Find the sum of all three-digit numbers which are multiples of 12 but not 15.

4. In a certain A.P. the sum of 35 terms is equal to the sum of the next 15 terms. Given that the 8th term is 35. Find the first term and common difference.

5. Find the sum of all even numbers between 71 and 149.

6. Find the sum of all two-digit natural numbers which are not divisible by 3.

7. In the arithmetic series $5 + 7 + 9 + \ldots$, find the sum of the first n terms and the value of n for which $S_n = 192$.

8. How many terms of an A.P. $24, 20, 16, \ldots$ give a sum of 0?

9. In the arithmetic sequence $-4, -2, 0, 2, 4, \ldots$, find the least number of term so that the sum of the sequence is greater than 300.

10. The sum of first 5 terms and the sum of first 15 terms of an A.P. are 25 and 225 respectively. Find the sum of first n terms.

11. The sixth term of an A.P. is 22 and the tenth term is 34. Find the sum to first 16 terms of the A.P.

12. If the fourth term of an A.P is 9, find the sum of the first 7 terms. Given that the tenth term is 33, calculate the sum from the 8th term to the 15th term of that A.P.

13. The sum of four consecutive numbers in an A.P. is 28. The product of the second and third numbers exceeds that of the first and fourth numbers by 18. Find the numbers.

14. In an A.P. whose first term is -27, the tenth term is equal to the sum of the first 9 terms. Calculate the common difference.

15. A theater has 30 seats in the first row, 34 seats in the second row, 38 seats in the third row, and so on in the same increasing pattern. If 110 seats are in the last row, how many seats are in the theater?

16. The sum of the first n terms of an A.P. is 21. The common difference is 4 and the sum of first $2n$ terms is 78. Find the first term.

17. In an A.P. $5, 9, 13, 17, \ldots$, find the value of n for which the sum of first $2n$ terms will exceed the sum of the first n terms by 234.

18. How many bricks are there in a pile one brick in thickness, if there are 27 bricks in the bottom row, 25 in the second row etc., and 1 in the top row?

19. If there are 256 bricks in a pile arranged in the manner as in problem 18, how many bricks are there in the 3rd row from the bottom of the pile?

20. The sum of the first 4 terms of an A.P. is 26 and the sum of their square is 214. Find the first 4 terms.

21. For a certain A.P. $S_n = \frac{n}{2}(3n - 7)$, calculate S_1, S_2, S_3, S_4. Hence find the first 4 terms of the corresponding sequence and a formula for the nth term.

22. $7, x_1, x_2, \ldots, x_m, -23$ is an A.P. The ratio of x_m to x_1 is $-5 : 1$. Find the value of m.

4.4 Geometric Progression (G.P.)

A Geometric Progression or **Geometric Sequence** is a sequence in which the ratio of each term to the one before it is constant. This ratio is called the **common ratio** and is denoted by r. Here are some examples of geometric progressions

(i) $3, 6, 12, 24, 48, \ldots$ (ii) $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

The common ratio for (i) is 2 and for (ii) is $\frac{1}{2}$.

If the first term of a geometric progression is denoted by a, then

$u_1 = a$. Since $\frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \cdots = \frac{u_{n+1}}{u_n} = \cdots = r$,

$$\begin{aligned} u_2 &= u_1 r = ar \\ u_3 &= u_2 r = ar^2 \\ u_4 &= u_3 r = ar^3 \end{aligned}$$

and so on.

In general, the n^{th} **term** of a G.P. is given by

$$\boxed{u_n = ar^{n-1}}$$

where a is the first term and r is the common ratio.

Example 16.

The second term of a G.P. is 6 and the fifth term is four times the third term. Find the G.P. in which each term is positive.

Solution

$$\begin{aligned} u_2 &= 6 \\ ar &= 6 \qquad (1) \\ u_5 &= 4u_3 \\ ar^4 &= 4ar^2 \\ \frac{ar^4}{ar^2} &= 4 \\ r^2 &= 4 \\ r &= \pm 2 \end{aligned}$$

Since each term of the G.P. is positive, $r = 2$.
Substitute $r = 2$ in equation (1), we get $a = 3$.
Therefore the geometric progression is $3, 6, 12, 24, \ldots$.

Example 17.

In a G.P. whose terms are all positive, the third term exceeds the first term by 24 while the fifth term exceeds the first term by 240. Find the first term.

Solution

$$u_3 - u_1 = 24, \; u_5 - u_1 = 240.$$

Thus, $ar^4 - a = 240$ and $ar^2 - a = 24$

$$a(r^4 - 1) = 240 \text{ and } a(r^2 - 1) = 24$$

$$a(r^2 - 1)(r^2 + 1) = 240$$

$$24(r^2 + 1) = 240$$

$$r^2 + 1 = 10$$

$$r^2 = 9$$

$$r = \pm 3$$

Since all terms of the G.P. are positive, $r = 3$.

Substitute $r = 3$ in $a(r^2 - 1) = 24$

$$a(9 - 1) = 24$$

$$8a = 24$$

$$a = 3$$

Therefore the first term is 3.

Example 18.

The fourth term and last term of a G.P. are 8 and 512 respectively. If the common ratio is -2, find the number of terms.

Solution

$$u_4 = 8, \qquad r = -2$$

$$ar^3 = 8$$

$$a(-2)^3 = 8$$

$$a = -1$$

Let n be the number of terms.

$$u_n = 512$$

$$ar^{n-1} = 512$$

$$(-1)(-2)^{n-1} = 512$$

$$(-2)^{n-1} = -512 = (-2)^9$$

$$n - 1 = 9$$

$$n = 10.$$

Geometric Mean (G.M.)

In a finite geometric progression,

the **geometric mean** of two positive numbers x and y is $\sqrt{xy}$.

Then $x, \sqrt{xy}, y$ and $y, \sqrt{xy}, x$ are geometric progressions.

Example 19.

The A.M. between x and y is 20 and the G.M. between them is 16. Find the value of $x^2 + y^2$.

Solution

A.M. between x and y = 20

$$\frac{x+y}{2} = 20$$

$$\therefore x + y = 40.$$

G.M. between x and y = 16

$$\sqrt{xy} = 16$$

$$\therefore xy = 256.$$

$$\begin{aligned} x^2 + y^2 &= (x^2 + 2xy + y^2) - 2xy \\ &= (x+y)^2 - 2xy \\ &= 40^2 - 2(256) \\ &= 1088. \end{aligned}$$

Exercise 4.4

1. Find (a) the common ratio, (b) the 10^{th} term and (c) the n^{th} term of each of the following geometric progressions.

   (i) $6, 2, \frac{2}{3}, \frac{2}{9}, \ldots$ (ii) $-2, 4, -8, 16, \ldots$

   (iii) $2, 10, 50, 250, \ldots$ (iv) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$

2. Given that $x, 2, y$ are three terms of an A.P. and $2, x, y$ are three terms of a G.P. Find the values of x and y where $x \neq y$.

3. If a number is added to each of the numbers 2, 14 and 50, then the result forms the first three terms of a geometric progression. Find that the number and the fifth term of that geometric progression.

4. The second term of a G.P. is 24 and the fifth term is 192. Find the n^{th} term.

5. The second term of a G.P. is 64 and the fifth term is 27. Find the first six terms of the G.P.

6. If $3, x, y, z, w, 3072$ is a G.P., find the values of x, y, z and w.

7. The fourth term of a G.P. is 3 and the sixth term is 147. Find the first 3 terms of the two possible geometric progressions.

8. The product of the first 3 terms of a G.P. is 1 and the product of the third, fourth and fifth terms is $11\frac{25}{64}$. Find the fifth term of the G.P.

9. Find two numbers inserting them between the roots of the equation $x^2 - 84x + 243 = 0$ forms a geometric progression.

10. If a, b, c, d is a G.P., show that $a^2 - b^2, b^2 - c^2, c^2 - d^2$ is also a G.P.

11. The finite geometric progression has ten terms. The sum of all odd terms is 1023 and the sum of all even terms is 2046. Find the common ratio and the fifth term of the geometric progression.

12. The sum of three numbers in G.P. is 112. If we substract 7, 21, 51 from these numbers in that order, we obtain an A.P. Find the numbers.

13. The ratio of the sum of the first, second and third terms of a geometric progression to the sum of the third, fourth and fifth terms is $4 : 9$. Find the tenth term of the progression if the sixth term is $15\frac{3}{16}$.

14. The length of the sides of a triangle form a G.P. If the shortest side is 9 cm and the perimeter is 37 cm, find the length of the other two sides.

15. In a G.P, the product of three consecutive terms is 512. When 8 is added to the first term and 6 to the second, then the terms form an A.P. Find the terms of the G.P.

16. Given that $6, x$ and y are consecutive terms of a G.P. while x, y and 36 are consecutive terms of an A.P. Find x and y.

17. The ratio of two positive numbers is 9:1. If the sum of the arithmetic mean and geometric mean between the two numbers is 96, find the two numbers.

18. If the arithmetic mean between x and y is 15 and the geometric mean is 9, find x and y.

19. The sum of the first n terms of an arithmetic progression is given by $S_n = n^2 + 2n$. If three terms of this progression u_2, u_m, u_{22} are consecutive terms in a geometric progression, find m.

4.5 Geometric Series

A geometric series is the sum of the terms in a geometric progression.
Here are some geometric progressions and the corresponding series.

geometric progression	geometric series
2, 4, 8, 16, 32	$2+4+8+16+32$
$4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$	$4+2+1+\frac{1}{2}+\frac{1}{4}+\ldots$

Let S_n denote the sum of the n terms of the geometric progression $a, ar, ar^2, \ldots, ar^{n-1}$ where $r \neq 1$. Then

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \qquad (1)$$

When equation (1) is multiplied throughout by r, we get

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n \qquad (2)$$

By substracting equation (2) from equation (1), $S_n - rS_n = (a - ar^n)$

$$S_n = \frac{a(1-r^n)}{1-r}$$

If we know the first term is a and the common ratio is r, then

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}, r \neq 1} \text{ or equivalently, } \boxed{S_n = \frac{a(r^n-1)}{r-1}, r \neq 1}$$

If $r = 1$, then $S_n = \underbrace{a + a + a + \ldots + a}_{n \text{ times}} = na$.

Example 20.

Find the sum of the first 6 terms of the G.P. $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$

Solution

$$a = 4,\ r = \frac{1}{2},\ n = 6$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_6 = \frac{4\{1-(\frac{1}{2})^6\}}{1-\frac{1}{2}} = \frac{63}{8}.$$

Example 21.

Find x if the numbers $x+3, 5x-3$ and $7x+3$ are three consecutive terms of a G.P. of positive numbers. With this value of x and given that $x+3$ is the third term of the G.P., find the sum of the first 8 terms of the progression.

Solution

$x+3, 5x-3,\ 7x+3$ is a G.P.

$$\therefore \frac{5x-3}{x+3} = \frac{7x+3}{5x-3}$$

$$(5x-3)^2 = (x+3)(7x+3)$$

$$25x^2 - 30x + 9 = 7x^2 + 24x + 9$$

$$18x^2 - 54x = 0$$

$$18x(x-3) = 0$$

$$\therefore x = 0 \text{ or } x = 3$$

Since the terms of G.P. are positive, $x = 3$.

$$r = \frac{5x-3}{x+3} = \frac{12}{6} = 2$$

$$u_3 = x+3$$

$$\therefore ar^2 = x+3$$

$$a(2^2) = 3+3 = 6$$

$$a = \frac{6}{4} = \frac{3}{2}$$

$$S_8 = \frac{\frac{3}{2}(2^8-1)}{2-1} = \frac{3}{2}(256-1) = \frac{765}{2}.$$

Exercise 4.5

1. Find the sum of the first 10 terms for each of the following G.P.

   (i) $4, 2, 1, \frac{1}{2}, \ldots$ (ii) $-2, 4, -8, 16, \ldots$

2. If the sum of the first three terms of a G.P. is 21 and the sum of the next three terms is 168, then find the first term and the common ratio.

3. In the geometric sequence $2, 4, 8, 16, 32, \ldots$, find the least number of term so that the sum of the sequence is greater than 2000?

4. If $1 + 3 + 3^2 + 3^3 + \cdots + 3^n = 1093$, find n.

5. Given a sequence in which $u_1 = 1$ and $u_n = \sqrt{2}\, u_{n-1}$, for $n \geq 2$. Find u_n in terms of n and hence calculate the sum of the first 12 terms.

6. The sum of the fourth and sixth terms of a G.P. is 90 and the sum of the seventh and ninth terms is 2430. Find the sum of the first 17 terms of the G.P.

7. Given that $x + 1, x + 5$ and $2x + 4$ are three positive consecutive terms of a geometric progression calculate the value of x and the sum of the first six terms of the progression if $x + 1$ is the third term of the progression.

8. The sum of first two terms of a geometric progression is 10 and the third term exceeds the first term by 15. Find the common ratio and calculate the sum of first five terms.

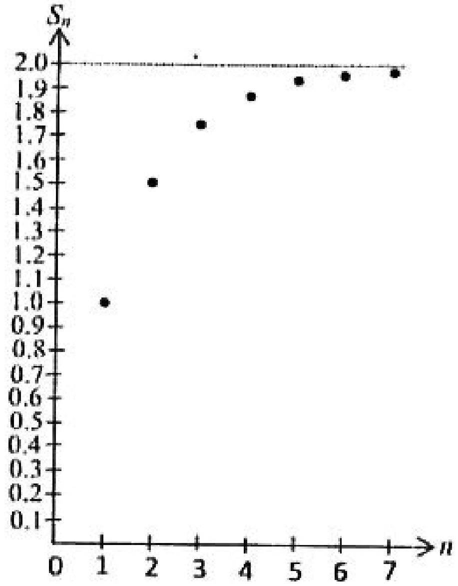
9. The sum of the first 5 terms of a G.P. is 8 and the sum of the terms from the fourth to the eighth inclusive is $15\frac{5}{8}$. Find the common ratio and the sixth term.

10. The sum of the first four terms of a G.P. is 5 times the sum of the first 2 terms. If the fifth term is 256, find the values of the eighth term.

4.6 Infinite Geometric Series

An infinite geometric series is the sum of terms in an infinite geometric progression. If we keep adding terms of an infinite geometric series, the sum becomes increasingly large and do not approach anywhere, then this series is said to be **divergent**. If the sum gets closer and closer to a real number, then we say that series is **convergent**. For example, the following graph shows the values of S_n for a geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ with first term $u_1 = 1$ and common ratio $r = \frac{1}{2}$.

n	S_n
1	1
2	$\frac{3}{2}$ or 1.5
3	$\frac{7}{4}$ or 1.75
4	$\frac{15}{8}$ or 1.875
5	$\frac{31}{16}$ or 1.9375
6	$\frac{63}{32}$ or 1.96875



From the above figure, we can see that n becomes larger and larger, the value of S_n is getting closer and closer to 2. Thus we say that the series converges to 2 and this series is convergent.

A geometric series may converge or diverge depending on the value of common ratio r. To determine this, consider the geomteric progression $a, ar, ar^2, ar^3, \dots$ and the sum of the first n terms

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Case 1. If $|r| < 1$ (i.e., $-1 < r < 1$), then r^n approaches zero and the value of S_n will closer and closer to $\frac{a}{1-r}$ as n gets larger and larger. In this case, $\frac{a}{1-r}$ is called sum to infinity of G.P. and denoted by S.

Case 2. If $|r| \geq 1$, the value of S_n does not approach any number as n increases. In this case, sum to infinity cannot be found.

If $|r| < 1$, then the sum to infinity S exists and

$$S = \frac{a}{1-r}.$$

Example 22.

Determine whether the sum to infinity for each of the following geometric series exists and find the sum to infinity when they exist.

(i) $3 + 0.3 + 0.03 + \ldots$

(ii) $\frac{1}{2} + \frac{2}{3} + \frac{8}{9} + \frac{32}{27} + \ldots$

(iii) $3, -\frac{2}{3} + \frac{4}{27} - \frac{8}{243} + \ldots$

Solution

(i) $r = \frac{0.3}{3} = 0.1$

$\therefore |r| = 0.1 < 1$ and hence the sum to infinity exists.

$$S = \frac{a}{1-r} = \frac{3}{1-0.1} = \frac{10}{3}$$

(ii) $r = \frac{2}{3} \times \frac{2}{1} = \frac{4}{3}$

$|r| = \frac{4}{3} > 1$ and hence the sum to infinity does not exist.

(iii) $r = -\frac{2}{3} \times \frac{1}{3} = -\frac{2}{9}$

$|r| = \frac{2}{9} < 1$ and hence the sum to infinity exists.

$$S = \frac{a}{1-r} = \frac{3}{1-(-\frac{2}{9})} = \frac{27}{11}.$$

Example 23.

The second term of a geometric progression is $\frac{5}{16}$ and the sum to infinity is -1. Find the common ratio.

Solution

$$u_2 = \frac{5}{16}$$

$$ar = \frac{5}{16}$$

$$a = \frac{5}{16r}$$

$$S = -1$$

$$\frac{a}{1-r} = -1$$

$$\frac{5}{16r(1-r)} = -1$$

$$-16r + 16r^2 = 5$$
$$16r^2 - 16r - 5 = 0$$
$$(4r - 5)(4r + 1) = 0$$
$$4r = 5, \quad 4r = -1$$
$$r = \frac{5}{4}, \quad r = -\frac{1}{4}$$

Since the sum to infinity exists for $|r| < 1$,

$$r = -\frac{1}{4}.$$

Example 24.

A geometric progression is defined by $u_n = \frac{1}{3^n}$. Find S_n and the smallest value of n for which the sum of n terms and the sum to infinity differ by less than $\frac{1}{100}$.

Solution

$$u_n = \frac{1}{3^n}$$
$$u_1 = \frac{1}{3}, u_2 = \frac{1}{9}$$
$$\therefore r = \frac{1}{9} \times \frac{3}{1} = \frac{1}{3}$$
$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{\frac{1}{3}(1 - (\frac{1}{3})^n)}{1 - \frac{1}{3}} = \frac{1}{2}(1 - \frac{1}{3^n})$$

and

$$S = \frac{a}{1 - r} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}$$
$$S - S_n < \frac{1}{100}$$
$$\frac{1}{2} - \frac{1}{2}(1 - \frac{1}{3^n}) < \frac{1}{100}$$
$$\frac{1}{3^n} < \frac{1}{50}$$
$$3^n > 50$$
$$3^1 \not> 50, 3^2 \not> 50, 3^3 \not> 50$$

But $\quad 3^4 = 81 > 50.$

Therefore the smallest value of n is 4.

Exercise 4.6

1. Find the sum to infinity of each of the following geometric series.
   (i) $3 + 1 + \frac{1}{3} + \frac{1}{9} + \cdots$ (ii) $2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots$
   (iii) $\frac{7}{2} + 3 + \frac{18}{7} + \frac{108}{49} + \cdots$ (iv) $81 - 27 + 9 - 3 + \cdots$

2. Determine the values of x, so that the sum to infinity of geometric series $5 + \frac{5}{3}(x+1) + \frac{5}{3^2}(x+1)^2 + \cdots$ exists.

3. The second term of a G.P. is 2 and its sum to infinity is 9. Find the sum of the first 4 terms of the two possible geometric progressions.

4. Three consecutive terms of a G.P. are $\frac{1}{3^{2x-1}}, \frac{1}{9^x}, \frac{1}{243}$. Find (i) the value of x (ii) the sum to infinity if the first term of the G.P. is 3.

5. The sum of the first three terms of a G.P. is 27 and the sum of the fourth, fifth and sixth terms is -1. Find the common ratio and the sum to infinity of the G.P.

6. In a G.P. the ratio of the sum of the first 3 terms to the sum to infinity of the G.P. is 19:27. Find the common ratio.

7. In a G.P., the first term is 16 and the sum to infinity is 24. Given that each of the terms in the progression is squared to form a new G.P., find the sum to infinity of the new G.P.

8. Given that $x + 18, x + 4$ and $x - 8$ are the first three terms of a G.P., find the value of x. Hence, find
   (i) the common ratio (ii) the fifth term (iii) the sum to infinity.

9. The first term of a G.P. exceeds the second term by 2 and the sum to infinity is 50. Find the first term and the common ratio.

10. Find the smallest value of n for which the sum to n terms and the sum to infinity of a G.P. is $1, \frac{1}{5}, \frac{1}{25}, \cdots$ differ by less than $\frac{1}{1000}$.

11. The first and fourth terms of a geometric progression are 12 and $-\frac{4}{9}$ respectively. (i) Find the sum of the first n term of the progression. (ii) Find the sum to infinity of the progression.

Chapter 5

Matrices

The term **Matrix** was introduced by the 19th-century English Mathematician James Joseph Sylvester. It did not long before mathematicians realised that matrices are a convenient device for extending the common notions of numbers. Sir William Rowan Hamilton and Arthur Caylay made further contributions to the theory of matrices. It was the latest way to solve the system of linear equations. It is undoubtedly the most powerful tool in mathematics. The theory of matrices is, in the main, a part of algebra. It becomes clear that matrices posessed a utility that extended beyond the domain of algebra and into other regions of mathematics. Matrices are used in wide applications in engineering, physics, economics, and statistics as well as in various branches of mathematics.

5.1 Matrix Notation and Definitions

A matrix is a rectangular array of numbers arranged in rows and columns enclosed by a round or square bracket. The rows of a matrix are the arrays of numbers that go across the page. The columns are those that go down the page. In a common notation, capital letters are usually used to represent matrices, and the corresponding small letter with a double subscript describes an element of the matrix. The element a_{ij}, located in the i^{th} row and j^{th} column of the matrix A, is a scalar quantity or a single valued expression. The order or dimensions of a matrix is given by the number of rows followed by the number of columns.

In general, A matrix A of order $m \times n$ can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & \cdots & a_{mn} \end{pmatrix}.$$

Here $a_{11}, a_{12}, a_{13}, a_{14}, \dots \dots$ are called the elements or entries of the matrix. In this matrix, a_{13} refers to the element in row 1, column 3 of matrix A.

> A **matrix** A which has order $m \times n$ is written as
> $A = (a_{ij})_{m\times n}$ where $i = 1, 2, 3, \dots \dots, m$
> $j = 1, 2, 3, \dots \dots, n$
> and a_{ij} is the **element** in row i and column j.
> An $m \times n$ matrix has m rows and n columns.
> $m \times n$ specifies the **order** of a matrix.

For example:

$A = \begin{pmatrix} 2 & 4 & 8 & -1 \\ 1 & 6 & 7 & -9 \\ 9 & 5 & 3 & -5 \end{pmatrix}$ is a 3×4 matrix. In this matrix, a_{23} refers to the element in row 2, column 3, which is the number 7.

A matrix which has only one row is called a **row matrix**.
For example:

$$B = (9 \quad 10 \quad 11 \quad 8)$$

B has 1 row and 4 columns. B is a row matrix of order 1×4.

A matrix which has only one column is called a **column matrix**.
For example:

$$C = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

C has 3 rows and 1 column. C is a column matrix of order 3×1.

An ordinary number can be regarded as a 1×1 matrix.
For example:

3 can be thought of as the matrix (3).

A matrix which has the same number of rows and columns is called a **square matrix**.
For example:

$$D = \begin{pmatrix} 3 & 2 & 5 \\ 6 & 1 & 4 \\ 4 & 0 & 2 \end{pmatrix}$$

Note:

- A $1 \times n$ matrix which has exactly one row is called a row matrix.
- An $m \times 1$ matrix which has exactly one column is called a column matrix.
- An $n \times n$ matrix is a square matrix of order n.

You have been using matrices for many years without realising it. Here you may learn how the data values represent in the matrix.

This table shows the number of Small(S), Medium(M) and Large(L) T-Shirts on sale at an online shop.

	Small(S)	Medium(M)	Large(L)
Blue(B)	1	2	5
Red(R)	2	3	6
White(W)	4	0	7

We can write this table as matrix by extracting the numbers and placing them in round or square bracket. Matrices help organize information.

$$\begin{array}{c} \\ B \\ R \\ W \end{array}\begin{array}{c} \;S\;\;M\;\;L \\ \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ 4 & 0 & 7 \end{pmatrix} \end{array}, \qquad \text{simply} \qquad \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ 4 & 0 & 7 \end{pmatrix}$$

If the columns of this matrix represent sizes (Small, Medium, Large) and the rows represent colours (Blue, Red, White), then 7 represents the number of White T-Shirt in Large size.

Matrix Equality

Two matrices are equal if they have the same order and their corresponding entries are equal.

Suppose two matrices be $A=(a_{ij})$ and $B=(b_{ij})$. $A = B \iff a_{ij} = b_{ij}$ for all i, j.

For example, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then $a = 1$, $b = 2$, $c = 3$ and $d = 4$.

Two matrices cannot be equal unless they have the same order.

Thus $\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, since their orders are different.

Example 1.

What are the values of x and y in the matrix equation?

$$\begin{pmatrix} 12 & 0 \\ 2y & 9 \end{pmatrix} = \begin{pmatrix} 3x & 0 \\ 6 & 9 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 12 & 0 \\ 2y & 9 \end{pmatrix} = \begin{pmatrix} 3x & 0 \\ 6 & 9 \end{pmatrix}$$

By matrix equality, $3x = 12$ and $2y = 6$.

Therefore $x = 4$ and $y = 3$.

Example 2.

Find x and y in this matrix equation $\begin{pmatrix} x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$.

Solution

$\begin{pmatrix} x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

$x + y = 4 \quad \dots \quad \dots \quad (1)$

$x - y = 6 \quad \dots \quad \dots \quad (2)$

Adding (1) and (2), we get $2x = 10$.

So $x = 5$ and hence $y = -1$.

Transpose of a Matrix

Let A be a matrix of order $m \times n$. A matrix of order $n \times m$ whose rows are columns and whose columns are rows of A is called the **transpose** of A and is denoted by A^T (read as A transpose).

From a given matrix, $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$.

Exercise 5.1

1. Determine the order of the following matrices.

   (a) $\begin{pmatrix} 2 & 0 & 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$ (d) $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

2. Answer questions (a) to (d) for the matrix

$$\begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 2 & 6 & 9 & 12 \end{pmatrix}$$

   (a) State (i) the number of rows (ii) the number of columns.
   (b) List the elements in the second row.
   (c) List the elements in the third column.
   (d) Write down the entry in the third row and fourth column.

3. For each of the following matrices, state the order of matrix and the entry in the second row and first column.

   (a) $\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 5 & 6 & 7 \end{pmatrix}$ (c) $\begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix}$ (d) $\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$

4. Write down examples of matrices with numerical elements arranged in
   (a) 1 row and 3 columns (b) 2 rows and 3 columns
   (c) 3 rows and 2 columns (d) 5 rows and 2 columns.

5. Ma Ma goes shopping at a store to buy 2 loaves of bread at 1800 kyats for each, 3 liters of milk at 1200 kyats per liter, and 1 tub of butter at 2200 kyats. Represent the quantities purchased in a row matrix, and the costs per each quantity in a column matrix.

6. What is the order of each matrix? List any equalities for pairs of the following matrices.

$A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 2 \end{pmatrix}$ $\quad C = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad E = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$ $\quad F = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$G = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\quad H = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ $\quad I = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$J = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad K = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$ $\quad L = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

7. Find x and y if:

(a) $\begin{pmatrix} 3x & -y \end{pmatrix} = \begin{pmatrix} 12 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} x+2y \\ 2x-y \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$ (c) $\begin{pmatrix} x^2 & x^3 \\ -1 & -8 \end{pmatrix} = \begin{pmatrix} 4 & -8 \\ y+1 & y^3 \end{pmatrix}$

8. Let $P = \begin{pmatrix} x & 5 \\ -3 & y \end{pmatrix}$ and $Q = \begin{pmatrix} 9 & -3 \\ 5 & -7 \end{pmatrix}$. Find x and y, given that $P = Q^T$.

9. If $B = \begin{pmatrix} -3 & 2 & 4 & -1 \\ -4 & 1 & 5 & 0 \end{pmatrix}$, find $(B^T)^T$.

5.2 Matrix Operations

Matrix Addition

To add two matrices, they must have the same order, and add corresponding entries. For example, in a long holiday, a baker produced some buns, pies and rolls. His production levels for buns, pies and rolls in dozen are given by the matrix:

$$\begin{array}{r} \\ \text{buns} \\ \text{pies} \\ \text{rolls} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Sun} & \text{Mon} & \text{Sat} \end{array} \\ \begin{pmatrix} 60 & 55 & 45 \\ 50 & 20 & 35 \\ 40 & 30 & 25 \end{pmatrix} \end{array}$$

Some newly ordered items have added to bake. 30 buns, 20 pies, and 15 rolls must be added to the production levels of each day. His new order is given by the matrix:

$$\begin{pmatrix} 30 & 30 & 30 \\ 20 & 20 & 20 \\ 15 & 15 & 15 \end{pmatrix}$$

Clearly, a baker produced in total:

$$\begin{pmatrix}60 & 55 & 45\\50 & 20 & 35\\40 & 30 & 25\end{pmatrix}+\begin{pmatrix}30 & 30 & 30\\20 & 20 & 20\\15 & 15 & 15\end{pmatrix}=\begin{pmatrix}60+30 & 55+30 & 45+30\\50+20 & 20+20 & 35+20\\40+15 & 30+15 & 25+15\end{pmatrix}=\begin{pmatrix}60 & 85 & 75\\70 & 40 & 55\\55 & 45 & 40\end{pmatrix}$$

Note:
- We can only add matrices of the same orders.
- The resultant matrix is also a same order.
- We add corresponding entries.
  $A+B=(a_{ij})+(b_{ij})=(a_{ij}+b_{ij})$

Example 3.

If $P=\begin{pmatrix}-1 & 6\\-2 & 4\end{pmatrix}$, $Q=\begin{pmatrix}2 & 5\\1 & 8\end{pmatrix}$, find $P+Q$.

Solution

$$P+Q=\begin{pmatrix}-1 & 6\\-2 & 4\end{pmatrix}+\begin{pmatrix}2 & 5\\1 & 8\end{pmatrix}=\begin{pmatrix}-1+2 & 6+5\\-2+1 & 4+8\end{pmatrix}=\begin{pmatrix}1 & 11\\-1 & 12\end{pmatrix}$$

Zero Matrix

A zero matrix is a matrix whose elements are all zero. It is denoted by O.

For example, $O=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$ is the 2×2 zero matrix, and $O=\begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\end{pmatrix}$ is the 2×3 zero matrix.

Example 4.

Given that $O=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$ and $A=\begin{pmatrix}a & b\\c & d\end{pmatrix}$.

Show that $O+A=A+O=A$.

Solution

$$O+A=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}+\begin{pmatrix}a & b\\c & d\end{pmatrix}=\begin{pmatrix}a & b\\c & d\end{pmatrix}=A$$

$$A+O=\begin{pmatrix}a & b\\c & d\end{pmatrix}+\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}=\begin{pmatrix}a & b\\c & d\end{pmatrix}=A.$$

Therefore $O+A=A+O=A$.

Note: The zero matrix is the identity element for addition of matrices.

Example 5.

Given that $A=\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}$ and $B=\begin{pmatrix}-2 & -3\\-4 & -5\end{pmatrix}$, find $A+B$ and $B+A$ and hence show that $A+B=B+A=O$.

Solution

$$A+B = \begin{pmatrix}2&3\\4&5\end{pmatrix}+\begin{pmatrix}-2&-3\\-4&-5\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}=O$$

$$B+A = \begin{pmatrix}-2&-3\\-4&-5\end{pmatrix}+\begin{pmatrix}2&3\\4&5\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}=O$$

Hence $A+B=B+A=O$.

The example provides us with some ideas about the negative of a matrix.

Each entry in B is the negative of the corresponding entry in A. For this reason, B is called **the negative of** A and is written $-A$.

It means that the negative matrix of $A=(a_{ij})$ is $-A=(-a_{ij})$.

For example:

If $B=\begin{pmatrix}1&-2&3\\-4&1&-5\end{pmatrix}$, then $-B=\begin{pmatrix}-1&-(-2)&-3\\-(-4)&-1&-(-5)\end{pmatrix}=\begin{pmatrix}-1&2&-3\\4&-1&5\end{pmatrix}$

Note: Since $A+(-A)=(-A)+A=O$, we can say that the additive inverse of $-A$ is A that is $-(-A)=A$.

Example 6.

If $A=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}$, $B=\begin{pmatrix}0&2&5\\7&3&1\end{pmatrix}$, $C=\begin{pmatrix}8&2&9\\4&3&7\end{pmatrix}$, and $D=\begin{pmatrix}9&4\\6&3\end{pmatrix}$, find:

(a) $A+B$ (b) $B+A$ (c) $A+D$

(d) $B+C$ (e) $(A+B)+C$ (f) $A+(B+C)$

Solution

(a) $A+B=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}+\begin{pmatrix}0&2&5\\7&3&1\end{pmatrix}=\begin{pmatrix}1+0&2+2&3+5\\4+7&5+3&6+1\end{pmatrix}=\begin{pmatrix}1&4&8\\11&8&7\end{pmatrix}$

(b) $B+A=\begin{pmatrix}0&2&5\\7&3&1\end{pmatrix}+\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}=\begin{pmatrix}0+1&2+2&5+3\\7+4&3+5&1+6\end{pmatrix}=\begin{pmatrix}1&4&8\\11&8&7\end{pmatrix}$

(c) A and D do not have the same order. Hence $A+D$ cannot be operated.

(d) $B+C=\begin{pmatrix}0&2&5\\7&3&1\end{pmatrix}+\begin{pmatrix}8&2&9\\4&3&7\end{pmatrix}=\begin{pmatrix}0+8&2+2&5+9\\7+4&3+3&1+7\end{pmatrix}=\begin{pmatrix}8&4&14\\11&6&8\end{pmatrix}$

(e) $(A+B)+C=\begin{pmatrix}1&4&8\\11&8&7\end{pmatrix}+\begin{pmatrix}8&2&9\\4&3&7\end{pmatrix}=\begin{pmatrix}1+8&4+2&8+9\\11+4&8+3&7+7\end{pmatrix}=\begin{pmatrix}9&6&17\\15&11&14\end{pmatrix}$

(f) $A+(B+C)=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}+\begin{pmatrix}8&4&14\\11&6&8\end{pmatrix}=\begin{pmatrix}1+8&2+4&3+14\\4+11&5+6&6+8\end{pmatrix}=\begin{pmatrix}9&6&17\\15&11&14\end{pmatrix}$

What law for addition of matrices does these results suggest?

The matrix addition is both commutative and associative which are like those of ordinary number addition.

Exercise 5.2

1. Perform the following if they are possible, if not, write "not possible".

   (i) $(1)+\begin{pmatrix}1\\0\end{pmatrix}$ (ii) $(9\ \ 6)+(7\ \ 8)$ (iii) $\begin{pmatrix}2&4\\6&8\end{pmatrix}-\begin{pmatrix}3&5\\7&9\end{pmatrix}$

   (iv) $(1\ \ 2\ \ 3)+\begin{pmatrix}2\\4\\6\end{pmatrix}$ (v) $\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$ (vi) $\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}+\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

2. For $A=\begin{pmatrix}5&5\\4&9\end{pmatrix}, B=\begin{pmatrix}2&3\\1&4\end{pmatrix}, C=\begin{pmatrix}1&7\\4&-3\end{pmatrix}$, is it true that $(A+B)+C=A+(B+C)$?

3. Write down the negative of each of the following matrices.

   (a) $(3\ \ 2)$ (b) $\begin{pmatrix}1\\2\\3\end{pmatrix}$ (c) $\begin{pmatrix}5&-8\\4&-7\end{pmatrix}$ (d) $\begin{pmatrix}-4&2&1\\3&-1&0\end{pmatrix}$

4. In each of the following cases: find the matrix A which satisfies the given equation.

   (a) $A+\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}-4\\3\end{pmatrix}$ (b) $A+\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}0&2\\2&1\end{pmatrix}$

5. Solve each of the following equations for the 2×2 matrix X.

   (a) $X+\begin{pmatrix}-2&1\\0&-3\end{pmatrix}=\begin{pmatrix}3&-1\\5&4\end{pmatrix}$ (b) $\begin{pmatrix}2&4\\3&5\end{pmatrix}+X=\begin{pmatrix}4&-2\\3&1\end{pmatrix}$

6. Solve the following equation for the 2×2 matrix A.

   $A^T-\begin{pmatrix}3&1\\4&2\end{pmatrix}=\begin{pmatrix}9&5\\0&8\end{pmatrix}$

5.3 Matrix Multiplication

Scalar Multiplication

The term scalar multiplication refers to the product of a real number and a matrix. In this case, each entries in the matrix is multiplied by the scalar.

Multiplication of a matrix by a scalar:
If $A=(a_{ij})$ has order $m\times n$, and k is a scalar, then we define the product $B=kA$ by $(b_{ij})=(ka_{ij})$. B is also a matrix of order $m\times n$.

For example:

Suppose in the refrigerator there are 8 cans of coca cola, 4 cans of coffee, and 2 cans of coconut juice. We represent this in matrix form as $A=\begin{pmatrix}8\\4\\2\end{pmatrix}$.

If we doubled cans in the refrigerator, it is reasonable to denote $A + A$ by $2A$, we have
$2A = 2\begin{pmatrix}8\\4\\2\end{pmatrix} = \begin{pmatrix}2\times 8\\2\times 4\\2\times 2\end{pmatrix} = \begin{pmatrix}16\\8\\4\end{pmatrix}$.
Notice that to get $2A$ from A we simply multiply each entry in the matrix A by 2.

Likewise, halving them gives $\frac{1}{2}A = \begin{pmatrix}\frac{1}{2}\times 8\\\frac{1}{2}\times 4\\\frac{1}{2}\times 2\end{pmatrix} = \begin{pmatrix}4\\2\\1\end{pmatrix}$.

Example 7.

Given that $A = \begin{pmatrix}2 & 1\\4 & 3\end{pmatrix}$ and $B = \begin{pmatrix}1 & 5\\0 & 2\end{pmatrix}$. Find:

(a) $5A$ (b) $(-1)A$ (c) $3A - 2B$

Solution

(a) $5A = 5\begin{pmatrix}2 & 1\\4 & 3\end{pmatrix} = \begin{pmatrix}10 & 5\\20 & 15\end{pmatrix}$

(b) $(-1)A = (-1)\begin{pmatrix}2 & 1\\4 & 3\end{pmatrix} = \begin{pmatrix}-2 & -1\\-4 & -3\end{pmatrix}$

(c) $3A = 3\begin{pmatrix}2 & 1\\4 & 3\end{pmatrix} = \begin{pmatrix}6 & 3\\12 & 9\end{pmatrix}$

$2B = 2\begin{pmatrix}1 & 5\\0 & 2\end{pmatrix} = \begin{pmatrix}2 & 10\\0 & 4\end{pmatrix}$

$3A - 2B = \begin{pmatrix}6 & 3\\12 & 9\end{pmatrix} - \begin{pmatrix}2 & 10\\0 & 4\end{pmatrix} = \begin{pmatrix}6-2 & 3-10\\12-0 & 9-4\end{pmatrix} = \begin{pmatrix}4 & -7\\12 & 5\end{pmatrix}$

Note: From the meaning of kA, it readily follows that (i) $0A = O$, (ii) $kO = O$ where O is a zero matrix of suitable order and (iii) $(-1)k = (-k)$.

Example 8.

Solve $5\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} - 3X = 4\begin{pmatrix}-4 & 7\\3 & 8\end{pmatrix}$ for the 2×2 matrix X.

Solution

$$
\begin{aligned}
5\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} - 3X &= 4\begin{pmatrix}-4 & 7\\3 & 8\end{pmatrix}\\
\begin{pmatrix}5 & 10\\15 & 20\end{pmatrix} - 3X &= \begin{pmatrix}-16 & 28\\12 & 32\end{pmatrix}\\
\begin{pmatrix}-5 & -10\\-15 & -20\end{pmatrix} + \begin{pmatrix}5 & 10\\15 & 20\end{pmatrix} - 3X &= \begin{pmatrix}-5 & -10\\-15 & -20\end{pmatrix} + \begin{pmatrix}-16 & 28\\12 & 32\end{pmatrix}\\
\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} - 3X &= \begin{pmatrix}-21 & 18\\-3 & 12\end{pmatrix}\\
O - 3X &= \begin{pmatrix}-21 & 18\\-3 & 12\end{pmatrix}
\end{aligned}
$$

$$-3X = \begin{pmatrix} -21 & 18 \\ -3 & 12 \end{pmatrix}$$

$$(-\frac{1}{3})(-3X) = -\frac{1}{3}\begin{pmatrix} -21 & 18 \\ -3 & 12 \end{pmatrix}$$

$$X = \begin{pmatrix} 7 & -6 \\ 1 & -4 \end{pmatrix}$$

Exercise 5.3

1. If $A = \begin{pmatrix} 9 & 12 \\ 6 & 9 \end{pmatrix}$, find:
   (a) $2A$ (b) $-3A$ (c) $\frac{1}{3}A$ (d) $\frac{2}{3}A$ (e) $0A$

2. If $A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 0 & 2 \\ 4 & -2 & 1 \end{pmatrix}$ find:
   (a) $A + B$ (b) $A - C$ (c) $2A + B$ (d) $2B - C$

3. A builder builds a block of 12 identical flats. Each flat is to contain 1 table, 4 chairs, 2 beds, and 1 wardrobe.

   Let $P = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 1 \end{pmatrix}$ be the matrix representing the furniture in one flat.

   In terms of P, what is the matrix representing the furniture in all flats? Evaluate this matrix.

4. Solve the following equations:
   (a) $a\begin{pmatrix} 2 \\ 2 \end{pmatrix} + b\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$ (b) $3\begin{pmatrix} 2x \\ y \end{pmatrix} + 3\begin{pmatrix} x \\ 3y \end{pmatrix} = \begin{pmatrix} 18 \\ 36 \end{pmatrix}$
   (c) $\begin{pmatrix} 2 \\ -5 \end{pmatrix} + 3\begin{pmatrix} 7 \\ a \end{pmatrix} = \begin{pmatrix} 2a \\ b \end{pmatrix}$ (d) $2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 5 \\ c & 6 \end{pmatrix} = \begin{pmatrix} a & b \\ 7 & d \end{pmatrix}$

5. Solve each of the following equations:
   (a) $2X - \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 5 \\ 2 & 8 \end{pmatrix}$ (b) $4X - \begin{pmatrix} 3 & 1 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 0 & 13 \end{pmatrix}$
   (c) $2\begin{pmatrix} 1 & -1 & 3 \\ 2 & -7 & 5 \end{pmatrix} + X = 3\begin{pmatrix} 1 & 2 & -4 \\ 3 & -5 & 1 \end{pmatrix}$

6. Given that $A = \begin{pmatrix} 2 & -2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 5 & a \\ c & 4 \end{pmatrix}$ and $C = \begin{pmatrix} b & 6 \\ 4 & d \end{pmatrix}$,
   find the values of a, b, c and d if:
   (a) $2A + B = C$ (b) $3A - 2B = 4C$

Multiplication of Matrices

Can we multiply one matrix by another matrix? When we do matrix multiplication? The matrices can be multiplied when the number of columns in the first matrix is the same as the number of rows in the second matrix. In fact, we can multiply each entry in the 1×3 **row matrix** by the corresponding entry in the 3×1 **column matrix**, and found the sum of these products as a 1×1 **product matrix**. Notice that we can write the row matrix first and the column matrix second.

In general, $\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} d \\ e \\ f \end{pmatrix} = (ad + be + cf)$.

What does the matrix multiplication mean? When does it use? The following illustration will suggest an answer to these questions. The student will need to observe some cases in studying this section as the multiplication is more exciting than the process of addition.

Suppose a school librarian, decides to buy 5 laptops, 10 tablets, and 4 computers.

The prices are:	laptop	tablet	computer
	\$1700	\$450	\$700

We can represent this by the quantity matrix $Q = \begin{pmatrix} 5 & 10 & 4 \end{pmatrix}$ and rate matrix

$R = \begin{pmatrix} 1700 \\ 450 \\ 700 \end{pmatrix}$.

We can calculate the total cost of these items by multiplying the number of each item by its cost, and then adding the products:

$$5 \times 1700 + 10 \times 450 + 4 \times 700 = 15800$$

This equation is formed by multiplying the elements of row of Q by the corresponding elements of each column of R and finding the sum of the products. This method is similar to multiplying two matrices. So that we can evaluate the total cost by the **matrix multiplication.**

$$\begin{aligned} QR &= \begin{pmatrix} 5 & 10 & 4 \end{pmatrix} \begin{pmatrix} 1700 \\ 450 \\ 700 \end{pmatrix} \\ &= \begin{pmatrix} 5 \times 1700 + 10 \times 450 + 4 \times 700 \end{pmatrix} \\ &= \begin{pmatrix} 15800 \end{pmatrix} \end{aligned}$$

Next you may learn more complicated matrix multiplications.

Thida goes shopping at King Store to buy 2 shirts, 4 blouses, and 3 skirts costing \$15, \$38, and \$25 each respectively.

We represent this by the quantities matrix $Q = \begin{pmatrix} 2 & 4 & 3 \end{pmatrix}$ and the rate matrix in King

Store $= \begin{pmatrix} 15 \\ 38 \\ 25 \end{pmatrix}$.

When Thida goes to a different store, Queen Store, she finds the prices for the same items are \$12 for shirts, \$40 for blouses, and \$20 for skirts.

The rate matrix in Queen Store is $\begin{pmatrix}12\\40\\20\end{pmatrix}$.

The price per item for each store can be represented by two columns in the rate matrix

$$R=\begin{pmatrix}15 & 12\\38 & 40\\25 & 20\end{pmatrix}.$$

To find the total cost of these items in each store, we can multiply the number of items by their respective price.

The total cost at King Store is $2\times 15+4\times 38+3\times 25=257$,
and at Queen Store is $2\times 12+4\times 40+3\times 20=244$.

Matrix multiplication can perform this in the following form:

$$QR=\begin{pmatrix}2 & 4 & 3\end{pmatrix}_{1\times 3}\begin{pmatrix}15 & 12\\38 & 40\\25 & 20\end{pmatrix}_{3\times 2}$$

the same
$1\times ③$ $③\times 2$

$$=\begin{pmatrix}2\times 15+4\times 38+3\times 25 & 2\times 12+4\times 40+3\times 20\end{pmatrix}_{1\times 2}$$
$$=\begin{pmatrix}257 & 244\end{pmatrix}$$

Thida's total cost at King Store is \$257, and at Queen Store is \$244.

Now suppose Hsu Hsu needs 1 shirt, 2 blouses, and 1 skirt.

The quantities matrix for both Thida and Hsu Hsu:

$$\begin{array}{r} \\ \text{Thida} \\ \text{Hsu Hsu}\end{array}\begin{array}{c}\begin{array}{ccc}\text{shirt} & \text{blouse} & \text{skirt}\end{array}\\ \begin{pmatrix}2 & 4 & 3\\1 & 2 & 1\end{pmatrix}\end{array}$$

Hsu Hsu's total cost at King Store is $1\times 15+2\times 38+1\times 25=116$.
and at Queen Store is $1\times 12+2\times 40+1\times 20=112$.

Since matrix multiplication can do if the number of columns of the first matrix is equal to the number of rows of the second matrix, the answer has the same number of rows as the first matrix and the same number of columns as the second matrix.

Matrix multiplication for Thida and Hsu Hsu can be formed the following:

$$\begin{pmatrix}2 & 4 & 3\\1 & 2 & 1\end{pmatrix}\begin{pmatrix}15 & 12\\38 & 40\\25 & 20\end{pmatrix}=\begin{pmatrix}2\times 15+4\times 38+3\times 25 & 2\times 12+4\times 40+3\times 20\\1\times 15+2\times 38+1\times 25 & 1\times 12+2\times 40+1\times 20\end{pmatrix}$$

$2\times ③$ $③\times 2$
the same

row 1 × col 1 ↓ row 1 × col 2 ↓

$$=\begin{pmatrix}257 & 244\\116 & 112\end{pmatrix}$$

row 2 × col 1 ↑ row 2 × col 2 ↑

F11A

The above example shows multiplication of 2×3 matrix and 3×2 matrix gives 2×2 matrix.

By the observation, we now define matrix multiplication more formally.

Suppose $A = (a_{ij})$ and $B = (b_{ij})$. The product AB of an $m \times p$ matrix A and an $p \times n$ matrix B is the $m \times n$ matrix C whose entry in the ith row and jth column is the sum of the products of corresponding entries in the ith row of A and the jth column of B. We define AB by $C = (c_{ij})$ where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{ip}b_{pj}$.

That is,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ . & . & \cdots & . \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ . & . & \cdots & . & \cdots & . \\ . & . & \cdots & . & \cdots & . \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2n} \\ . & . & \cdots & . & \cdots & . \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ . & . & \cdots & . & \cdots & . \\ . & . & \cdots & . & \cdots & . \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mn} \end{pmatrix}$$

> **Multiplication of matrices:**
> If $C = AB$ where $(c_{ij}) = (a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{ip}b_{pj})$,
> for each pair i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$.

Note:

- The product AB is not defined unless the number of columns of A and the number of rows of B are the same.
- Matrix multiplication is associative, but not commutative. It is not like an ordinary arithmetic multiplication. The positions of the matrices in the product are very important.

Example 9.

Given $X = \begin{pmatrix} 3 & 2 \\ 5 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$, find XY.

Solution

$$\begin{aligned} XY &= \begin{pmatrix} 3 & 2 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \\ &= \begin{pmatrix} (3 \times 1) + (2 \times 2) & (3 \times 3) + (2 \times 4) & (3 \times 5) + (2 \times 6) \\ (5 \times 1) + (0 \times 2) & (5 \times 3) + (0 \times 4) & (5 \times 5) + (0 \times 6) \end{pmatrix} \\ &= \begin{pmatrix} 7 & 17 & 27 \\ 5 & 15 & 25 \end{pmatrix} \end{aligned}$$

Example 10.

Given $P = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 4 & 5 \\ 2 & 0 \end{pmatrix}$, find PQ and QP.

Solution

$$PQ = \begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}\begin{pmatrix}4 & 5\\ 2 & 0\end{pmatrix} = \begin{pmatrix}4+4 & 5+0\\ 12+2 & 15+0\end{pmatrix} = \begin{pmatrix}8 & 5\\ 14 & 15\end{pmatrix}$$

$$QP = \begin{pmatrix}4 & 5\\ 2 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix} = \begin{pmatrix}4+15 & 8+5\\ 2+0 & 4+0\end{pmatrix} = \begin{pmatrix}19 & 13\\ 2 & 4\end{pmatrix}.$$

Notice that $PQ \neq QP$, so multiplication of matrices is not commutative.

Example 11.

If $A = (3 \quad 5)$ and $B = \begin{pmatrix}2 & -1\\ -1 & 0\end{pmatrix}$ which of the products AB, BA are possible? Simplify those products that exist.

Solution

order of A order of B

$1 \times ②$ and $② \times 2$. $\therefore AB$ exists and order of AB is 1×2.

↑ the same ↑

$$\begin{aligned} AB &= (3 \quad 5)\begin{pmatrix}2 & -1\\ -1 & 0\end{pmatrix} \\ &= (6-5 \quad -3+0) \\ &= (1 \quad -3) \end{aligned}$$

order of B order of A

$2 \times ②$ and $① \times 2$. $\therefore BA$ does not exist.

↑ different ↑

Example 12.

Given $\begin{pmatrix}3 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}9\\ 1\end{pmatrix}$, find a system of equations in x and y. Hence find x and y.

Solution

$$\begin{aligned} \begin{pmatrix}3 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} &= \begin{pmatrix}9\\ 1\end{pmatrix} \\ \begin{pmatrix}3x+y\\ 2x-y\end{pmatrix} &= \begin{pmatrix}9\\ 1\end{pmatrix} \end{aligned}$$

the system of equations in x and y is

$$\begin{aligned} 3x+y &= 9 \\ 2x-y &= 1. \end{aligned}$$

Adding these two equations, we get

$$5x = 10$$
$$x = 2.$$

Substituting $x = 2$ in one of these equations, we have $y = 3$.

Unit Matrix

The **unit matrix or identity matrix** I is a square matrix having 1's on the **main diagonal**, and 0's everywhere else. The unit matrix of order 2 is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It behaves like unity in the real number system. If A is a square matrix, then $IA = AI = A$.

Example 13.

Find the products of $\begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix}$. What do you notice that?

Solution

$$\begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7+0 & 0+5 \\ 6+0 & 0+4 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 7+0 & 5+0 \\ 0+6 & 0+4 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix}$$

$$\text{So} \quad \begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix}$$

Example 14.

If $A = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$, find AB and BA.

Solution

$$AB = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2+(-2) & 2+(-2) \\ -6+6 & -6+6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

$$BA = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} 2+(-6) & -4+12 \\ 1+(-3) & -2+6 \end{pmatrix} = \begin{pmatrix} -4 & 8 \\ -2 & 4 \end{pmatrix}$$

Note:

- $AB = O$ does not necessarily mean that $A = O$ or $B = O$
- Powers of a square matrix A are defined as follows:

  $A^2 = AA, A^3 = A^2A, A^4 = A^3A$ and so on.

Example 15.

$A = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}$. Find A^2, A^3 and A^4 and hence deduce a formula for A^n, where n is a positive integer.

Solution

$$\begin{aligned} A^2 &= AA = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}\begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix} = \begin{pmatrix} k^2+0 & k+k \\ 0+0 & 0+k^2 \end{pmatrix} = \begin{pmatrix} k^2 & 2k \\ 0 & k^2 \end{pmatrix} \\ A^3 &= A^2A = \begin{pmatrix} k^2 & 2k \\ 0 & k^2 \end{pmatrix}\begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix} = \begin{pmatrix} k^3 & 3k^2 \\ 0 & k^3 \end{pmatrix} \\ A^4 &= A^3A = \begin{pmatrix} k^3 & 3k^2 \\ 0 & k^3 \end{pmatrix}\begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix} = \begin{pmatrix} k^4 & 4k^3 \\ 0 & k^4 \end{pmatrix} \end{aligned}$$

Hence $A^n = \begin{pmatrix} k^n & nk^{n-1} \\ 0 & k^n \end{pmatrix}$, where n is a positive integer.

Example 16.

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find p, q such that $A^2 = pA + qI$.

Solution

$$\begin{aligned} A^2 &= pA + qI \\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} &= p\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + q\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{pmatrix} &= \begin{pmatrix} p & 2p \\ 3p & 4p \end{pmatrix} + \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \\ \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} &= \begin{pmatrix} p+q & 2p \\ 3p & 4p+q \end{pmatrix} \end{aligned}$$

Hence

$$\left.\begin{aligned} p+q &= 7 \\ 2p &= 10 \\ 3p &= 15 \\ 4p+q &= 22 \end{aligned}\right\} \text{ from which } p = 5, q = 2.$$

Example 17.

Find the two matrices of the form $X = \begin{pmatrix} x & 1 \\ 0 & y \end{pmatrix}$ such that $X^2 = I$.

Solution

$$\begin{aligned} X^2 &= XX = \begin{pmatrix} x & 1 \\ 0 & y \end{pmatrix}\begin{pmatrix} x & 1 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x^2 & x+y \\ 0 & y^2 \end{pmatrix} \\ X^2 &= I \\ \begin{pmatrix} x^2 & x+y \\ 0 & y^2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \therefore x^2 &= 1, & x+y &= 0, & y^2 &= 1, \\ \therefore x &= \pm 1, & y &= -x, & y &= \pm 1. \end{aligned}$$

Since $y = -x$; $x = 1, y = -1$ (or) $x = -1, y = 1$

Thus the required two matrices are $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$.

Exercise 5.4

1. Find the following matrix products.

   (a) $\begin{pmatrix} 2 & -3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

   (d) $\begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 & 6 \\ 0 & 4 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 5 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix}$

2. Obtain the matrix products of the following where possible.

   (a) $\begin{pmatrix} 1 \\ 7 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 3 & 4 \\ 1 & 7 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

   (d) $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 2 \\ 3 & 1 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 5 \end{pmatrix}$ (f) $\begin{pmatrix} 4 & 3 \\ 1 & 5 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 7 \end{pmatrix}$

   (g) $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 4 & 2 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ (h) $\begin{pmatrix} \cos x & \sin x \end{pmatrix} \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$

3. In each of the following, find a system of equations in x and y. Hence find x and y.

   (a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

4. Find the values of a and b if $\begin{pmatrix} a & 2a \\ 2b & b \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 33 \\ 50 \end{pmatrix}$.

5. If $A = \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 2 & k \end{pmatrix}$, find

   (i) AB (ii) BA (iii) the value of k if $AB = BA$.

6. Given that $2\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find the values of a, b, c and d.

7. The matrices $A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ are such that $AB = A + B$. Find the values of a, b and c.

8. If $A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 4 \\ -2 & 5 \end{pmatrix}$, find:
   (a) $(AB)C$ (b) $A(BC)$ (c) $(CB)A$ (d) $C(BA)$
   What law appears to hold?

9. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$
   (a) is $A(B + C) = AB + AC$? Can you give the name of this law?
   (b) is $A + (BC) = (A + B)(A + C)$?

10. $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$, find:
   (a) $A + B$ (b) $A - B$ (c) $(A + B)(A - B)$
   (d) A^2 (e) B^2
   Is it true that $(A + B)(A - B) = A^2 - B^2$?

11. For the matrices A and B given in problem (10), find
   (a) $(A + B)^2$
   (b) $A^2 + 2AB + B^2$. Is it true that $(A + B)^2 = A^2 + 2AB + B^2$?

12. $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, verify that $A^2 - 2A + I = O$, where I is the identity matrix of order 2.

13. Show that the matrix $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ satisfies the equation $A^2 - 4A - 5I = O$.

14. Given that $D = \begin{pmatrix} 2 & -3 \\ -2 & 1 \end{pmatrix}$ and that $D^2 - 3D - kI = O$, find the value of k.

15. Evaluate $(2A - B)C$ where $A = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 7 & 2 \\ -1 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

5.4 The Inverse of a Square Matrix of Order 2

The real numbers 3 and $\frac{1}{3}$ are called **multiplicative inverses** because their product is $3 \times \frac{1}{3} = \frac{1}{3} \times 3 = 1$, which is the multiplicative identity. For the matrices $A = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix}$;

pre-multiplying B by A, $AB = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

post-multiplying B by A, $BA = \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$\therefore AB = BA = I$

We say that A and B are **multiplicative inverses** of each other.

If $AB = BA = I$, then the inverse of A, A^{-1} is B, and the inverse of B, B^{-1} is A.

Example 18.

If $A = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}$, show that A and B are inverses of each other.

Solution

We have to show that $AB = I = BA$.

$$AB = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$BA = \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Since $AB = I = BA$, A and B are inverses of each other.

Note:

- The inverse matrix could be found by interchanging the entries in the main diagonal, and changing the signs of the entries in the other diagonal if the difference of the product of entries in the two diagonals was 1.

More about Inverse of Square Matrices of Order 2

We will now determine how to find the inverse of a matrix. Does every 2×2 matrix have an inverse?

To answer this question, suppose A and B be the matrices:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \text{ provided } ad - bc \neq 0$$

Multiplying these matrices gives:

$$AB = \frac{1}{ad - bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$BA = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Hence $AB = I = BA$.

The value $ad - bc$ is called the **determinant** of a matrix A, denoted by $\det A$.

Matrix A does not have a multiplicative inverse when $ad - bc = 0$.

If $\det A \neq 0$ then A^{-1} exists.

Thus $A^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

> For the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:
> - The value $ad - bc$ is called the **determinant** of a matrix A, denoted by $\det A$.
> - If $\det A \neq 0$ then A has an inverse, i.e., A^{-1} exists.
>
>   In this case, A is said to be **non-singular**, and $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
> - If $\det A = 0$, then A has no inverse, and it is said to be a **singular**.

Example 19.

Find, if it exists, the inverse matrix of $\begin{pmatrix} -2 & 4 \\ 1 & -1 \end{pmatrix}$ and of $\begin{pmatrix} 6 & 3 \\ -4 & -2 \end{pmatrix}$.

Solution

Let $M = \begin{pmatrix} -2 & 4 \\ 1 & -1 \end{pmatrix}$.

$$\begin{aligned} \det M &= (-2)(-1) - (1)(4) = 2 - 4 = -2 \neq 0, \quad \text{so } M^{-1} \text{ exists.} \\ M^{-1} &= \frac{1}{\det M}\begin{pmatrix} -1 & -4 \\ -1 & -2 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -1 & -4 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 2 \\ \frac{1}{2} & 1 \end{pmatrix}. \end{aligned}$$

Let $N = \begin{pmatrix} 6 & 3 \\ -4 & -2 \end{pmatrix}$.

$\det N = (6)(-2) - (-4)(3) = -12 + 12 = 0$

$\therefore N^{-1}$ does not exists.

Example 20.

Solve the matrix equation $\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix} X = \begin{pmatrix} 0 & 7 \\ 9 & 2 \end{pmatrix}$ for 2×2 matrix X.

Solution

Let $A = \begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 7 \\ 9 & 2 \end{pmatrix}$.

$\det A = 3(2) - 3(1) = 6 - 3 = 3$.

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}.$$

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \\ \text{So,}\ X &= A^{-1}B. \end{aligned}$$

Therefore

$$X = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 0 & 7 \\ 9 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -9 & 12 \\ 27 & -15 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 9 & -5 \end{pmatrix}.$$

Example 21.

Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$. Find the inverse matrix of A and of B. Hence use your results to find the matrices P and Q such that (i) $AP = B$ (ii) $QA = B$.

Solution

$$\begin{aligned}
\det A &= 4 - 3 = 1 \\
A^{-1} &= \frac{1}{\det A}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \\
\det B &= 2 - 12 = -10 \\
B^{-1} &= \frac{1}{\det B}\begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix} = \frac{1}{-10}\begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{10} \end{pmatrix}
\end{aligned}$$

(i)

$$\begin{aligned}
AP &= B \\
A^{-1}AP &= A^{-1}B \\
IP &= A^{-1}B \\
P &= A^{-1}B = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -7 & 2 \\ 5 & 0 \end{pmatrix}
\end{aligned}$$

(ii)

$$\begin{aligned}
QA &= B \\
QAA^{-1} &= BA^{-1} \\
QI &= BA^{-1} \\
Q &= BA^{-1} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 4 & -5 \end{pmatrix}
\end{aligned}$$

Exercise 5.5

In question 1 to 4, show that each matrix is the inverse of the other.

1. $\begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix}$
2. $\begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & -3 \\ -4 & 7 \end{pmatrix}$
3. $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$
4. $A = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{pmatrix}$
5. Find the inverse of $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

6. For $A = \begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}$, find:
(a) $\det A$ (b) $\det(-A)$ (c) $\det(2A)$ (d) A^{-1} (e) $(A^{-1})^{-1}$

7. Find the inverse of the following matrices where possible.
(a) $\begin{pmatrix} 2 & -1 \\ 3 & -\frac{3}{2} \end{pmatrix}$ (b) $\begin{pmatrix} -8 & -4 \\ -4 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 3 \\ -2 & 6 \end{pmatrix}$

8. For each of the following matrices A, find A^{-1} and state the values of k for which A^{-1} exists.
(a) $\begin{pmatrix} k+1 & 2 \\ 1 & k \end{pmatrix}$ (b) $\begin{pmatrix} k^2 & k-1 \\ 2k & 1 \end{pmatrix}$ (c) $\begin{pmatrix} k+1 & 2 \\ k^2+2 & 3k \end{pmatrix}$

9. Suppose that $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} \frac{1}{2} & k \\ 0 & 2a \end{pmatrix}$ and $C = \begin{pmatrix} 6 & 2 \\ -3 & h \end{pmatrix}$.
   (a) If $AB = I$, find the values of k and a.
   (b) Find the value of h for which $\det A = \det C$.
   (c) If $\det B = \det C$, find the value of h when $a = 3$.

10. Given that $A = \begin{pmatrix} 2 & 1 \\ -5 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

11. If $A = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}$, prove that $A^2 + 2A^{-1} = 3I$ where I is the unit matrix of order 2.

12. Solve each of the following matrix equations for 2×2 matrix X.
(a) $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 2 \\ 1 & 5 \end{pmatrix} X = \begin{pmatrix} 2 & 5 \\ 3 & -1 \end{pmatrix}$

13. Given that $A = \begin{pmatrix} 7 & 5 \\ 8 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$, write down the inverse matrix of B and use it to find the matrices P and Q such that
(a) $PB = A$, (b) $BQ = 2A$.

14. Given that $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix}$, write down the matrix A^{-1}, and use it to solve the following equations:
(a) $AX = B - A$ (b) $YA = 3B + 2A$

15. Suppose $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$, and $AXB = C$. Find X.

Chapter 6

Statistics

In this chapter, we explain the measures of variation such as the range, variance and standard deviation. We also discuss the measure of position (quartiles). Then we explain the cumulative frequency table and its curve to determine the median and quartiles. Finally, we describe the relation of two variables in data sets, and determine the strength of linear relationship between them.

6.1 Measure of Variation

We first recall the measures of centre of a data set. These measures include mean, median, and mode. For a set of data, the **mean** is defined by the sum of the values, divided by the number of values. If the numbers in the set are $x_1, x_2, \ldots, x_n$, then

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

where the symbol $\bar{x}$ is used for mean and n is the number of values. The **median** is the middle point of the data arranged in ascending order. If there is not just one middle value, the median is taken by the mean of two middle values. The **mode** is the value that occurs most often in a set of data.

However, the measure of centre of data is not enough to describe the whole data set. Therefore we must know how the data values are dispersed. To determine the spread of the data values, we use **measures of variation**, or **measures of dispersion**. These measures include the range, variance, and standard deviation.

When we have a value for the centre of a data set, we can determine the distance between this centre and the other data values. This distance from the mean is called the spread of the data. The simplest measure of spread is the **range** R which is the difference between the largest value and the smallest value.

$$\boxed{R = \text{largest value} - \text{smallest value}}$$

The range is sensitive according to the outlier (smallest and largest values). To have a more meaningful statistics to measure variability, we use variance and standard deviation. We first consider **data variation** which is based on the difference of each data value from the mean. This difference is called a **deviation**, in symbol $x_i - \bar{x}$. If we sum the deviation for all data values about the mean, this result will always be zero, i.e., $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$. To overcome this problem, we use the square distances instead of the actual distances.

The **variance** σ^2 is the mean of squares of the deviations.

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

where x_i is individual value, $\bar{x}$ is the mean and n is the number of values.

The **standard deviation** σ is the square root of the variance.

$$\sigma = \sqrt{\text{variance}}$$

Example 1.

The marks scored in a test by seven students are 3, 4, 5, 2, 8, 8, 5. Find the mean and standard deviation.

Solution

$$\text{Mean} = \bar{x} = \frac{3+4+5+2+8+8+5}{7} = \frac{35}{7} = 5$$

x_i	3	4	5	2	8	8	5
$x_i - \bar{x}$	-2	-1	0	-3	3	3	0
$(x_i - \bar{x})^2$	4	1	0	9	9	9	0

Then

$$\text{variance} = \sigma^2 = \frac{4+1+0+9+9+9+0}{7} = \frac{32}{7} = 4.5714$$

$$\text{standard deviation} = \sigma = \sqrt{4.5714} = 2.1381$$

Students can find means, variances, and standard deviations directly by inputting the given data into a calculator.

From now on, we will use x and $\sum x$ instead of x_i and $\sum_{i=1}^{n} x_i$ for simplicity.

We can compute the similar formula for the variance as follow:

$$\begin{aligned}
\sigma^2 &= \frac{\sum(x-\bar{x})^2}{n} \\
&= \frac{\sum(x^2 - 2x\bar{x} + (\bar{x})^2)}{n} \\
&= \frac{\sum x^2}{n} - \frac{\sum 2x\bar{x}}{n} + \frac{\sum(\bar{x})^2}{n} \\
&= \frac{\sum x^2}{n} - 2\bar{x}\frac{\sum x}{n} + \frac{n(\bar{x})^2}{n} \\
&= \frac{\sum x^2}{n} - 2\bar{x}\bar{x} + (\bar{x})^2 \\
&= \frac{\sum x^2}{n} - (\bar{x})^2
\end{aligned}$$

Example 2.

Ten students are weighted (w kg). The summary data for the weights are

$$\sum w = 440, \quad \sum w^2 = 19490.$$

Find the mean and standard deviation of the students' weight.

Solution

$$\text{The mean} = \bar{w} = \frac{\sum w}{n} = \frac{440}{10} = 44.$$

$$\sigma^2 = \frac{\sum w^2}{n} - (\bar{w})^2 = \frac{19490}{10} - (44)^2 = 1949 - 1936 = 13$$

then the standard deviation is $\sigma = \sqrt{13} = 3.6056$.

Measures of Position

In addition to measure of centre and measure of variation, there are measures of position. These measures include standard scores, percentiles, deciles, and quartiles. In this course, we only focus on quartiles. **Quartiles** Q_1, Q_2 and Q_3 divide the data arranged in ascending order into four equal groups. Here, Q_2 is called the **median** of the whole data set. Q_1 and Q_3 are called the **lower quartile** and the **upper quartile**.

The **interquartile range** is a measure of variability.

$$\text{interquartile range} = \text{upper quartile} - \text{lower quartile} = Q_3 - Q_1$$

Example 3.

Find the median and interquartile range for each of the set of data below:

(a) 7, 9, 4, 6, 3, 2, 8, 1, 10, 15, 11

(b) 6, 5, 1, 14, 3, 7, 8, 2, 13

(c) 2, 15, 13, 6, 5, 12, 50, 22, 18, 52

(d) 5, 3, 5, 7, 8, 2, 4, 1, 2, 2, 3, 6

Solution

(a) Arrange the data in ascending order first.

$$1, 2, \underset{\substack{\uparrow \\ Q_1}}{3}, 4, 6, \underset{\substack{\uparrow \\ Q_2}}{7}, 8, 9, \underset{\substack{\uparrow \\ Q_3}}{10}, 11, 15$$

median $Q_2 = 7$, $Q_1 = 3$, $Q_3 = 10$.
interquartile range $= Q_3 - Q_1 = 10 - 3 = 7$.

(b) Arrange the data in ascending order first.

$$1, \underset{\substack{\uparrow \\ Q_1}}{\underbrace{2, 3}}, 5, \underset{\substack{\uparrow \\ Q_2}}{6}, 7, \underset{\substack{\uparrow \\ Q_3}}{\underbrace{8, 13}}, 14$$

median $Q_2 = 6$, $Q_1 = \dfrac{2+3}{2} = 2.5$, $Q_3 = \dfrac{8+13}{2} = 10.5$.
interquartile range $= Q_3 - Q_1 = 10.5 - 2.5 = 8$.

(c) Arrange the data in ascending order.

$$2, 5, \underset{\substack{\uparrow \\ Q_1}}{6}, 12, \underset{\substack{\uparrow \\ Q_2}}{\underbrace{13, 15}}, 18, \underset{\substack{\uparrow \\ Q_3}}{22}, 50, 52$$

median $Q_2 = \dfrac{13+15}{2} = 14$, $Q_1 = 6$, $Q_3 = 22$.
interquartile range $= Q_3 - Q_1 = 22 - 6 = 16$.

(d) Arrange the data in ascending order.

$$1, 2, \underset{\substack{\uparrow \\ Q_1}}{\underbrace{2, 2}}, 3, \underset{\substack{\uparrow \\ Q_2}}{\underbrace{3, 4}}, 5, \underset{\substack{\uparrow \\ Q_3}}{\underbrace{5, 6}}, 7, 8$$

median $Q_2 = \dfrac{3+4}{2} = 3.5$, $Q_1 = \dfrac{2+2}{2} = 2$, $Q_3 = \dfrac{5+6}{2} = 5.5$.
interquartile range $= Q_3 - Q_1 = 5.5 - 2 = 3.5$.

Frequency Tables and Grouped Data

If we are given a list of large amounts of data values, we can summarize them as a frequency table. If we are given a frequency table, we can always expand it into a list of all data values. For example, suppose we are given the following exact data value:

x	10	20	30
Frequency	3	2	4

We can write this out as a list: 10, 10, 10, 20, 20, 30, 30, 30, 30

From this list we can calculate the statistics such as the mean and standard deviation as before.

From the frequency table, we can find the mean

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i \cdot x_i}{\sum_{i=1}^{n} f_i}$$

where f_i is the frequency of x_i, and $\sum_{i=1}^{n} f_i$, the sum of all frequencies, is the number of data item.

When we are given the grouped data values to estimate the standard deviation, we replace each group by the mid-point value. For example,

x	[50, 100)	[100, 200)	[200, 300)	[300, 400)
Frequency	15	20	42	8

is replaced by

Midpoint	75	150	250	350
f	15	20	42	8

Here $[a, b) = \{x \mid a \leq x < b\}$.

We can now calculate the variance and standard deviation for frequency table.

The formulae for the variance and standard deviation for grouped data are

$$\text{variance} = \sigma^2 = \frac{\sum_{i=1}^{n} \left(f_i \cdot (x_i - \bar{x})^2\right)}{\sum_{i=1}^{n} f_i}$$

and

$$\text{standard deviation} = \sigma = \sqrt{\text{variance}}.$$

We have the similar formula for the variance of group data as follow:

$$\sigma^2 = \frac{\sum(f_i \cdot x_i^2)}{\sum f_i} - (\bar{x})^2$$

Example 4.

Find the mean and standard deviation for the following frequency table:

x	5	6	7	8	9
Frequency(f)	9	9	10	12	10

Solution

We construct the table as

x	f	$f \cdot x$	x^2	$f \cdot x^2$
5	9	45	25	225
6	9	54	36	324
7	10	70	49	490
8	12	96	64	768
9	10	90	81	810
	$\sum f = 50$	$\sum(f \cdot x) = 355$		$\sum(f \cdot x^2) = 2617$

$$\text{Mean} = \bar{x} = \frac{\sum(f \cdot x)}{\sum f} = \frac{355}{50} = 7.1$$

$$\text{Variance} = \sigma^2 = \frac{\sum(f \cdot x^2)}{\sum f} - (\bar{x})^2 = \frac{2617}{50} - (7.1)^2 = 1.93$$

$$\text{Standard deviation} = \sigma = \sqrt{1.93} = 1.3892$$

Students can find means, variances, and standard deviations directly by inputting the given data into a calculator.

Example 5.

A man recorded the length, in minutes, of each telephone call he made for a month. These data are summarized in the table below. Find the mean and standard deviation for the data.

Length of telephone call (minute)	$(0, 10]$	$(10, 20]$	$(20, 30]$	$(30, 40]$	$(40, 50]$
Frequency	4	15	5	12	10

Here $(a, b] = \{x \mid a < x \leq b\}$.

Solution

We construct the table as

Frequency(f)	midpoint (x)	$f \cdot x$	x^2	$f \cdot x^2$
4	5	20	25	100
15	15	225	225	3375
5	25	125	625	3125
12	35	420	1225	14700
10	45	450	2025	20250
$\sum f = 46$		$\sum(f \cdot x) = 1240$		$\sum(f \cdot x^2) = 41550$

$$\text{Mean} = \bar{x} = \frac{\sum(f \cdot x)}{\sum f} = \frac{1240}{46} = 26.9565$$

$$\text{Variance} = \sigma^2 = \frac{\sum(f \cdot x^2)}{\sum f} - (\bar{x})^2 = \frac{41550}{46} - (\frac{1240}{46})^2 = 176.6068$$

$$\text{Standard deviation} = \sigma = \sqrt{176.6068} = 13.2893$$

Exercise 6.1

1. Find the median and the interquartile range for the following sets of data:
   (a) 16, 18, 22, 19, 3, 21, 17, 20 (b) 33, 38, 43, 30, 29, 40, 51
   (c) 24, 32, 54, 31, 16, 18, 19, 14, 17, 20 (d) 14, 16, 27, 18, 13, 19, 36, 15, 20

2. Fifteen students do a mathematics test. Their marks are as follows:
   $$7, 4, 9, 7, 6, 10, 12, 11, 3, 8, 5, 9, 8, 7, 3$$
   Find the median and the interquartile range.

3. Find the mean, variance and the standard deviation of the following data sets.
   (a) 1, 2, 3, 4, 5, 6, 7 (b) 5, 6, 7, 3, 2, 9, 11 (c) 4, 3, 2, 7, 0, 9
   (d) 5, 5, 5, 4, 2, 7, 7, 7, 7 (e) 1, 3, 5, 5, 7, 9, 10, 15, 17

4. The numbers of errors, x, on each of 200 pages of typescript was monitored. The results when summarized showed that
   $$\sum x = 1000, \qquad \sum x^2 = 5500.$$
   Calculate the mean and the standard deviation of the number of errors per page.

5. In a student group, a record was kept for the number of absent days each student had over one particular term. Calculate the mean and standard deviation for these data shown in the table below.

Number of absent days (x)	0	1	2	3	4
Number of students (f)	15	10	20	8	2

6. A moth trap was set every night for five weeks. The number of moths caught in the trap was recorded. Calculate the mean and standard deviation for these data shown in the table below.

Number of moths (x)	7	8	9	10	11
Frequency (f)	3	8	9	5	9

7. The lifetimes of 80 batteries, to the nearest hour (h), is shown in the following table. Calculate the mean and standard deviation for these data.

Lifetime (h)	$(6, 10]$	$(10, 15]$	$(15, 20]$	$(20, 25]$	$(25, 30]$
Number of batteries	12	8	15	30	15

8. The table summarizes the distances travelled by 150 students to college each day. Calculate the mean and standard deviation for these data.

Distance (s)	$(0, 2]$	$(2, 4]$	$(4, 6]$	$(6, 8]$	$(8, 10]$	$(10, 12]$
Number of students	8	22	55	35	18	12

6.2 Cumulative Frequency

In this section we describe how to find the median and quartiles by using **cumulative frequency** which is a new way to represent the data. Cumulative frequency is a count of the total number of data items up to a certain value.

Example 6.

The following table shows the number of sweets in a pod, for a sample of 100 pods, taken at random. Construct a cumulative frequency table.

Number of sweets per pod	2	3	4	5	6	7	8
Number of pods	5	10	10	14	19	27	15

Solution

The cumulative frequency table is obtained as follows:

Number of sweets per pod(x)	2	3	4	5	6	7	8
Cumulative frequency	5	15	25	39	58	85	100

Example 7.

The frequency table shows the marks of 84 students in a Mathematics test.

Mark (x)	$(0, 20]$	$(20, 40]$	$(40, 60]$	$(60, 80]$	$(80, 100]$
Frequency	12	20	34	14	4

(a) Construct a cumulative frequency table and draw a cumulative frequency curve for the Mathematics marks.

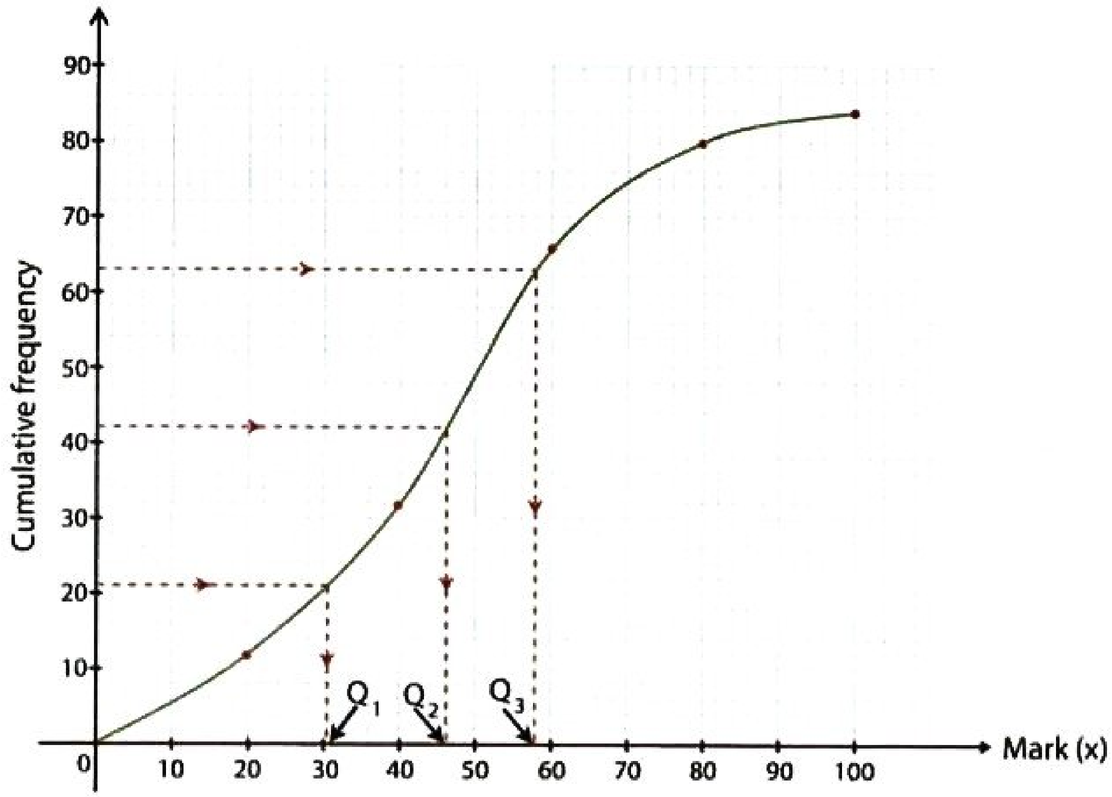
(b) Find the median and interquartile range.

Solution

(a) The cumulative frequency table is obtained as follow:

Mark (x)	$x \leq 20$	$x \leq 40$	$x \leq 60$	$x \leq 80$	$x \leq 100$
Cumulative frequency	12	32	66	80	84

Here the first row of the cumulative frequency table consists of the upper boundaries of the data groups.



We notice that nobody got mark 0, so $(0, 0)$ is the leftmost point in the diagram. Therefore we plot the five points $(20, 12), (40, 32), \ldots, (100, 84)$ from the cumulative frequency table plus the additional point $(0, 0)$ and join them up with a smooth curve.

(b) The total frequency is 84, so draw lines across from the y-axis at

$0.25 \times 84 = 21$ for the lower quartile,

$0.5 \times 84 = 42$ for the median and

$0.75 \times 84 = 63$ for the upper quartile.

Therefore, we estimate the values Q_1, Q_2 and Q_3 on the x-axis from the graph. So,

lower quartile $Q_1 \approx 30$, ($\approx$ means approximately equal)

the median $Q_2 \approx 46$

upper quartile $Q_3 \approx 58$

the interquartile range $= Q_3 - Q_1 = 58 - 30 = 28$

Example 8.

The table shows the mass, in kilograms, of 65 workers of a factory.

Mass (x)	$50 < x \leq 55$	$55 < x \leq 60$	$60 < x \leq 65$	$65 < x \leq 70$	$70 < x \leq 75$
Number of workers	10	20	20	10	5

(a) Construct a cumulative frequency table and draw a cumulative frequency curve from the given information.

(b) Find the median and the interquartile range.

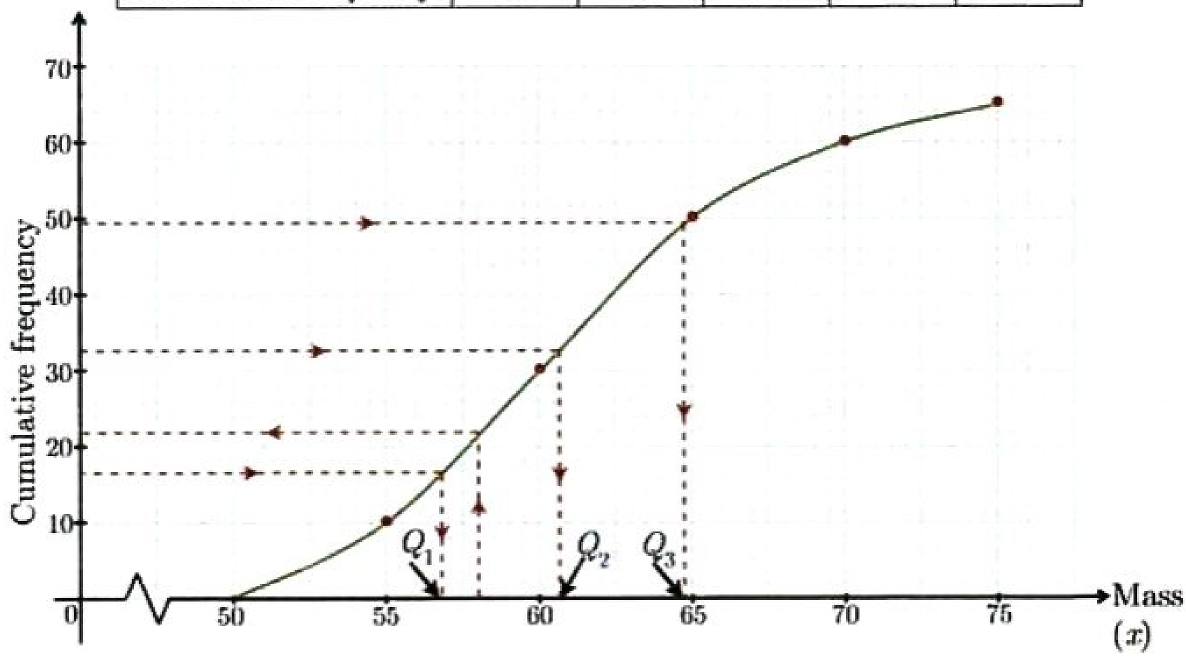
(c) How many workers have mass 58 kg or less?

(d) How many workers have mass of more than 58 kg?

Solution

(a)

Mass (x)	$x \leq 55$	$x \leq 60$	$x \leq 65$	$x \leq 70$	$x \leq 75$
Cumulative frequency	10	30	50	60	65



(b) The total frequency is 65, so draw lines across from the y-axis at

$0.25 \times 65 = 16.25$ for the lower quartile,
$0.5 \times 65 = 32.5$ for the median,
$0.75 \times 65 = 48.75$ for the upper quartile.

Therefore, we estimate the values Q_1, Q_2, Q_3 on the x-axis on the graph. Thus

lower quartile $Q_1 \approx 57$
the median $Q_2 \approx 61$
upper quartile $Q_3 \approx 65$
the interquartile range$= Q_3 - Q_1 = 65 - 57 = 8$

(c) Number of workers whose mass less than 58 kg $= 22$
(d) Number of workers whose mass greater than 58 kg $= 65 - 22 = 43$.

Exercise 6.2

1. A random sample of 120 students was asked how long it took them to complete their homework the previous night. The time was recorded and summarized in the table below:

time (m min)	$25 < m \leq 30$	$30 < m \leq 35$	$35 < m \leq 40$	$40 < m \leq 50$	$50 < m \leq 80$
Frequency	40	30	20	10	20

   (a) Draw a cumulative frequency curve to illustrate this information.

   (b) Find the median and interquartile range.

2. The lengths, in centimetres, of 400 fish in a fish farm were measured. The results are shown in the table below:

Length(x cm)	$30 < x \leq 40$	$40 < x \leq 50$	$50 < x \leq 60$	$60 < x \leq 70$	$70 < x \leq 80$
Number of fish	60	100	120	80	40

   (a) Draw a cumulative frequency curve for the information.

   (b) Find the median and interquartile range.

3. The table shows the marks obtained by 100 students in a mathematics examination.

Mark (m)	$(0, 20]$	$(20, 40]$	$(40, 60]$	$(60, 80]$	$(80, 100]$
Number of students	5	12	28	36	19

   (a) Construct a cumulative frequency table and draw a cumulative frequency curve from the given information.

   (b) Estimate (i) the number of students obtaining 50 marks or less, and
   (ii) the number of students obtaining more than 88 marks.

4. The masses, in kilograms, of 500 boys in a school are recorded and shown in the table below:

Mass(x kg)	(45, 50]	(50, 55]	(55, 60]	(60, 65]	(65, 70]	(70, 75]	(75, 80]
Frequency	20	50	150	140	80	40	20

   (a) Draw a cumulative frequency curve for the information.

   (b) Find the median and interquartile range.

5. A sample of 240 eggs from a farm were weighted. The results are shown in the table below:

Mass (x g)	(35, 40]	(40, 45]	(45, 50]	(50, 55]	(55, 60]	(60, 65]
Number of eggs	25	55	80	50	25	5

   (a) Construct a cumulative frequency table and draw a cumulative frequency curve from the given information.

   (b) How many eggs have mass 48 g or less?

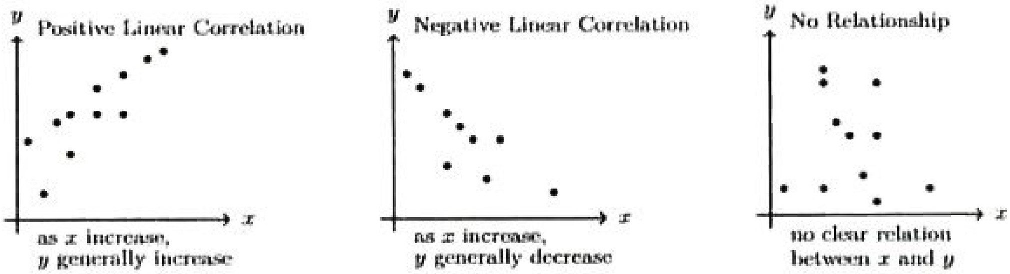
   (c) How many eggs have mass of more than 48 g?

6.3 Correlation

In this section we only explain the linear relation between independent and dependent variables. Therefore we collect two variables to see whether a relationship exists between them. This can be illustrated by a scatter diagram.

> **A scatter diagram** is a graph of the ordered pairs (x, y) of numbers consisting of the independent variable x and the dependent variable y.

A linear correlation is the statistical method used to determine a relationship between dependent and independent variables.



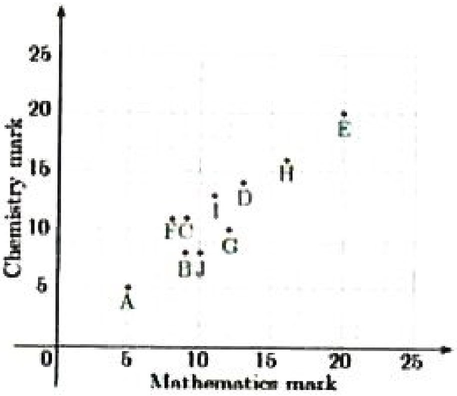
A positive correlation is a relationship between two variables when both variables increase or decrease at the same time. A negative correlation is a relationship between two variables when one variable increases, the other variable decreases, and vice versa.

Example 9.

Ten students sat a Mathematics test and a Chemistry test. Both tests were marked out of 20. Their marks are as shown below. Draw a scatter diagram to represent these data.

Students	A	B	C	D	E	F	G	H	I	J
Mathematics mark	5	9	9	13	20	8	12	16	11	10
Chemistry mark	5	8	11	14	20	11	10	16	13	8

Solution



From the diagram we see that the points lie approximately on a straight line, the higher the Mathematics mark the higher the Chemistry mark. It shows positive linear relationship.

Correlation coefficient

A **correlation coefficient** r is used to determine the strength of the linear relationship between two variables. The value of r varies between -1 and 1.

Formula for the Correlation Coefficient r

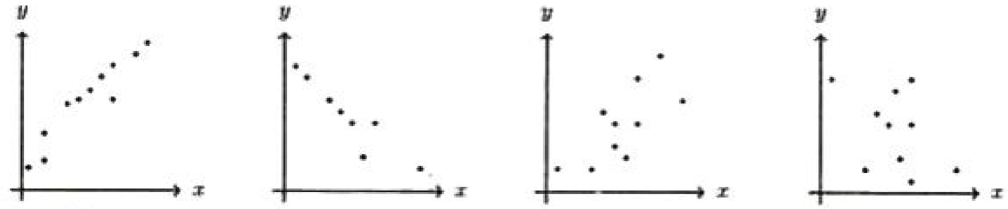
$$r = \frac{n\sum xy - \sum x \sum y}{\sqrt{(n\sum x^2 - (\sum x)^2)(n\sum y^2 - (\sum y)^2)}}$$

Value of r	Interpretation
$r \approx 1$	Strong positive linear correlation
$r \approx 0$	No linear correlation
$r \approx -1$	Strong negative linear correlation

Note. The value $r = 0$ does not mean that there is no relationship between the two variables – only that there is no linear relationship.

Example 10.

The scatter diagrams show various degrees of correlation.



(a) (b) (c) (d)

Match the diagrams with the correlation coefficients below.

(i) $r = -0.31$ (ii) $r = -0.94$ (iii) $r = 0.55$ (iv) $r = 0.97$.

Solution

Figure (a) is closed to strong positive linear correlation. So figure (a) is matched with (iv).
Figure (b) is closed to strong negative linear correlation. So figure (b) is matched with (ii).
Figure (c) is matched with (iii).
Figure (d) is matched with (i).

Example 11.

Calculate the correlation coefficient of the following summary data:

$$\sum x = 15, \quad \sum y = 30, \quad \sum x^2 = 55, \quad \sum y^2 = 190, \quad \sum xy = 100, \quad n = 5.$$

Solution

$$\begin{aligned} n\sum x^2 - \left(\sum x\right)^2 &= 5 \times 55 - (15)^2 = 50. \\ n\sum y^2 - \left(\sum y\right)^2 &= 5 \times 190 - (30)^2 = 50. \\ n\sum xy - \sum x \sum y &= 5 \times 100 - (15)(30) = 50. \end{aligned}$$

Therefore, $r = \dfrac{50}{\sqrt{50 \times 50}} = 1$. We have a perfect positive correlation.

Example 12.

Calculate the correlation coefficient of the following summary data:

$$\sum x = 57, \quad \sum y = 511, \quad \sum x^2 = 579, \quad \sum y^2 = 38993, \quad \sum xy = 3745, \quad n = 7.$$

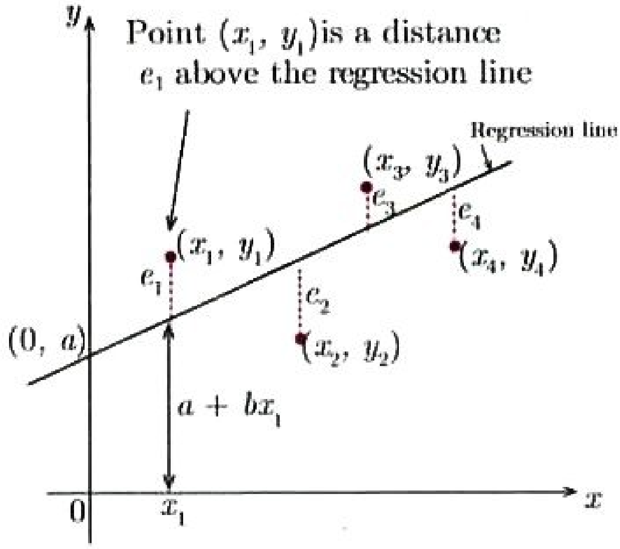
Solution

$$n\sum x^2 - \left(\sum x\right)^2 = 7 \times 579 - (57)^2 = 804.$$
$$n\sum y^2 - \left(\sum y\right)^2 = 7 \times 38993 - (511)^2 = 11830.$$
$$n\sum xy - \sum x \sum y = 7 \times 3745 - (57)(511) = -2912.$$

Therefore, $r = \dfrac{-2912}{\sqrt{804 \times 11830}} = -0.9442$. We have a negative correlation.

Linear Regression

We now are ready to consider the nature of relationship between variables. If the value of correlation coefficient is significant, we can find the equation of the **regression line,** also known as **line of best fit**. It is a statistical method used to determine the strength and character of the relationship between two variables. The purpose of regression line is to make predictions on the basis of the data.



For each point on a scatter diagram, we can express y in terms of x as $y_i = (a + bx_i) \pm e_i$, where e_i is the vertical distance from the regression line. If the points are close to the line, (i.e., e_i closes to 0), we get the better prediction on the data.

The equation of the regression line is

$$y = a + bx$$

where $a = \dfrac{(\sum y)(\sum x^2) - (\sum x)(\sum xy)}{n\sum x^2 - (\sum x)^2}$ and $b = \dfrac{n\sum xy - (\sum x)(\sum y)}{n\sum x^2 - (\sum x)^2}$.

Example 13.

The results from an experiment in which different masses were placed on a spring and the resulting length of the spring measured, are shown below. Find the equation of the regression line of the data.

Mass (x kg)	20	40	60	80	100
Length (y cm)	40	50	60	60	70

Solution

$\sum x = 300, \quad \sum x^2 = 22000, \quad \sum xy = 18200, \quad \sum y = 280$

We compute that

$$n\sum x^2 - \left(\sum x\right)^2 = 5 \times 22000 - (300)^2 = 20000$$

$$n\sum xy - \sum x \sum y = 5 \times 18200 - (300)(280) = 7000$$

Therefore, we obtain

$$a = \frac{(\sum y)(\sum x^2) - (\sum x)(\sum xy)}{n\sum x^2 - \left(\sum x\right)^2} = 35$$

and

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2} = 0.35$$

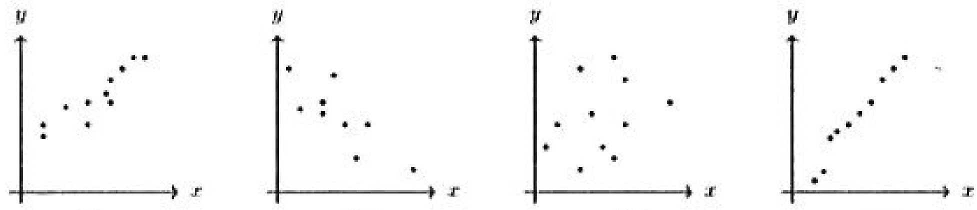
The regression line is $y = 35 + 0.35x$

Students can get the regression line and the relative correlation coefficient by using a calculator with inputting the given data into the calculator.

Exercise 6.3

1. A student works out the correlation coefficient between the heights of a group of fathers and the heights of their sons to be 0.954. Write down what this tells you about the relationship between their heights.

2. Some scatter diagrams are shown below.



(i) (ii) (iii) (iv)

Write down which scatter diagram shows:

(a) a correlation of +0.99,

(b) a correlation that could be described as positive correlation,

(c) a correlation of -0.97,

(d) a correlation that shows almost no correlation.

3. Given the following summary data,
$\sum x = 367$, $\sum y = 270$, $\sum x^2 = 33845$, $\sum y^2 = 12976$, $\sum xy = 17135$, $n = 6$,
calculate the correlation coefficient.

4. Ten children had their IQ measured and then took a general knowledge test. Their IQ, (x), and their marks, (y), for the test were summarized as follows:
$\sum x = 150$, $\sum y = 100$, $\sum x^2 = 20000$, $\sum y^2 = 9000$, $\sum xy = 10000$

   (a) Calculate the correlation coefficient.

   (b) Describe and interpret the correlation coefficient between IQ and general knowledge.

5. In a training scheme for young people, the average time taken for each age group to reach a certain level of proficiency was measured. The data are shown in the table.

Age (x years)	14	15	16	17	18	19	20	21	22	23
Average time (y hours)	8	6	7	9	8	11	9	10	11	12

   (a) Calculate the correlation coefficient.

   (b) Describe and interpret the relationship between average time and age.

6. An accountant monitors the number of items (x) produced per month by a company together with the total production costs (y). The table shows these data.

Number of items (x)	20	30	45	50	60	70	20	35	57	85	32	70
Production costs (y)	40	60	70	45	89	96	37	53	83	40	80	75

Find the equation of the regression line in the form $y = a + bx$.

7. The relationship between the number of coats of paint applied to a boat and the resulting weather resistance was tested in a laboratory. Find the equation of the regression line and correlation coefficient for data shown in the table below.

Coats of paint (x)	1	2	3	4	5
Protection (y years)	1.6	3.1	3.9	4.8	7.6

Chapter 7

Circles

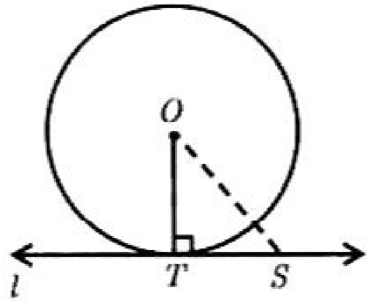
In the Grade 10 Text book, we learned about properties of angles in a circle and properties of chords. Now, we will learn properties of tangents and properties of points to be concyclic.

7.1 Properties of Tangents

In this section, properties of tangents and the relation between lengths of tangents and secants will be studied.

In a circle with centre O and radius r, if P is a point such that $OP < r$, then P is inside the circle. If $OP = r$, then P is on the circle. If $OP > r$, then P is outside the circle. By using this facts we will prove the following theorem.

Theorem 7.1. A line perpendicular to the radius of a circle at the point of contact is a tangent to the circle.

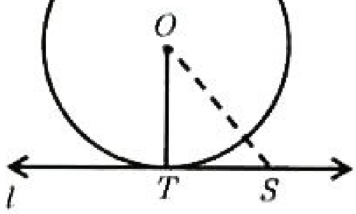


Given : The line l is perpendicular to the radius OT at T.

To prove : l is a tangent to $\odot O$.

Proof : Let S be any point on l distinct from T.
Then OS is the hypotenuse of the right triangle OTS.
Hence $OS > OT$.
Since OT is the radius of the circle, S is outside the circle.
Therefore l passes through the circle only at the point T,
i.e., l is a tangent to $\odot O$.

Theorem 7.2. A tangent to a circle is perpendicular to the radius at the point of contact.

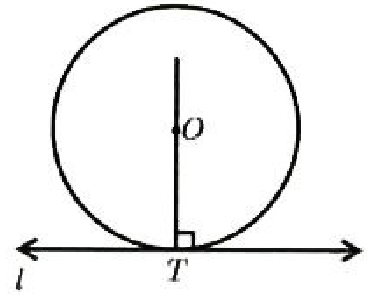


Given : The line l is the tangent to $\odot O$ at T

To prove : $OT \perp l$

Proof : Take any point S on l distinct from T.
Then S is outside the circle O and $OS > OT$.
Hence OT is the shortest distance from O to the line l.
Therefore $OT \perp l$.

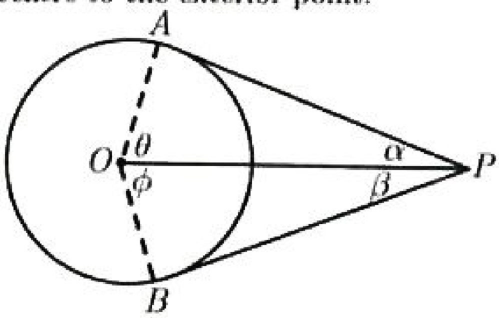
Corollary 7.3. If a line is perpendicular to a tangent to a circle at the point of contact, it passes through the centre of the circle.



Tangent Segment

A **tangent segment** of a circle is a segment of a tangent line whose endpoints are the point of contact and an exterior point of the circle.

Theorem 7.4. Two tangent segments drawn from an exterior point to the circle are equal in length. They subtend equal angles at the centre. They are equally inclined to the segment joining the centre to the exterior point.



Given : $\odot O$, and tangent segments PA and PB.
To prove : $PA = PB$, $\theta = \phi$ and $\alpha = \beta$.
Proof : Join OA, OB.

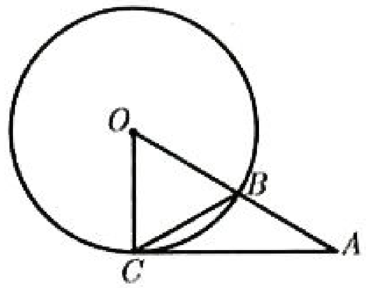
In $\triangle OAP$ and $\triangle OBP$

$$\begin{aligned} \angle OAP &= \angle OBP = 90^\circ && \text{(radius} \perp \text{tangent)} \\ OP &= OP && \text{(common side)} \\ OA &= OB && \text{(radii)} \\ \therefore \triangle OAP &\cong \triangle OBP && \text{(RHS)} \end{aligned}$$

$\therefore PA = PB$, $\theta = \phi$ and $\alpha = \beta$

Example 1.

In the given $\odot O$, $AB = OB$ and $\angle BOC = 60^\circ$. Find $\angle A$. Is AC a tangent segment?



Solution

$$\begin{aligned} OB &= OC \quad \text{(radii)} \\ \therefore \angle OCB &= \angle OBC \\ \angle OBC &= \angle OCB = \frac{180^\circ - 60^\circ}{2} = 60^\circ. \end{aligned}$$

So $\triangle OBC$ is equilateral.

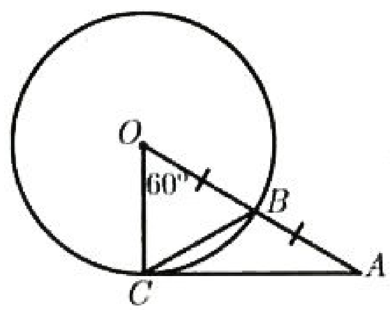
$$\begin{aligned} \therefore BC &= OB \\ AB &= OB \quad \text{(given)} \\ \therefore BC &= AB \\ \therefore \angle A &= \angle BCA \end{aligned}$$

In $\triangle ABC$,

$$\begin{aligned} \angle A + \angle BCA &= \angle OBC = 60^\circ \\ 2\angle BCA &= 60^\circ \quad (\because \angle BCA = \angle A) \\ \angle BCA &= 30^\circ \\ \angle OCA &= \angle OCB + \angle BCA = 60^\circ + 30^\circ = 90^\circ \end{aligned}$$

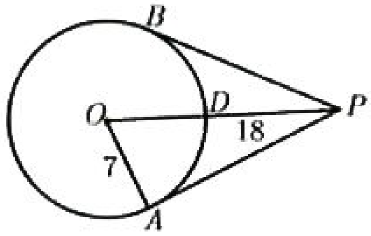
$\therefore OC \perp AC$.

Therefore AC is a tangent.



Example 2.

In the given $\odot O$, PA and PB are tangent segments, $OA = 7$ and $DP = 18$. Find PA and PB.



Solution

$$\begin{aligned} OD &= OA \quad \text{(radii)} \\ \therefore OD &= 7 \\ OP &= OD + DP = 7 + 18 = 25 \\ \angle OAP &= 90^\circ \quad \text{(radius} \perp \text{tangent)} \end{aligned}$$

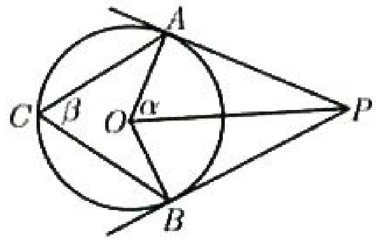
$$\begin{aligned} \therefore OP^2 &= OA^2 + PA^2 \\ 25^2 &= 7^2 + PA^2 \\ PA^2 &= 625 - 49 = 576 \\ PA &= 24 \end{aligned}$$

$PB = PA$ (tangent segments from the same external point)

$\therefore PB = 24.$

Example 3.

In the figure, PA and PB are tangents to $\odot O$ and $\angle APB = 60^\circ$. Find α and β.



Solution

$$\begin{aligned} \angle OAP &= 90^\circ \quad \text{(radius} \perp \text{tangent)} \\ \angle APO &= \frac{1}{2}\angle APB \quad (\because \angle APO = \angle BPO). \\ \angle APO &= \frac{1}{2} \times 60^\circ = 30^\circ \end{aligned}$$

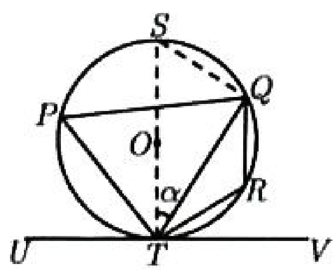
In right ΔOAP,

$$\alpha + \angle APO = 90^\circ$$
$$\alpha + 30^\circ = 90^\circ$$
$$\therefore \alpha = 60^\circ$$

PA and PB subtend equal angles at the centre.

$$\therefore \angle POB = \alpha$$
$$\therefore \angle POB = 60^\circ$$
$$\angle AOB = \alpha + \angle POB = 60^\circ + 60^\circ = 120^\circ$$
$$\beta = \frac{1}{2}\angle AOB \quad \text{(subtended by arc } AB\text{)}$$
$$\therefore \beta = \frac{1}{2} \times 120^\circ = 60^\circ.$$

Theorem 7.5. An angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment of the circle.



Given : In $\odot O, UV$ is the tangent to the circle at T, and QT is a chord.

To prove : $\angle QTV = \angle P$ and $\angle QTU = \angle R$.

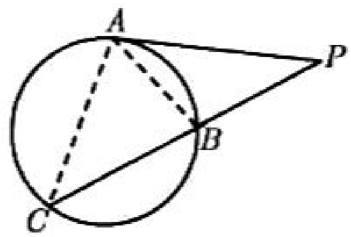
Proof : Draw a diameter TS. Join S and Q.

$$\angle S = \angle P \quad \text{(subtended by arc } QT\text{)}$$
$$\angle SQT = 90^\circ \quad (\because ST \text{ is a diameter})$$
$$\therefore \angle S + \alpha = 90^\circ$$
$$\angle STV = 90^\circ \quad \text{(radius} \perp \text{tangent)}$$
$$\angle QTV + \alpha = 90^\circ$$
$$\therefore \angle S + \alpha = \angle QTV + \alpha$$
$$\therefore \angle S = \angle QTV$$
$$\therefore \angle QTV = \angle P$$
$$\angle P + \angle R = 180^\circ \quad (PQRT \text{ is a cyclic quadrilateral})$$
$$\angle QTU + \angle QTV = 180^\circ$$
$$\therefore \angle QTU + \angle QTV = \angle P + \angle R$$
$$\therefore \angle QTU = \angle R.$$

Theorem 7.6. (Converse of Theorem 7.5) If a line drawn through an endpoint of a chord of a circle forms an angle that is equal to an angle in the alternate segment, then the line is a tangent to the circle.

The following theorem shows the product property of lengths of tangent and secant segments.

Theorem 7.7. If a tangent and a secant of a circle are drawn from a point outside the circle, the square of the length of the tangent segment is equal to the product of the lengths of the secant segment and its external segment.



Given : PA is a tangent segment and PBC is a secant segment.

To prove : $PA^2 = PC \cdot PB$

Proof : Join AB, AC.

In $\triangle PAB$ and $\triangle PCA$

$\angle P = \angle P$ (common angle)

$\angle PAB = \angle C$ (PA is a tangent and AB is a chord)

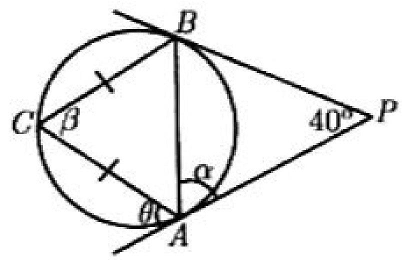
$\therefore \triangle PAB \sim \triangle PCA$ (AA Similarity)

$$\therefore \frac{PA}{PC} = \frac{PB}{PA}$$

$$\therefore PA^2 = PC \cdot PB.$$

Example 4.

In the figure, $AC = BC$, and PA and PB are tangent to the circle at A and B respectively. Find the values of α, β and θ.



Solution

In $\triangle PAB$,

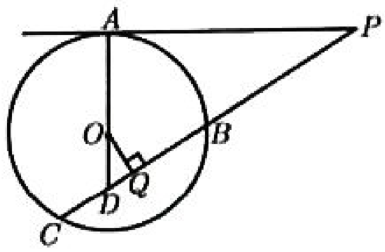
$$\begin{aligned} PA &= PB && \text{(tangent segments from an external point)} \\ \therefore \alpha &= \angle PBA \\ \alpha &= \angle PBA = \frac{180^\circ - 40^\circ}{2} = 70^\circ \\ \alpha &= \beta && \text{(}PA\text{ is a tangent and }AB\text{ is a chord)} \\ \therefore \beta &= 70^\circ \end{aligned}$$

In $\triangle ABC$,

$$\begin{aligned} \angle CBA &= \angle CAB && (\because AC = BC) \\ \therefore \angle CBA &= \angle CAB = \frac{180^\circ - 70^\circ}{2} = 55^\circ \\ \theta &= \angle CBA && \text{(}PA\text{ is a tangent and }CA\text{ is a chord)} \\ \therefore \theta &= 55^\circ. \end{aligned}$$

Example 5.

In the figure, PA is tangent to $\odot O$, $PA = 10\sqrt{3}$, $AD = 10$ and $CD = 5$. Find PB and BQ.

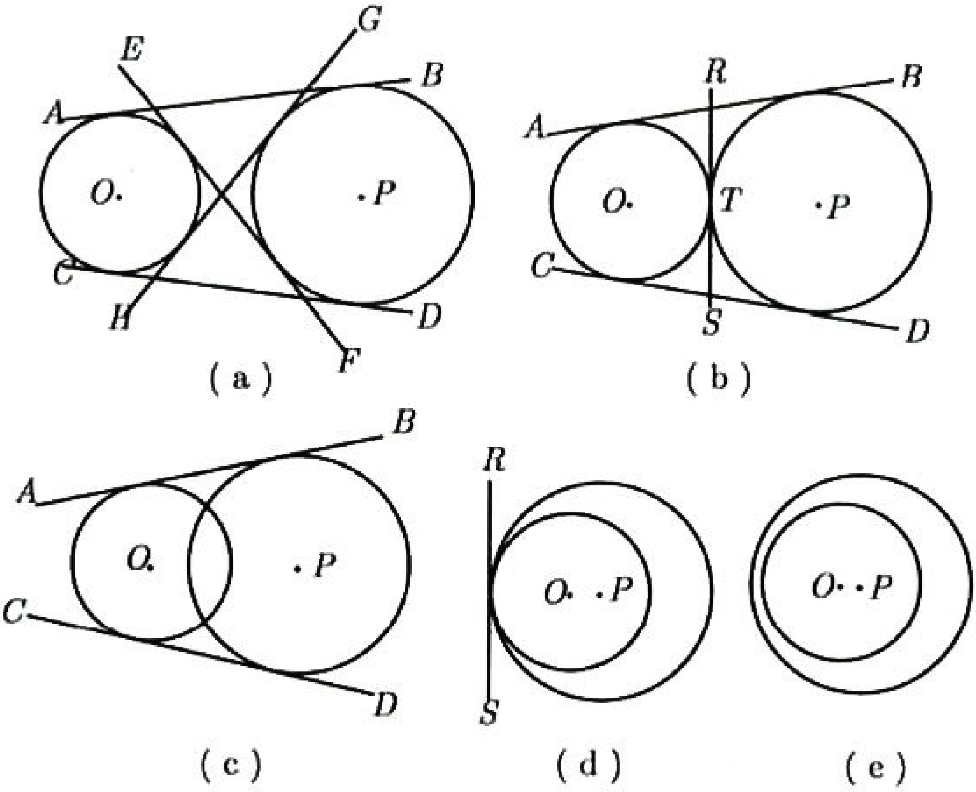


Solution

$$\begin{aligned} \angle DAP &= 90^\circ && \text{(radius} \perp \text{tangent)} \\ \therefore PD^2 &= PA^2 + AD^2 \\ &= (10\sqrt{3})^2 + 10^2 \\ &= 300 + 100 = 400 \\ \therefore PD &= 20 \\ PC &= PD + DC = 20 + 5 = 25 \\ \therefore PC &= 25 \\ PA^2 &= PB \cdot PC \\ 300 &= PB \times 25 \\ PB &= 12 \\ \therefore BC &= PC - PB = 25 - 12 = 13 \\ BQ &= \frac{1}{2}BC && (OQ \perp BC) \\ \therefore BQ &= \frac{13}{2} \end{aligned}$$

Common Tangents

A line which is tangent to more than one circle is called a **common tangent**. A common tangent is called a **common external tangent** if the circles are on the same side of the tangent. A common tangent is called a **common internal tangent** if the circles are on opposite sides of the tangent.



Figure(a) has four common tangents; two external tangents AB, CD and two internal tangents EF, GH.

Figure(b) has three common tangents; two external tangents AB, CD and one internal tangent RS.

Figure(c) has two common external tangents AB, CD.

Figure(d) has one common external tangent RS.

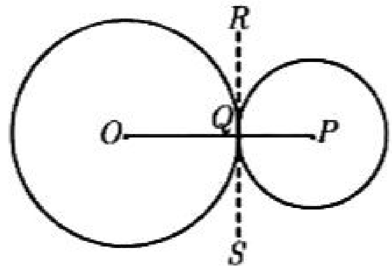
Figure(e) has no common tangent.

Two circles are said to be **tangent internally** if exactly one common external tangent can be drawn to these circles. Two circles are said to be **tangent externally** if exactly one common internal tangent can be drawn to these circles.

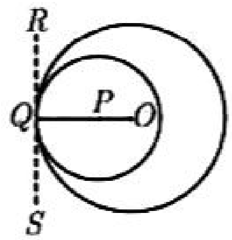
In Figure(b), the two circles are tangent externally.

In Figure(d), the two circles are tangent internally.

Theorem 7.8. When two circles are tangent externally or internally, their centres and the point of contact are collinear.



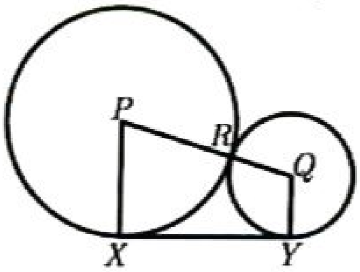
Given : $\odot O$ and $\odot P$ are tangent externally at Q.
To prove : O, Q and P are collinear.
Proof : Draw a common tangent RS at Q.
Join OQ , PQ.
Then $OQ \perp RS$ and $PQ \perp RS$ (radius $\perp$ tangent)
$\angle OQS + \angle PQS = 180°$
Therefore OQP is a straight line.
Therefore O, Q and P are collinear.



Given : $\odot O$ and $\odot P$ are tangent internally at Q
To prove : O, P and Q are collinear.
Proof : Draw a common tangent RS at Q
Join OQ and PQ
Then $OQ \perp RS$ and $PQ \perp RS$ (radius $\perp$ tangent)
If O were not on PQ,
then $\angle OQS > \angle PQS$ or $\angle OQS < \angle PQS$.
These are impossible since $\angle OQS = \angle PQS = 90°$.
Hence O lies on PQ.
Therefore Q, P and O are collinear.

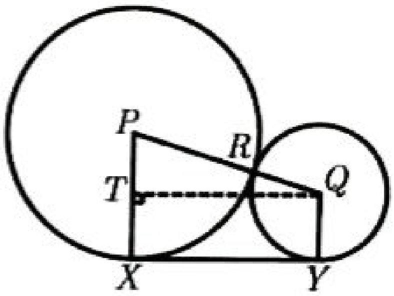
Example 6.

$\odot P$ and $\odot Q$ are tangent externally at R. If the radius of $\odot P$ is 16 and the radius of $\odot Q$ is 9, find the length of the common external tangent segment XY.



Solution

Since $\odot P$ and $\odot Q$ are tangent externally at R, P, R, Q are collinear.
$\angle PXY = \angle QYX = 90°$. (radius $\perp$ tangent)
Draw QT which is perpendicular to PX at T.
Then $QTXY$ is a rectangle.



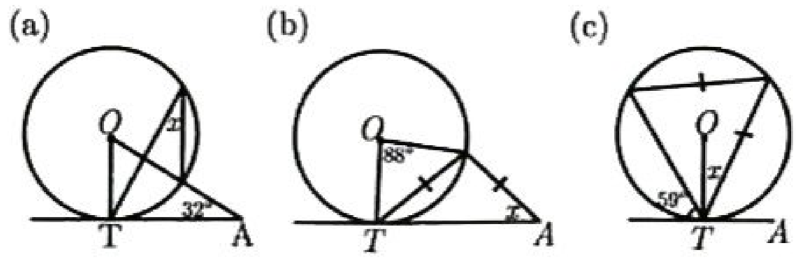
$$\begin{aligned}
\therefore TX &= QY \\
TX &= 9 \\
PT &= PX - TX = 16 - 9 = 7. \\
PQ &= PR + RQ = 16 + 9 = 25.
\end{aligned}$$

In right triangle PTQ,

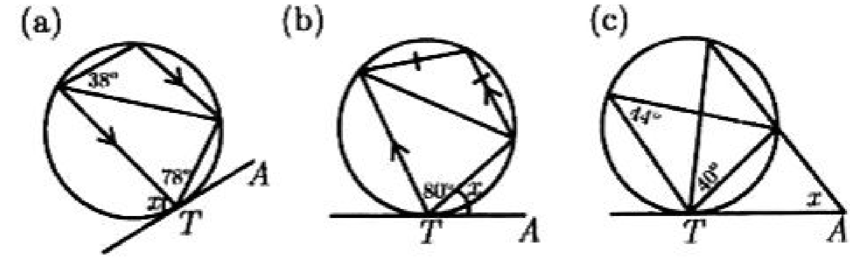
$$\begin{aligned}
PQ^2 &= PT^2 + TQ^2 \\
\therefore TQ &= \sqrt{PQ^2 - PT^2} = \sqrt{25^2 - 7^2} = 24 \\
XY &= TQ \quad \text{(opposite sides of rectangle } QTXY) \\
\therefore XY &= 24.
\end{aligned}$$

Exercise 8.1

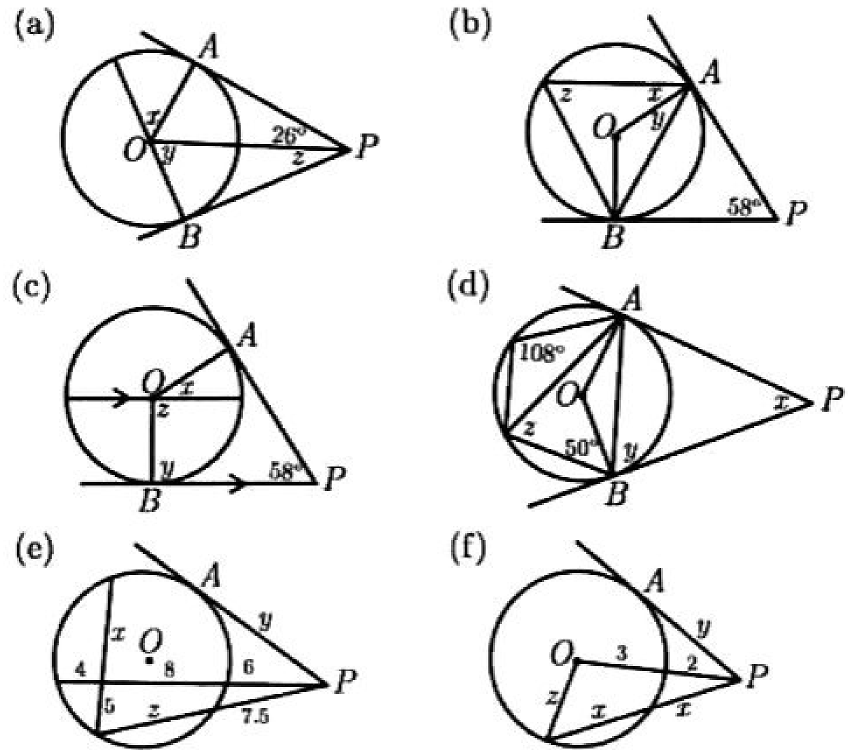
1. In the following figures, O is the centre and AT is a tangent. Find the value of x.



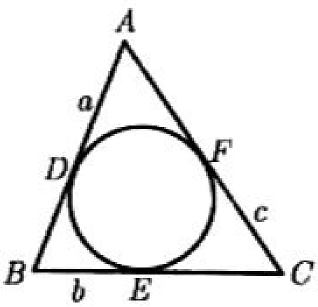
2. In the following figures, AT is a tangent. Find the value of x.



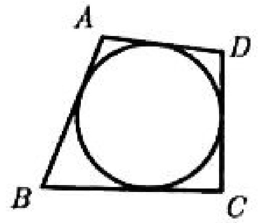
3. In the following figures, PA, PB are tangents to $\odot O$ at A and B respectively. Find the values of x, y and z.



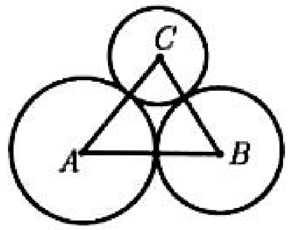
4. In the figure, the circle is inscribed in $\triangle ABC$. If $AB = 12$, $BC = 16$ and $AC = 20$, find a, b and c.



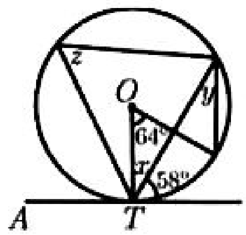
5. In the figure the circle is inscribed in the quadrilateral $ABCD$. If $AB = 20$, $DC = 14$ and $AD = 12$, find BC.



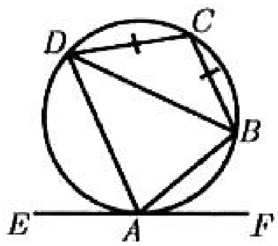
6. In the figure, $\odot A$, $\odot B$ and $\odot C$ are tangent to each other. If $AB = 18$, $BC = 10$ and $AC = 14$, find the radius of each circle.



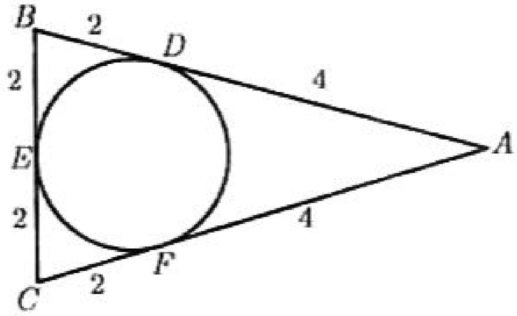
7. In the figure, AT is a tangent at T. Find x, y and z.



8. In the figure, EAF is tangent at A, $BC = CD$, $\angle BAF = \frac{1}{2}\angle BAD$. Prove that $AD \parallel BC$.



9. In the figure, circle is inscribed in $\triangle ABC$. Find the radius of the circle.



10. Two unequal circles are tangent externally at O. AB is a chord of the first circle. AB is tangent to the second circle at C, and AO meets the circle at E. Prove that $\angle BOC = \angle COE$.

11. Circles P and Q are congruent and tangent externally at C. A straight line ACB is drawn meeting one circle at A and the other at B. Prove that $AC = BC$.

12. Draw a circle and a tangent TAS meeting it at A. Draw a chord AB making $\angle TAB = 60°$ and another chord $BC \parallel TS$. Prove that $\triangle ABC$ is equilateral.

13. Two circle intersect at A, B. At A, a tangent is drawn to each circle meeting the circles again at P and Q respectively. Prove that $\angle ABP = \angle ABQ$.

14. Two circles touch internally at P. Through P two straight lines AB, CD are drawn meeting one circle at A, C and the other at B, D respectively. Prove that $AC \parallel DB$.

15. Prove Theorem 7.6: If a line drawn through an endpoint of a chord of a circle forms an angle that is equal to an angle in the alternate segment, then the line is a tangent to the circle.

16. Two circles touch externally at P. Through P two straight lines AB, CD are drawn meeting one circle at A, C and the other at B, D respectively and such that CB is a common external tangent. Prove that the circle CBP is tangent to AC and DB.

17. P,Q and R are points on a circle. The tangent at P meets RQ produced at T, and PC bisecting $\angle RPQ$ meets RQ at C. Prove that $\triangle TPC$ is isosceles.

18. AB and AC are equal chords of a circle. Prove that the tangent at A bisects the exterior angle between AB and AC.

19. PT is tangent to the circle at T, and PQR is a secant. $\odot T$ with radius TQ meets QR again at S. Prove that $\angle RTS = \angle RPT$.

20. Two circles intersect at A, B. The tangent to the first at A meets the second again at C, and the tangent to the second at B meets the first again at D. Prove that AD and CB are parallel.

21. Two unequal circles are tangent internally at A. BC, a chord of the larger circle, is tangent to the smaller circle at D. Prove that AD bisects $\angle BAC$.

22. M is the midpoint of a chord AB of a given circle. C is any point on the major arc AB, and CM meets the circle at D. The circle tangent to AB at A and passing through C intersects CD at E. Prove that $DM = ME$.

7.2 Concyclic Points

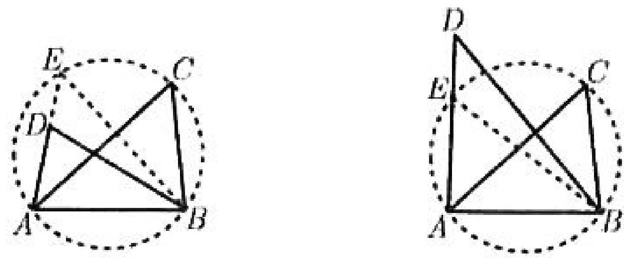
If three points A, B and C on a circle were collinear, the perpendicular bisectors of the chords AB and BC would be parallel. This is impossible because these two perpendicular bisectors pass through the centre. Therefore no circle contains three collinear points. In other words, three non-collinear points lie on a circle. Equivalently we can say that **every triangle is cyclic**. Moreover, any three non-collinear points determine a **unique** circle. The vertices of a polygon that all lie on one circle are called **concyclic points**.

We have known that

- angles inscribed in the same segment of a circle are equal;
- an inscribed angle subtended by a diameter is a right angle;
- opposite angles of a cyclic quadrilateral are supplementary;
- the exterior angle of a cyclic quadrilateral is equal to the opposite interior angle of the quadrilateral;
- if two chords of a circle intersect in the circle, the product of the lengths of segments of one chord is equal to the product of the lengths of segments of the other chord;
- if two secants are drawn to a circle from an external point, the product of the lengths of one secant and its external segment is equal to the product of the lengths of the other secant and its external segment.

The converses of the above statements are stated in the following theorems and a corollary.

Theorem 7.9. If a line segment subtends equal angles at two other points on the same side of it, then the two points and the endpoints of the segment are concyclic.

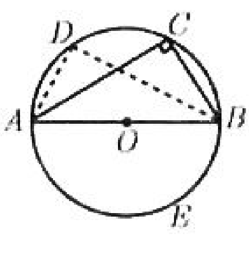


Given : Equal angles $\angle ACB$ and $\angle ADB$ are subtended by AB on the same side of AB.

To prove : A, B, C, D are concyclic.

Proof : Let the circle through A, B, C be drawn.
Suppose D is not on circle ABC.
Then D must be either inside or outside the circle.
If D is inside the circle, produce AD to meet the circle at E.
If D is outside the circle, let E be the intersection point of AD and the circle.
Join B and E.
In both cases, $\angle ACB = \angle AEB$.
But $\angle ACB = \angle ADB$.
$\therefore \angle ADB = \angle AEB$.
This is impossible because BD is not parallel to BE.
$\therefore D$ must be on circle ABC.
$\therefore A, B, C, D$ are concyclic.

Theorem 7.10. The circle described on the hypotenuse of a right triangle as diameter passes through the opposite vertex.



Given : $\triangle ABC$ is a right triangle with $\angle ACB = 90°$.
Circle AEB is described on AB as diameter.

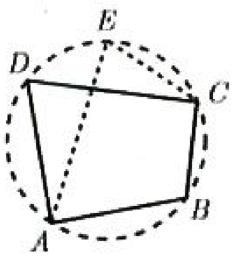
To prove : Circle AEB passes through C.

Proof : On the circle, take any point D on the same side of AB as C.
Join AD, BD.

$$\begin{aligned} \therefore \angle ADB &= 90^\circ && \text{(angle subtended by a diameter)} \\ \angle ACB &= 90^\circ && \text{(given)} \\ \therefore \angle ACB &= \angle ADB \end{aligned}$$

So A, B, C, D are concyclic.
Therefore circle AEB passes through C.

Theorem 7.11. If a pair of opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

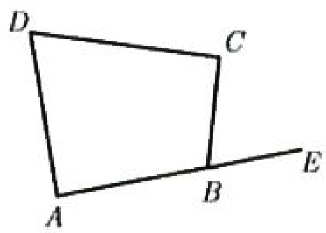


Given : $ABCD$ is a quadrilateral and $\angle B + \angle D = 180^\circ$.
To prove : $ABCD$ is cyclic.
Proof : Draw circle ABC and take any point E on the circle.
Join AE, CE.
Then A, B, C, E are concyclic.

$$\begin{aligned} \therefore \angle B + \angle E &= 180^\circ \\ \angle B + \angle D &= 180^\circ && \text{(given)} \\ \therefore \angle D &= \angle E \end{aligned}$$

So A, D, E, C are concyclic.
Hence D lies on circle AEC.
Since, B also lies on circle AEC,
then A, B, C, D lie on the same circle.
Therefore $ABCD$ is cyclic.

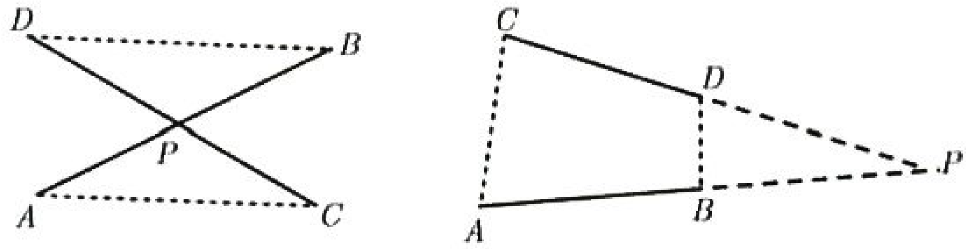
Corollary 7.12. If an exterior angle of a quadrilateral is equal to the opposite interior angle, the quadrilateral is cyclic.



Given : $ABCD$ is a quadrilateral with $\angle CBE = \angle D$.
To prove : $ABCD$ is cyclic.
Proof : $\angle CBE + \angle CBA = 180^\circ$
Since $\angle CBE = \angle D$, then $\angle D + \angle CBA = 180^\circ$.
Therefore $ABCD$ is cyclic.

We say that two line segments **intersect internally** at a point if the intersection point is between the endpoints of the segments. We say that two segments **intersect externally** at a point if the intersection point is outside the endpoints of the segments when produced.

Theorem 7.13. If two segments intersect (internally or externally), so that the product of two new parts of one segment is equal to the product of two new parts of the other segment, then the endpoints of the original two segments are concyclic.



Given : AB and CD intersect (internally or externally) at P.
$AP \cdot PB = CP \cdot PD$

To prove : A, B, C, D are concyclic.

Proof : Join AC and BD.
In $\triangle APC$ and $\triangle BPD$

$$\angle APC = \angle BPD$$

$$\frac{AP}{PD} = \frac{CP}{PB} \quad (\because AP \cdot PB = CP \cdot PD)$$

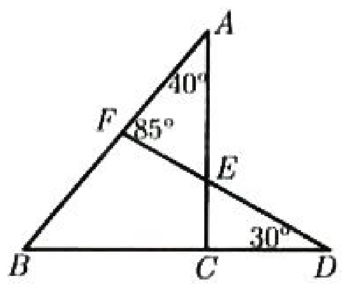
$$\therefore \triangle APC \sim \triangle DPB \quad \text{(SAS)}$$

$$\therefore \angle PAC = \angle PDB$$

$\therefore A, B, C, D$ are concyclic.

Example 7.

In the given figure, $\angle A = 40°$, $\angle D = 30°$ and $\angle AFE = 85°$. Prove that $BCEF$ is a cyclic quadrilateral.



Given : $\angle A = 40^\circ$, $\angle D = 30^\circ$ and $\angle AFE = 85^\circ$.
To prove : $BCEF$ is a cyclic quadrilateral.
Proof : In $\triangle BFD$,

$$\angle B + \angle D = \angle AFD$$
$$\therefore \angle B = \angle AFD - \angle D$$
$$= 85^\circ - 30^\circ = 55^\circ$$

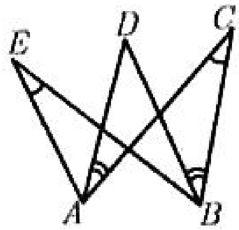
In $\triangle AEF$,

$$\angle CEF = \angle A + \angle AFE$$
$$= 40^\circ + 85^\circ = 125^\circ$$

$\therefore \angle B + \angle CEF = 55^\circ + 125^\circ = 180^\circ$.
$\therefore BCEF$ is a cyclic quadrilateral.

Example 8.

In the given figure, $\angle DAC = \angle DBC$ and $\angle AEB = \angle ACB$. Prove that A, B, C, D and E all lie on one circle.



Given : $\angle DAC = \angle DBC$ and $\angle AEB = \angle ACB$
To prove : A, B, C, D and E all lie on one circle.
Proof : $\angle DAC = \angle DBC$ (given)
So A, B, C, D are concyclic.
$\angle AEB = \angle ACB$ (given)
So A, B, C, E are concyclic.
Therefore D and E lie on unique circle ABC.
Hence A, B, C, D and E all lie on one circle.

Exercise 8.2

1. ABC is a triangle inscribed in a circle and DE the tangent at A. A line drawn parallel to DE meets AB, AC at F, G respectively. Prove that $BFGC$ is cyclic.

2. $\triangle ABC$ is a scalene acute triangle such that $AB < AC$. P, Q and R are midpoints of the respective sides. If S is the foot of the perpendicular from A to BC, prove that $PQRS$ is cyclic.

3. Two incongruent circles P and Q intersect at A and D. A line BDC is drawn to cut the circle P at B and circle Q at C, and such that $\angle BAC = 90^\circ$. Prove that $APDQ$ is cyclic.

4. ABC is a triangle in which $AB = AC$. P is a point inside the triangle such that $\angle PAB = \angle PBC$. Q is a point on BP produced such that $AQ = AP$. Prove that $ABCQ$ is cyclic.

5. Two circles intersect at A and B. A straight line CAD is drawn meeting one circle at C and the other at D. CB and DB are joined and produced to meet the circles at E and F respectively. If CF produced and DE produced meet at G, prove that the points B, F, G, E are concyclic.

6. Two incongruent circles P and Q intersect at A and D. A straight line BDC is drawn to meet the circle P at B and circle Q at C, and such that $\angle BAC = 90°$. Prove that $APDQ$ is cyclic.

7. Two circles intersect at A and B. A point P is taken on one so that PA produced meets the other at Q. The tangent at Q and the tangent at P meet in S. Prove that $PBQS$ is cyclic.

8. AB is a diameter of a circle and CBD is tangent to the circle at B. AC and AD cut the circle at G and H respectively. Prove that C, D, G, H are concyclic.

9. In $\triangle ABC$, $AB = AC$. P is any point on BC and Y any point on AP. The circles BPY and CPY cut AB and AC respectively at X and Z. Prove $XZ \parallel BC$.

10. Two circles intersect at A and B. Through B two straight lines PBX and QBY are drawn meeting one circle at P, Q and the other at X, Y, and such that $\angle PAB = \angle YAB$. Prove that $PB \cdot BX = QB \cdot BY$.

11. ABC is a triangle in which AX, BY, CZ are the perpendicular from vertices to the opposite sides. If the perpendiculars meet at O, prove that $AO \cdot OX = BO \cdot OY = CO \cdot OZ$.

12. AB is a diameter of a circle and AE is a chord of the circle. From any point C on AB produced, the line perpendicular to AB meets AE produced at D. Prove that $AE \cdot AD = AB \cdot AC$.

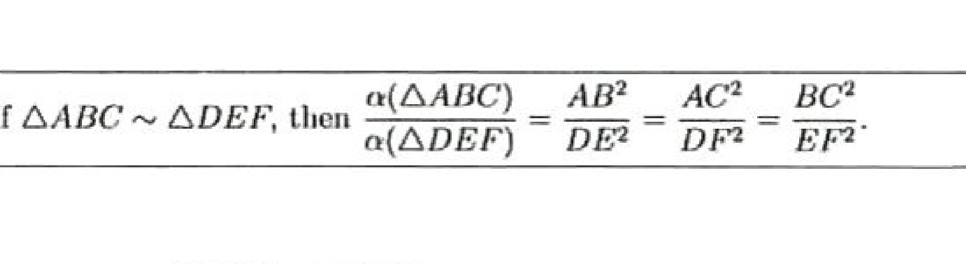
Chapter 8

Areas of Similar Triangles

8.1 Areas of Similar Triangles

In the Grade 10 Text book, we learned the properties of similar triangles. In this chapter we will learn the relationship between the areas of two similar triangles. Here we use the symbol "α" for the area of a triangle.

Theorem 8.1. The areas of two similar triangles have the same ratio as the squares of the lengths of any two corresponding sides.



If $\triangle ABC \sim \triangle DEF$, then $\dfrac{\alpha(\triangle ABC)}{\alpha(\triangle DEF)} = \dfrac{AB^2}{DE^2} = \dfrac{AC^2}{DF^2} = \dfrac{BC^2}{EF^2}$.

Given: $\triangle ABC \sim \triangle DEF$

To Prove: $\alpha(\triangle ABC) : \alpha(\triangle DEF) = BC^2 : EF^2$

Proof: $\triangle ABC \sim \triangle DEF$(given)

Therefore $\angle B = \angle E, \dfrac{AB}{DE} = \dfrac{BC}{EF}$

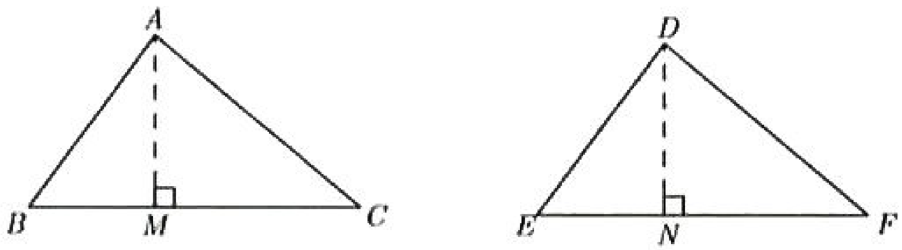
Draw $AM \perp BC$ and $DN \perp EF$.

$$\frac{\alpha(\triangle ABC)}{\alpha(\triangle DEF)} = \frac{\frac{1}{2}BC \cdot AM}{\frac{1}{2}EF \cdot DN} = \frac{BC}{EF} \cdot \frac{AM}{DN}$$

In $\triangle ABM$ and $\triangle DEN$

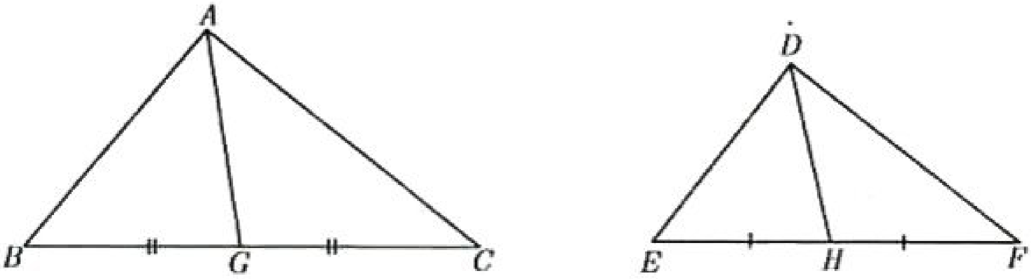
$$\begin{aligned}
\angle B &= \angle E && \text{(proved)}\\
\angle AMB &= \angle DNE && \text{(rt.angle)}\\
\triangle ABM &\sim \triangle DEN && \text{(AA corollary)}\\
\therefore \frac{AM}{DN} &= \frac{AB}{DE}\\
&= \frac{BC}{EF}\\
\therefore \frac{\alpha(\triangle ABC)}{\alpha(\triangle DEF)} &= \frac{BC}{EF} \cdot \frac{BC}{EF} = \frac{BC^2}{EF^2}\\
\text{i.e.,}\alpha(\triangle ABC) : \alpha(\triangle DEF) &= BC^2 : EF^2.
\end{aligned}$$

Corollary 8.2. The ratio of the areas of two similar triangles equals to the ratio of the squares of any two corresponding altitudes.



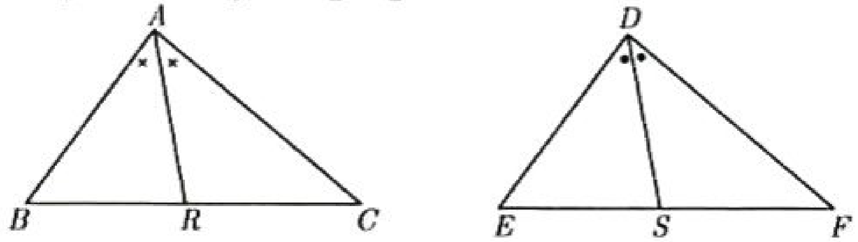
If $\triangle ABC \sim \triangle DEF$, then $\dfrac{\alpha(\triangle ABC)}{\alpha(\triangle DEF)} = \dfrac{AM^2}{DN^2}$.

Corollary 8.3. The ratio of the areas of two similar triangles equals to the ratio of the squares of any two corresponding medians.



If $\triangle ABC \sim \triangle DEF$, then $\dfrac{\alpha(\triangle ABC)}{\alpha(\triangle DEF)} = \dfrac{AG^2}{DH^2}$.

Corollary 8.4. The ratio of the areas of two similar triangles equals to the ratio of the square of any two corresponding angle bisectors.



If $\triangle ABC \sim \triangle DEF$, then $\dfrac{\alpha(\triangle ABC)}{\alpha(\triangle DEF)} = \dfrac{AR^2}{DS^2}$.

Example 1.

The bases of two similar triangles are 2.5 cm and 3.5 cm, respectively. The area of the smaller triangle is 3.75 cm^2. Find the area of the larger triangle.

Solution

Two given triangles are similar.

$$\frac{\alpha(\text{smaller triangle})}{\alpha(\text{larger triangle})} = \frac{(2.5)^2}{(3.5)^2} = \frac{5^2}{7^2} = \frac{25}{49}.$$

But $\alpha(\text{smaller triangle}) = 3.75 \text{ cm}^2$.

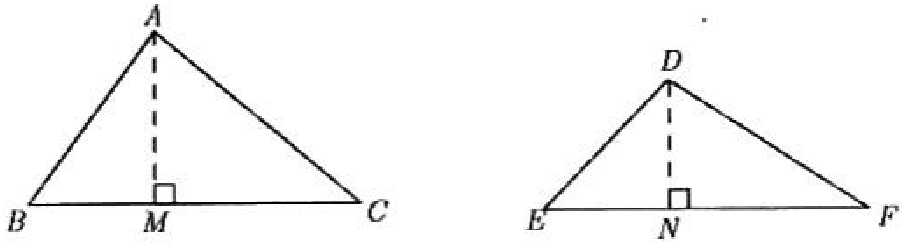
$$\therefore \frac{3.75}{\alpha(\text{larger triangle})} = \frac{25}{49}$$

$$\alpha(\text{larger triangle}) = \frac{3.75 \times 49}{25} = 7.35 \text{ cm}^2.$$

Example 2.

The area of two similar triangles are 56.25 cm^2 and 42.25 cm^2 respectively. Find the ratio of their corresponding altitudes.

Solution



Let $\triangle ABC \sim \triangle DEF$. $AM \perp BC$ and $DN \perp EF$.

$$\therefore \frac{\alpha(\triangle ABC)}{\alpha(\triangle DEF)} = \frac{AM^2}{DN^2}$$

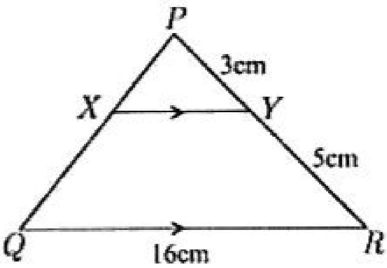
$$\frac{56.25}{42.25} = \frac{AM^2}{DN^2}; \quad \frac{AM^2}{DN^2} = \frac{225}{169}; \quad \frac{AM}{DN} = \frac{15}{13}$$

$$\therefore AM : DN = 15 : 13.$$

Example 3.

In $\triangle PQR$, $QR = 16$ cm. The point Y on PR is such that $PY = 3$cm, $YR = 5$ cm. The point X on PQ is such that $XY \parallel QR$. Find the length of XY.
If $\alpha(\triangle PXY) = 6$ cm^2, then find $\alpha(QXYR)$.

Solution



Since $XY \parallel QR, \triangle PXY \sim \triangle PQR$.

$$\frac{PY}{PR} = \frac{XY}{QR}$$

$$\frac{3}{8} = \frac{XY}{16}$$

$$XY = \frac{3 \times 16}{8} = 6 \text{ cm.}$$

$$\frac{\alpha(\triangle PXY)}{\alpha(\triangle PQR)} = \frac{PY^2}{PR^2} \quad (\triangle PXY \sim \triangle PQR)$$

$$= \frac{3^2}{8^2} = \frac{9}{64}$$

$$\frac{\alpha(\triangle PXY)}{\alpha(\triangle PQR) - \alpha(\triangle PXY)} = \frac{9}{64 - 9}$$

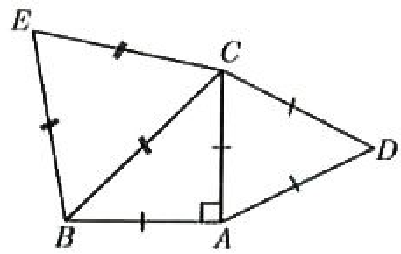
$$\frac{\alpha(\triangle PXY)}{\alpha(QXYR)} = \frac{9}{55}$$

$$\frac{6}{\alpha(QXYR)} = \frac{9}{55}$$

$$\therefore \alpha(QXYR) = \frac{6 \times 55}{9} = \frac{110}{3} = 36\frac{2}{3} \text{ cm}^2.$$

Example 4.

$\triangle ABC$ is an isosceles right triangle with $\angle A$ the right angle. E and D are points on opposite side of AC, with E on the same side of AC as B, such that $\triangle ACD$ and $\triangle BCE$ are both equilateral. Prove that $\alpha(\triangle BCE) = 2\alpha(\triangle ACD)$.



Given: $\triangle ABC$ is an isosceles right triangle with right angle at A. $\triangle ACD$ and $\triangle BCE$ are both equilateral.

To Prove: $\alpha(\triangle BCE) = 2\alpha(\triangle ACD)$

Proof: Since $\triangle ACD$ and $\triangle BCE$ are equilateral, they are similar.

$$\therefore \frac{\alpha(\triangle BCE)}{\alpha(\triangle ACD)} = \frac{BC^2}{AC^2}$$

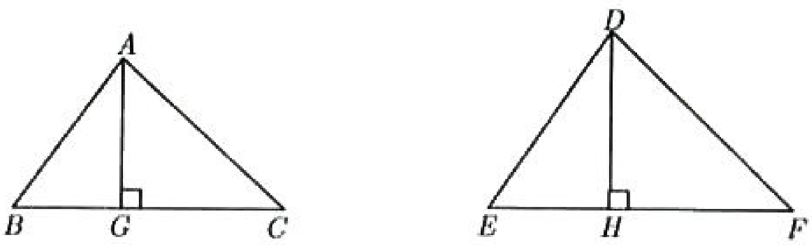
In the isosceles right $\triangle BAC$, $BC = \sqrt{2}AC$.

$$\therefore \frac{\alpha(\triangle BCE)}{\alpha(\triangle ACD)} = \frac{(\sqrt{2}AC)^2}{AC^2} = 2$$

Hence $\alpha(\triangle BCE) = 2\alpha(\triangle ACD)$.

Statement

The ratio of the areas of two triangles having equal bases (altitudes) equals the ratio of the corresponding altitudes (bases).



In $\triangle ABC$ and $\triangle DEF$, if bases $BC = EF$, then $\frac{\alpha(\triangle ABC)}{\alpha(\triangle DEF)} = \frac{AG}{DH}$.

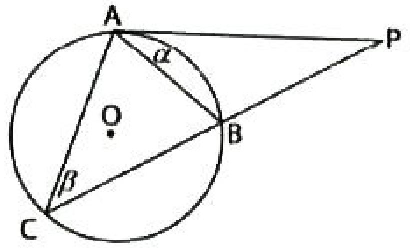
In $\triangle ABC$ and $\triangle DEF$, if altitudes $AG = DH$, then $\frac{\alpha(\triangle ABC)}{\alpha(\triangle DEF)} = \frac{BC}{EF}$.

Example 5.

In $\odot O$, PA is a tangent to the circle at A and PBC is a secant segment. Prove that $\frac{AB^2}{CA^2} = \frac{PB}{PC}$.

Given : In $\odot O$, PA is a tangent to the circle at A and PBC is a secant segment.

To Prove : $\dfrac{AB^2}{CA^2} = \dfrac{PB}{PC}$



Proof : In ΔPAB and ΔPAC

$\alpha = \beta$ (alternate segment)

$\angle P = \angle P$(common angle)

$\Delta PAB \sim \Delta PCA$ (AA corollary)

$$\frac{AB}{CA} = \frac{PB}{PA} = \frac{PA}{PC}$$

$$\frac{AB^2}{CA^2} = \frac{AB}{CA} \times \frac{AB}{CA} = \frac{PB}{PA} \times \frac{PA}{PC} = \frac{PB}{PC}$$

Exercise 8.1

1. ABC and BDE are two equilateral triangles such that D is the midpoint of BC. Find the ratio of the areas of triangles ABC and BDE.

2. ΔABC is bisected by a line PQ drawn parallel to its base BC. In what ratio does PQ divide the sides of the triangle?

3. A straight line drawn parallel to BC of ΔABC cuts the sides AB, AC in the ratio 2 : 3. Find the area of the triangle thus cut off, if the area of the whole triangle is $72.25\,cm^2$.

4. In trapezium $ABCD, AB$ is twice DC and $AB \parallel DC$. If AC and BD intersect at O, prove that $\alpha(\Delta AOB) = 4\alpha(\Delta COD)$.

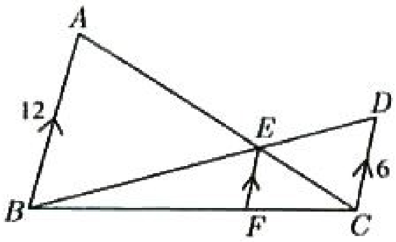
5. Two chords AC and BD of a circle intersect at O.
Prove that $\alpha(\Delta AOB) : \alpha(\Delta COD) = OA^2 : OD^2$.

6. PA and PB are the tangent segments at A and B to a circle whose centre is O.
Prove that $\alpha(\Delta PAB) : \alpha(\Delta OAB) = AP^2 : AO^2$.

7. In the figure $AB = 12$ cm, $DC = 6$ cm, and E is the intersection point of AC and BD. Find the ratios

   (a) $\alpha(\Delta EFC) : \alpha(\Delta ABC)$

   (b) $\alpha(\Delta BFE) : \alpha(\Delta BCD)$.

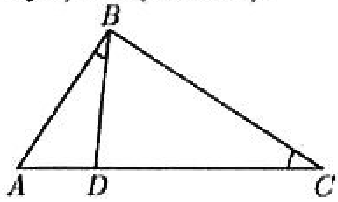


8. In $\triangle ABC, D$ is a point of AC such that $AD = 2CD$. E is a point on BC such that $DE \parallel AB$. Compare the areas of $\triangle CDE$ and $\triangle ABC$. If α (ABED) $= 40$, what is $\alpha(\triangle ABC)$?

9. In $\triangle PQR$, $\angle P = 90°$ and $PS \perp QR$. If $QR = 3PQ$, prove that $SR = 8QS$.

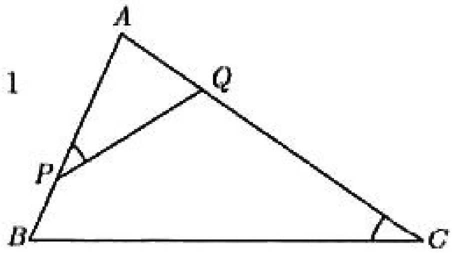
10. ABC is a triangle such that $BC : CA : AB = 3 : 4 : 5$. If BPC, CQA, ARB are equilateral triangles, prove that $\alpha(\triangle BPC) + \alpha(\triangle CQA) = \alpha(\triangle ARB)$.

11. In the figure, $AB = 6\,cm$, $AC = 9\,cm$, and D is a point on AC such that $\angle ABD = \angle ACB$. Calculate AD. Given that $\alpha(\triangle ABD) = 10\,cm^2$, calculate $\alpha(\triangle ABC)$.



12. $ABCD$ is a trapezium in which $AB \parallel CD$ and $\angle ADB = \angle C$. Prove that $AD^2 : BC^2 = AB : CD$.

13. In the figure, $\angle APQ = \angle C$, $AP : BP = 3 : 1$ and $AQ : QC = 1 : 2$. If $AQ = 2$, find the length of AP and the ratios of



(a) $\alpha(\triangle APQ) : \alpha(\triangle ABC)$

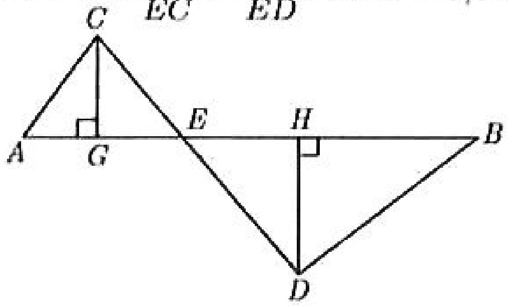
(b) $\alpha(\triangle APQ) : \alpha(BCQP)$.

14. ABC is a triangle. If BPC, CQA, ARB are equilateral triangles, and $\alpha(\triangle BPC) + \alpha(\triangle CQA) = \alpha(\triangle ARB)$ then prove that ABC is a right triangle.

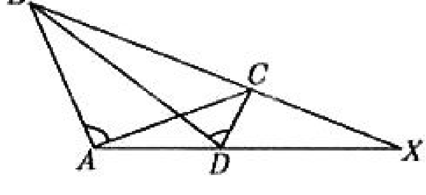
15. In the figure $CG \perp AE$, $DH \perp BE$ and $\frac{EA}{EC} = \frac{EB}{ED}$. If $CG = 6, DH = 8$ and $AB = 35$. Find

(a) $\frac{\alpha(\triangle ACE)}{\alpha(\triangle BDE)}$

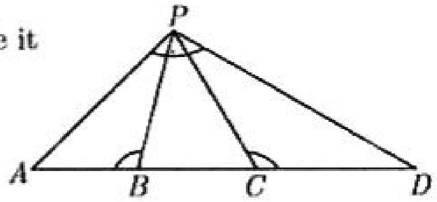
(b) $\alpha(\triangle BDE)$.



16. ADX and BCX are two segments such that $\angle BAC = \angle BDC$. Prove that $\frac{\alpha(\triangle ABX)}{\alpha(\triangle CDX)} = \frac{AB^2}{CD^2}$.



17. $ABCD$ is a segment and P a point outside it such that $\angle PBA = \angle PCD = \angle APD$. Prove that $\dfrac{\alpha(\Delta ABP)}{\alpha(\Delta PCD)} = \dfrac{AB^2}{BP^2}$.



18. ΔABC is inscribed in a circle. Straight lines are drawn through B and C parallel to CA and BA respectively, to meet the tangent at A in D and E. Prove that $\dfrac{DA}{AE} = \dfrac{AB}{EC} = \dfrac{AB^2}{AC^2}$.

19. In acute angle $\Delta ABC, AD$ and BE are altitudes. If $\alpha(\Delta DEC) = \dfrac{3}{4}\alpha(\Delta ABC)$, prove that $\angle ACB = 30°$.

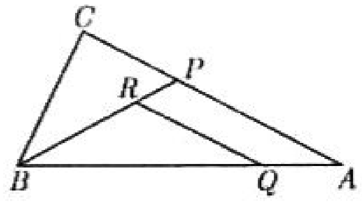
20. A, B, C, D are four points in order on a circle O, so that AB is a diameter and $\angle COD = 90°$. AD produced and BC produced meet at E. Prove that $\alpha(\Delta ECD) = \alpha(ABCD)$.

21. A, B, C, D are four points in order on a circle O, so that AB is a diameter. AD produced and BC produced meet at E. If $\alpha(\Delta ECD) = \alpha(ABCD)$, prove that $DC = \sqrt{2}AO$.

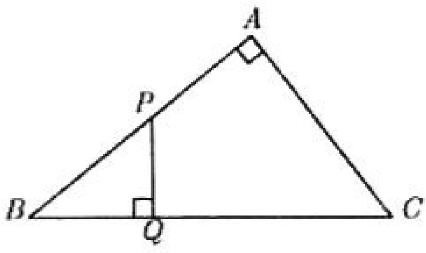
22. In $\Delta ABC, AD$ and BE are altitudes. If $\angle ACB = 45°$, prove that $\alpha(\Delta DEC) = \alpha(ABDE)$.

23. ABC is a right triangle with $\angle A$ the right angle. E and D are points on opposite side of AC, with E on the same side of AC as B, such that ΔACD and ΔBCE are both equilateral. If $\alpha(\Delta BCE) = 2\alpha(\Delta ACD)$, prove that ABC is an isosceles right triangle.

24. In the diagram, P is the point on AC, such that $AP = 2PC$, R is the point on BP such that $BR = 3RP$ and $QR \parallel AC$. Given that $\alpha(\Delta BPA) = 32\,cm^2$, calculate $\alpha(\Delta BPC)$ and $\alpha(\Delta BRQ)$.



25. In $\Delta ABC, \angle A = 90°$, PQ $\perp BC$. If $AC = 3, BC = 5$ and $CQ = 3$, then find $\alpha(\Delta BPQ) : \alpha(APQC)$.



Chapter 9

Introduction to Vectors

A scalar quantity is one which has only magnitude but a vector quantity is one which has both magnitude and direction. For example, distance and speed are scalar quantities, and displacement and velocity are vector quantities. In this chapter, the students will mainly study geometric vectors.

9.1 Geometric Vectors

A **geometric vector** is a directed line segment. A geometric vector can be represented as a line with an initial point and a terminal point.

Fig. 9.1 represents a geometric vector with initial point A and terminal point B. This geometric vector is denoted by $\overrightarrow{AB}$. The magnitude of $\overrightarrow{AB}$ is the length of the line segment AB and denoted by $\left|\overrightarrow{AB}\right|$ (simply by AB).

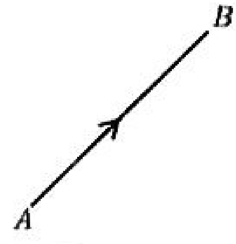


Fig. 9.1

Geometric vectors will also be denoted by small letters with over-right arrows, $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, etc. From now on, we will use the word "vector" instead of "geometric vector".

Definition. Vectors are said to be **parallel** if they have the same direction or opposite directions.

Definition. Two vectors are said to be **equal** if they have the same magnitude and the same direction.

The symbol '=' will be used to indicate this equal relationship.

In Fig. 9.2, $ABCD$ is a parallelogram. Therefore $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$.

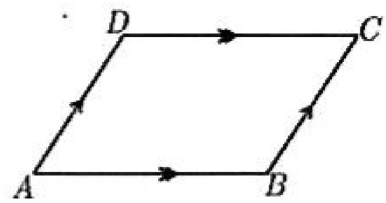


Fig. 9.2

Definition. A vector having the same magnitude as $\overrightarrow{a}$ but a direction opposite to that of $\overrightarrow{a}$ is called the **negative** of $\overrightarrow{a}$ and denoted by $-\overrightarrow{a}$.

$\overrightarrow{BA}$ and $\overrightarrow{AB}$ have the same magnitude but the direction of $\overrightarrow{BA}$ is opposite to that of $\overrightarrow{AB}$. Thus $\overrightarrow{BA} = -\overrightarrow{AB}$.

Definition. A vector whose magnitude is zero is called a **zero vector** and denoted by $\overrightarrow{0}$.

Addition of Vectors

Let $\overrightarrow{a}$ and $\overrightarrow{b}$ be two vectors. After placing the initial point of $\overrightarrow{b}$ on the terminal point of $\overrightarrow{a}$, the **sum** of $\overrightarrow{a}$ and $\overrightarrow{b}$, denoted by $\overrightarrow{a} + \overrightarrow{b}$, is a vector whose initial point is the initial point of $\overrightarrow{a}$ and terminal point is the terminal point of $\overrightarrow{b}$.

Triangle Rule:

If $\overrightarrow{a} = \overrightarrow{AB}$ and $\overrightarrow{b} = \overrightarrow{BC}$, then $\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{AC}$, See Fig. 9.3. That is,

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

Since a triangle is formed by $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{a} + \overrightarrow{b}$, we say that vectors are added according to **The Triangle Rule.**

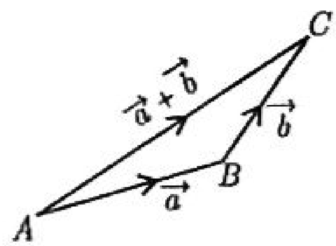


Fig. 9.3

Parallelogram Rule:

If $\overrightarrow{a} = \overrightarrow{AB}$ and $\overrightarrow{b} = \overrightarrow{AD}$, then $\overrightarrow{a}$ and $\overrightarrow{b}$ have the same initial point A. Produce a parallelogram $ABCD$ as shown in Fig. 9.4. By properties of parallelogram, $\overrightarrow{AD} = \overrightarrow{BC}$. Hence

$$\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC},$$

which is a diagonal of the parallelogram $ABCD$.

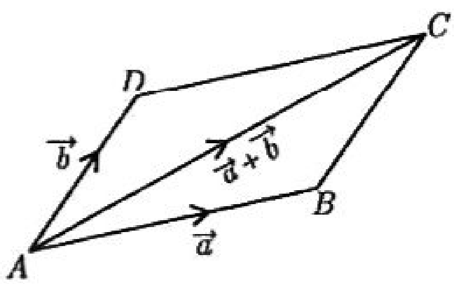


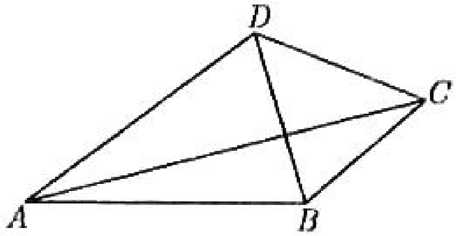
Fig. 9.4

Since the result $\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC}$ is formed by using parallelogram $ABCD$, we say that vectors are added according to **The Parallelogram Rule.**

Since $\overrightarrow{DC}=\overrightarrow{AB}$, then $\vec{b}+\vec{a}=\overrightarrow{AD}+\overrightarrow{AB}=\overrightarrow{AD}+\overrightarrow{DC}=\overrightarrow{AC}=\vec{a}+\vec{b}$.

Example 1.

$ABCD$ is a quadrilateral. Simplify the following.
(i) $\overrightarrow{AB}+\overrightarrow{BC}$, (ii) $(\overrightarrow{AB}+\overrightarrow{BC})+\overrightarrow{CD}$, (iii) $\overrightarrow{BC}+\overrightarrow{CD}$ (iv) $\overrightarrow{AB}+(\overrightarrow{BC}+\overrightarrow{CD})$.
Is $(\overrightarrow{AB}+\overrightarrow{BC})+\overrightarrow{CD}=\overrightarrow{AB}+(\overrightarrow{BC}+\overrightarrow{CD})$?



Solution

(i) $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$
(ii) $(\overrightarrow{AB}+\overrightarrow{BC})+\overrightarrow{CD}=\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}$
(iii) $\overrightarrow{BC}+\overrightarrow{CD}=\overrightarrow{BD}$
(iv) $\overrightarrow{AB}+(\overrightarrow{BC}+\overrightarrow{CD})=\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AD}$

By (ii) and (iv), $(\overrightarrow{AB}+\overrightarrow{BC})+\overrightarrow{CD}=\overrightarrow{AB}+(\overrightarrow{BC}+\overrightarrow{CD})$.

From the above example, we can drop off the brackets to find the sum of three vectors, i.e., the vector sum $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}=(\overrightarrow{AB}+\overrightarrow{BC})+\overrightarrow{CD}=\overrightarrow{AB}+(\overrightarrow{BC}+\overrightarrow{CD})=\overrightarrow{AD}$.
Thus we can deduce the following rule.

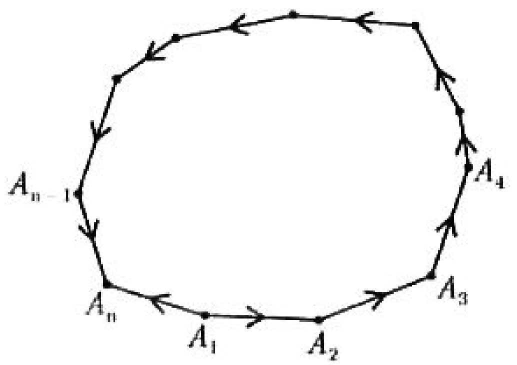
Polygon Rule:

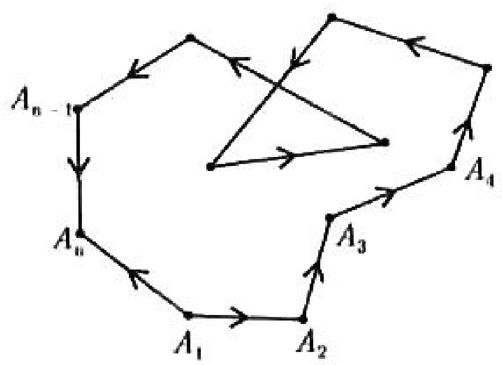
If $A_1A_2A_3\dots A_n$ is a closed polygon, then

$$\overrightarrow{A_1A_2}+\overrightarrow{A_2A_3}+\overrightarrow{A_3A_4}+\cdots+\overrightarrow{A_{n-1}A_n}=\overrightarrow{A_1A_n},$$

and

$$\overrightarrow{A_1A_2}+\overrightarrow{A_2A_3}+\overrightarrow{A_3A_4}+\cdots+\overrightarrow{A_{n-1}A_n}+\overrightarrow{A_nA_1}=\vec{0}.$$

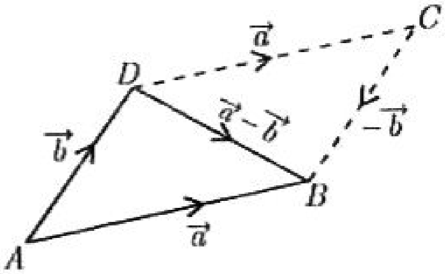




For vectors $\vec{a}$ and $\vec{b}$, $\vec{a} - \vec{b}$ is a vector defined by $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$. The vector $\vec{a} - \vec{b}$ is called the **difference** of $\vec{a}$ and $\vec{b}$.

Construct parallelogram $ABCD$ as shown in figure.

$$
\begin{aligned}
\therefore \overrightarrow{DB} &= \overrightarrow{DC} + \overrightarrow{CB} \\
&= \overrightarrow{AB} + \overrightarrow{DA} \\
&= \overrightarrow{AB} + (-\overrightarrow{AD}) \\
&= \overrightarrow{AB} - \overrightarrow{AD} \\
&= \vec{a} - \vec{b}
\end{aligned}
$$

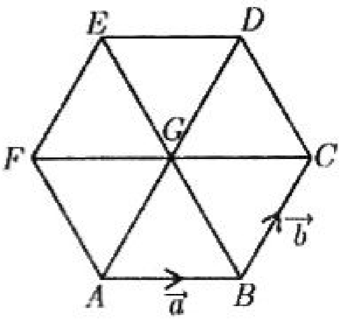


Example 2.

$ABCDEF$ is a regular hexagon and G is the common point of intersection of the diagonals. If $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$, express $\overrightarrow{CD}$, $\overrightarrow{EG}$ and $\overrightarrow{CA}$ in terms of $\vec{a}$ and $\vec{b}$.

Solution

$$
\begin{aligned}
\overrightarrow{CD} &= \overrightarrow{CG} + \overrightarrow{GD} \quad \text{(by the Triangle Rule)} \\
&= \overrightarrow{BA} + \overrightarrow{BC} \\
&= -\overrightarrow{AB} + \overrightarrow{BC} \\
&= -\vec{a} + \vec{b}
\end{aligned}
$$



$$
\begin{aligned}
\overrightarrow{EG} &= \overrightarrow{ED} + \overrightarrow{EF} \quad \text{(by the Parallelogram Rule)} \\
&= \overrightarrow{AB} + \overrightarrow{CB} \\
&= \overrightarrow{AB} + (-\overrightarrow{BC}) \\
&= \vec{a} - \vec{b}
\end{aligned}
$$

$$\overrightarrow{CA} = -\overrightarrow{AC} = -(\overrightarrow{AB} + \overrightarrow{BC}) = -(\vec{a} + \vec{b})$$

Properties of Vector Addition: Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be vectors.

(i) Vector addition is commutative, i.e., $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

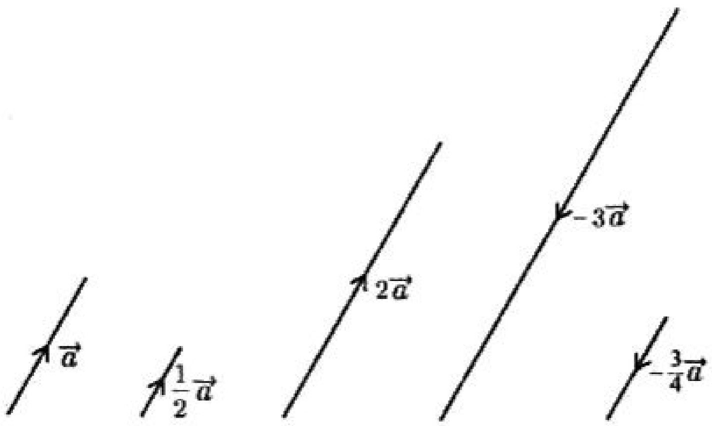
(ii) Vector addition is associative, i.e., $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$.

(iii) Zero vector is the additive identity, i.e., $\vec{a} + \vec{0} = \vec{a}$, for each vector $\vec{a}$.

(iv) $-\vec{a}$ is the **unique** additive inverse of $\vec{a}$, i.e., $\vec{a} + (-\vec{a}) = \vec{0}$.

Multiplication by a Scalar

The **product** of a vector $\vec{a}$ by a scalar k, denoted by $k\vec{a}$, is a vector whose magnitude is $|k|$ times that of $\vec{a}$, and whose direction is the same, or opposite to that of $\vec{a}$, according as k is positive or negative.



Note that

(i) $0\vec{a} = \vec{0}$, $(-1)\vec{a} = -\vec{a}$ and $k\vec{0} = \vec{0}$ for any scalar k,

(ii) two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} = k\vec{b}$, for some nonzero scalar k, and

(iii) the points A, B and C are collinear if and only if $\overrightarrow{AB} = h\overrightarrow{BC}$, for some nonzero scalar h.

Properties of Multiplication by a Scalar

If k, k_1 and k_2 are scalars, then

(i) $k_1(k_2\vec{a}) = (k_1k_2)\vec{a}$, (Associative Law)

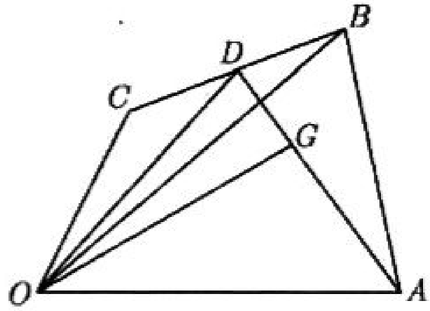
(ii) $(k_1 + k_2)\vec{a} = k_1\vec{a} + k_2\vec{a}$, (Distributive Law over Addition of Scalars)

(iii) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, (Distributive Law over Addition of Vectors)

Example 3.

In quadrilateral $OABC$, D is the midpoint of BC and G is a point on AD such that $AG : GD = 2 : 1$. If $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$, express $\overrightarrow{OD}$ and $\overrightarrow{OG}$ in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Solution



$$\begin{aligned}
\overrightarrow{OD} &= \overrightarrow{OC} + \overrightarrow{CD} \\
\overrightarrow{CD} &= \frac{1}{2}\overrightarrow{CB} \qquad (\because D \text{ is the midpoint of } BC) \\
&= \frac{1}{2}(\overrightarrow{CO} + \overrightarrow{OB}) \\
&= \frac{1}{2}\overrightarrow{CO} + \frac{1}{2}\overrightarrow{OB} = -\frac{1}{2}\overrightarrow{OC} + \frac{1}{2}\overrightarrow{OB} \\
\therefore \overrightarrow{OD} &= \overrightarrow{OC} - \frac{1}{2}\overrightarrow{OC} + \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\overrightarrow{OC} + \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\overrightarrow{c} + \frac{1}{2}\overrightarrow{b}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{OG} &= \overrightarrow{OA} + \overrightarrow{AG} \\
\overrightarrow{AG} &= \frac{2}{3}\overrightarrow{AD} \qquad (\because AG : GD = 2 : 1) \\
&= \frac{2}{3}(\overrightarrow{AO} + \overrightarrow{OD}) \\
&= \frac{2}{3}(-\overrightarrow{OA} + \overrightarrow{OD}) = -\frac{2}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OD} \\
\therefore \overrightarrow{OG} &= \overrightarrow{OA} - \frac{2}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}(\frac{1}{2}\overrightarrow{c} + \frac{1}{2}\overrightarrow{b}) = \frac{1}{3}\overrightarrow{a} + \frac{1}{3}\overrightarrow{c} + \frac{1}{3}\overrightarrow{b}
\end{aligned}$$

Theorem 9.1. Let $\overrightarrow{a}$ and $\overrightarrow{b}$ be nonzero and nonparallel vectors. If $h\overrightarrow{a} = k\overrightarrow{b}$, then $h = k = 0$.

Proof

Suppose that $h \neq 0$ and $k \neq 0$. Then $\frac{k}{h} \neq 0$ and $\overrightarrow{a} = \frac{k}{h}\overrightarrow{b}$. Hence $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel. This contradicts to the hypothesis of the theorem. Hence at least one of h and k must be zero, say $k = 0$. Then $h\overrightarrow{a} = k\overrightarrow{b} = 0\overrightarrow{b} = \overrightarrow{0}$. Since $\overrightarrow{a} \neq 0$, h must be zero. Therefore $h = k = 0$.

Corollary 9.2. Let $\overrightarrow{a}$ and $\overrightarrow{b}$ be nonzero and nonparallel vectors.
If $h\overrightarrow{a} + k\overrightarrow{b} = m\overrightarrow{a} + n\overrightarrow{b}$, then $h = m$ and $k = n$.

Proof

$$\begin{aligned}
h\overrightarrow{a} + k\overrightarrow{b} &= m\overrightarrow{a} + n\overrightarrow{b} \\
h\overrightarrow{a} - m\overrightarrow{a} &= n\overrightarrow{b} - k\overrightarrow{b} \\
(h - m)\overrightarrow{a} &= (n - k)\overrightarrow{b} \\
h - m = 0 \quad &\text{and} \quad n - k = 0 \qquad \text{(by Theorem 9.1)} \\
h = m \quad &\text{and} \quad n = k.
\end{aligned}$$

Example 4.

For nonzero and nonparallel vectors $\vec{a}$ and $\vec{b}$, it is given that $\vec{u} = 5\vec{a} + 4\vec{b}$, $\vec{v} = 3\vec{a} - \vec{b}$ and $\vec{w} = (2h-k)\vec{a} + (h+k+3)\vec{b}$. If $\vec{w} = 2\vec{u} - 3\vec{v}$, calculate the values of h and k.

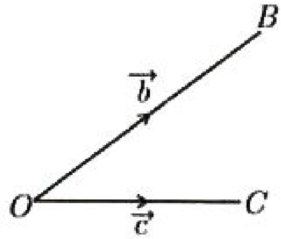
Solution

$$\begin{aligned} \vec{w} &= 2\vec{u} - 3\vec{v} \\ (2h-k)\vec{a} + (h+k+3)\vec{b} &= 2(5\vec{a} + 4\vec{b}) - 3(3\vec{a} - \vec{b}) \\ (2h-k)\vec{a} + (h+k+3)\vec{b} &= 10\vec{a} + 8\vec{b} - 9\vec{a} + 3\vec{b} \\ (2h-k)\vec{a} + (h+k+3)\vec{b} &= \vec{a} + 11\vec{b} \end{aligned}$$

By Corollary 9.2, $2h - k = 1$ and $h + k + 3 = 11$.
Solving these equations, we get $h = 3$ and $k = 5$.

Exercise 9.1

1. In the figure $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$. Make the point E and F such that $\overrightarrow{OE} = \frac{1}{2}\vec{b}$, $\overrightarrow{OF} = -2\vec{c}$. Find, in terms of $\vec{b}$ and $\vec{c}$, the vectors $\overrightarrow{EC}$, $\overrightarrow{BF}$ and $\overrightarrow{EF}$.



2. $ABCD$ is a parallelogram. Let O be the point of intersection of the diagonals and M be the midpoint of AB. If $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$, find $\overrightarrow{BC}$ and $\overrightarrow{BM}$ in terms of $\vec{a}$ and $\vec{b}$.

3. If P is a point inside a parallelogram $ABCD$, prove that $\overrightarrow{PA} + \overrightarrow{PC} = \overrightarrow{PB} + \overrightarrow{PD}$.

4. In $\triangle ABC$, P is the midpoint of the side BC and Q is a point on the side AC. Given that $2\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{BA} = 6\overrightarrow{PQ}$, show that $CQ = \frac{1}{3}CA$.

5. ABCDEF is a regular hexagon and G is the common point of intersection of the diagonals. Prove that $\overrightarrow{AB} + \overrightarrow{AF} = \frac{1}{2}\overrightarrow{AD}$.

6. $ABCDEFGH$ is a regular octagon. If $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$, find $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{EF}$, $\overrightarrow{FG}$, $\overrightarrow{GH}$, $\overrightarrow{HA}$ in terms of $\vec{a}$ and $\vec{b}$.

7. In $\triangle ABC$, E is the midpoint of AC and F is a point on median CD such that $CF : FD = 2 : 1$. If $\overrightarrow{AE} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$, find $\overrightarrow{AF}$, $\overrightarrow{EF}$ and $\overrightarrow{EB}$. Hence show that BFE is a straight line.

8. $OPQR$ is a parallelogram and OR is produced to S such that $OS = 3OR$. If Y is a point on OQ such that $\overrightarrow{OQ} = 4\overrightarrow{YQ}$, show that Y lies on PS.

9. It is given that $\vec{a}$ and $\vec{b}$ are nonzero and nonparallel vectors.
If $3\vec{a} + x(\vec{b} - \vec{a}) = y(\vec{a} + 2\vec{b})$, find the values of x and y.

10. For nonzero and nonparallel vectors $\vec{a}$ and $\vec{b}$, it is given that $\vec{p} = 3\vec{a} + 3\vec{b}$, $\vec{q} = 4\vec{a} - \vec{b}$ and $\vec{r} = h\vec{a} + (3h + k)\vec{b}$, calculate the value of h and of k.

9.2 Applications to Elementary Geometry

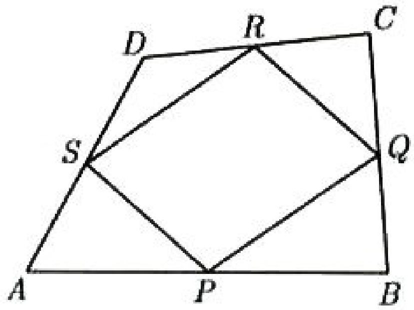
After we have studied the vector concepts, we can apply these to solve problems in plane geometry.

Example 5.

Show that the quadrilateral formed by joining the midpoints of the sides of a quadrilateral is a parallelogram.

Solution

Let $ABCD$ be a quadrilateral and P, Q, R, S be midpoints of AB, BC, CD, DA, respectively.
By polygon rule,



$$\begin{aligned}
\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} &= \vec{0} \\
\therefore \quad 2\overrightarrow{PB} + 2\overrightarrow{BQ} + 2\overrightarrow{RD} + 2\overrightarrow{DS} &= \vec{0} \\
\overrightarrow{PB} + \overrightarrow{BQ} + \overrightarrow{RD} + \overrightarrow{DS} &= \vec{0} \\
\overrightarrow{PQ} + \overrightarrow{RS} &= \vec{0} \\
\overrightarrow{PQ} &= -\overrightarrow{RS} \\
\overrightarrow{PQ} &= \overrightarrow{SR}
\end{aligned}$$

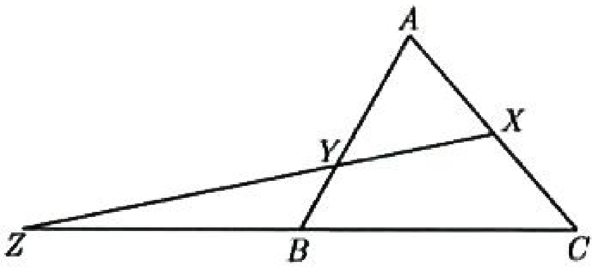
So, $PQ = SR$ and $PQ \parallel SR$.
Hence $PQRS$ is a parallelogram.

Example 6.

In $\triangle ABC$, CB is produced to Z such that $BC = BZ$. The point X is the midpoint of AC and the point Y is on AB such that $BY = \frac{1}{3}BA$. Prove that X, Y, Z are collinear.

Solution

$$
\begin{aligned}
\therefore \overrightarrow{XY} &= \overrightarrow{XA} + \overrightarrow{AY} \\
&= \frac{1}{2}\overrightarrow{CA} + \frac{2}{3}\overrightarrow{AB} \\
&= \frac{1}{2}(\overrightarrow{CB} + \overrightarrow{BA}) + \frac{2}{3}\overrightarrow{AB} \\
&= \frac{1}{2}\overrightarrow{CB} + \frac{1}{2}\overrightarrow{BA} + \frac{2}{3}\overrightarrow{AB} \\
&= \frac{1}{2}\overrightarrow{CB} - \frac{1}{2}\overrightarrow{AB} + \frac{2}{3}\overrightarrow{AB} \\
&= \frac{1}{2}\overrightarrow{CB} + \frac{1}{6}\overrightarrow{AB}
\end{aligned}
$$



$$
\begin{aligned}
\overrightarrow{YZ} &= \overrightarrow{YB} + \overrightarrow{BZ} \\
&= \frac{1}{3}\overrightarrow{AB} + \overrightarrow{CB} \\
&= \overrightarrow{CB} + \frac{1}{3}\overrightarrow{AB}
\end{aligned}
$$

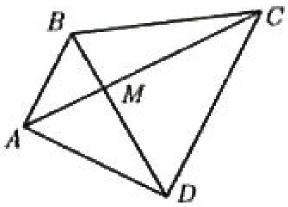
$$\therefore 2\overrightarrow{XY} = \overrightarrow{YZ}$$

So, X, Y, Z are collinear.

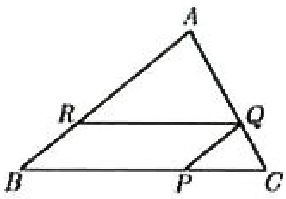
Exercise 9.2

1. In trapezium $PQRS$, $SR \parallel PQ$ and $SR = \frac{4}{3}PQ$. If T is a point on QS such that $QT : TS = 3 : 1$, prove that $PT \parallel QR$.

2. In the figure, $MC = 2MA$ and $MD = 2MB$. Prove that $DC \parallel AB$ and $DC = 2AB$.



3. In the figure, P, Q and R are points on the sides of $\triangle ABC$ such that $BP = 2PC$, $QA = 2CQ$ and $AR = 2RB$. Prove that $PQRB$ is a parallelogram.



4. P is the midpoint of the side CD of a parallelogram $ABCD$. If AP and BD intersect at Q, prove that $DQ = \frac{1}{3}DB$.

5. The median AD of ΔABC is produced to K so that, $DK = \frac{1}{3}AD$. If G is a point on AD such that $AG : GD = 2 : 1$, prove that $BKCG$ is a parallelogram.
6. Prove that the diagonals of a parallelogram bisect each other.
7. Prove that if the diagonals of a quadrilateral bisect each other, then this quadrilateral is a parallelogram.
8. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side, and of half its length.

9.3 Position Vectors

Definition. The position of the point P relative to the **origin** O is denoted by $\overrightarrow{OP}$. This vector $\overrightarrow{OP}$ is called the **position vector** of P.

In Fig. 9.5,

$$\begin{aligned}\overrightarrow{OA} + \overrightarrow{AB} &= \overrightarrow{OB}\\ \therefore \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\end{aligned}$$

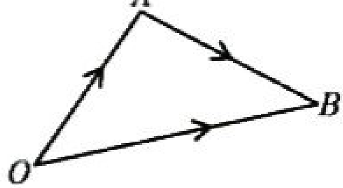


Fig. 9.5

Thus, for any points A and B, $\overrightarrow{AB}$ can be represented by position vectors, i.e.,

$$\boxed{\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}}$$

Example 7.

The position vectors of three points P, Q and R, relative to the origin O, are $2\vec{a} + \vec{b}$, $3\vec{a} - 2\vec{b}$ and $h\vec{a} + 5\vec{b}$ respectively. Express $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ in terms of $\vec{a}$ and $\vec{b}$. If P, Q and R are collinear, find the value of h.

Solution

$\overrightarrow{OP} = 2\vec{a} + \vec{b}$, $\overrightarrow{OQ} = 3\vec{a} - 2\vec{b}$, $\overrightarrow{OR} = h\vec{a} + 5\vec{b}$

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} = 3\vec{a} - 2\vec{b} - (2\vec{a} + \vec{b}) = \vec{a} - 3\vec{b}\\ \overrightarrow{PR} &= \overrightarrow{OR} - \overrightarrow{OP} = h\vec{a} + 5\vec{b} - (2\vec{a} + \vec{b}) = (h-2)\vec{a} + 4\vec{b}\end{aligned}$$

If P, Q and R are collinear,

$$\begin{aligned}\overrightarrow{PQ} &= k\overrightarrow{PR} \quad \text{for some nonzero scalar } k\\ \vec{a} - 3\vec{b} &= k\left[(h-2)\vec{a} + 4\vec{b}\right]\\ \vec{a} - 3\vec{b} &= k(h-2)\vec{a} + 4k\vec{b}\end{aligned}$$

By Corollary9.2, we have

$$k(h-2)=1 \text{ and } 4k=-3.$$

Solving these equations, we get

$$h=\frac{2}{3}.$$

Example 8.

The position vectors of A, B, P, relative to the origin O, are $\vec{a}$, $\vec{b}$ and $\vec{p}$. If $\vec{p}=\frac{1}{2}(3\vec{a}-\vec{b})$, calculate the ratio $AP:PB$.

Solution

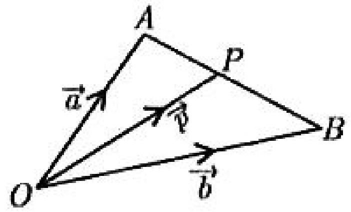
$$\begin{aligned}
\vec{p} &= \frac{1}{2}(3\vec{a}-\vec{b}) \\
2\vec{p} &= 3\vec{a}-\vec{b} \\
\vec{b} &= 3\vec{a}-2\vec{p} \\
\overrightarrow{PB} &= \overrightarrow{OB}-\overrightarrow{OP} \\
&= \vec{b}-\vec{p} \\
&= 3\vec{a}-2\vec{p}-\vec{p} \\
&= 3\vec{a}-3\vec{p}=3(\vec{a}-\vec{p})=3(\overrightarrow{OA}-\overrightarrow{OP})=3\overrightarrow{PA} \\
\therefore PB &= 3PA \\
\therefore\ AP:PB &= 1:3
\end{aligned}$$

Theorem 9.3 (The Section Formula). Let A, P, B be three collinear points with P divides AB such that $AP:PB=m:n$. If $\vec{a}$, $\vec{p}$, $\vec{b}$ are position vectors, relative to the origin O, then

$$\vec{p}=\frac{1}{m+n}(m\vec{b}+n\vec{a}).$$

Proof

$$\begin{aligned}
AP:PB &= m:n \\
nAP &= mPB \\
n\overrightarrow{AP} &= m\overrightarrow{PB} \qquad (\because A, P, B \text{ are collinear}) \\
n(\overrightarrow{OP}-\overrightarrow{OA}) &= m(\overrightarrow{OB}-\overrightarrow{OP}) \\
n(\vec{p}-\vec{a}) &= m(\vec{b}-\vec{p}) \\
n\vec{p}-n\vec{a} &= m\vec{b}-m\vec{p} \\
(m+n)\vec{p} &= m\vec{b}+n\vec{a} \\
\vec{p} &= \frac{1}{m+n}(m\vec{b}+n\vec{a})
\end{aligned}$$



Corollary 9.4 (Midpoint Formula). Let P be the midpoint of AB. If $\vec{a}$, $\vec{p}$, $\vec{b}$ are position vectors, relative to the origin O, then

$$\vec{p} = \frac{1}{2}(\vec{a} + \vec{b}).$$

Proof

Since $AP = PB$, we have $AP : PB = 1 : 1$.

$$\therefore \vec{p} = \frac{1}{1+1}(1\vec{b} + 1\vec{a}) = \frac{1}{2}(\vec{a} + \vec{b}).$$

Example 9.

The position vectors of points P and Q relative to an origin O are $2\vec{a} + 3\vec{b}$ and $3\vec{a} + 7\vec{b}$ respectively, and R is a point on PQ such that $PR : RQ = 2 : 3$. Find the position vector of R, relative to the origin O, in terms of $\vec{a}$ and $\vec{b}$.

Solution

$\overrightarrow{OP} = 2\vec{a} + 3\vec{b}$ and $\overrightarrow{OQ} = 3\vec{a} + 7\vec{b}$

$PR : RQ = 2 : 3$

By section formula,

$$\overrightarrow{OR} = \frac{1}{2+3}(2\overrightarrow{OQ}+3\overrightarrow{OP}) = \frac{1}{5}(6\vec{a}+14\vec{b}+6\vec{a}+9\vec{b}) = \frac{1}{5}(12\vec{a}+23\vec{b}) = \frac{12}{5}\vec{a}+\frac{23}{5}\vec{b}.$$

Example 10.

The position vectors of A, B and C, relative to the origin O, are $3\vec{p}+2\vec{q}$, $-5\vec{p}-3\vec{q}$ and $4\vec{p} - \vec{q}$ respectively. M is the midpoint of AB, and the point N is such that $\overrightarrow{AN} = \frac{1}{3}\overrightarrow{AC}$. Find $\overrightarrow{MN}$ in terms of $\vec{p}$ and $\vec{q}$.

Solution

$\overrightarrow{OA} = 3\vec{p} + 2\vec{q}$, $\overrightarrow{OB} = -5\vec{p} - 3\vec{q}$, $\overrightarrow{OC} = 4\vec{p} - \vec{q}$

Since M is the midpoint of AB,

$$\begin{aligned}
\overrightarrow{OM} &= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) = \frac{1}{2}(3\vec{p} + 2\vec{q} - 5\vec{p} - 3\vec{q}) = -\vec{p} - \frac{1}{2}\vec{q} \\
\overrightarrow{AN} &= \frac{1}{3}\overrightarrow{AC} = \frac{1}{3}(\overrightarrow{OC} - \overrightarrow{OA}) = \frac{1}{3}(4\vec{p} - \vec{q} - 3\vec{p} - 2\vec{q}) = \frac{1}{3}\vec{p} - \vec{q} \\
\overrightarrow{ON} - \overrightarrow{OA} &= \frac{1}{3}\vec{p} - \vec{q} \\
\therefore \overrightarrow{ON} &= \frac{1}{3}\vec{p} - \vec{q} + \overrightarrow{OA} = \frac{1}{3}\vec{p} - \vec{q} + 3\vec{p} + 2\vec{q} = \frac{10}{3}\vec{p} + \vec{q} \\
\therefore \overrightarrow{MN} &= \overrightarrow{ON} - \overrightarrow{OM} = \frac{10}{3}\vec{p} + \vec{q} + \vec{p} + \frac{1}{2}\vec{q} = \frac{13}{3}\vec{p} + \frac{3}{2}\vec{q}
\end{aligned}$$

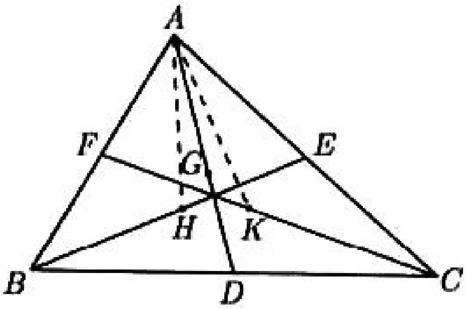
Example 11.

Prove by vector methods that the medians of a triangle are concurrent.

Solution

Let D, E, F be midpoints of BC, CA, AB, respectively.
Let G, H and K be points on AD, BE and CF, respectively such that

$$AG : GD = BH : HE = CK : KF = 2 : 1.$$



$$\overrightarrow{AG} = \frac{2}{3}\overrightarrow{AD} = \frac{2}{3}\left[\frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right)\right] = \frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$$

$$\overrightarrow{AH} = \frac{1}{1+2}\left(\overrightarrow{AB} + 2\overrightarrow{AE}\right) = \frac{1}{3}\left(\overrightarrow{AB} + \overrightarrow{AC}\right) = \frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$$

Similarly, $\overrightarrow{AK} = \frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$

$\therefore \overrightarrow{AG} = \overrightarrow{AH} = \overrightarrow{AK}$.
That is, G, H and K are identical.
Hence the three medians AD, BE and CF meet at one point.
Thus the medians of a triangle are concurrent.

Exercise 9.3

1. The position vectors of three points P, Q and R are $9\vec{a} - 4\vec{b}$, $-3\vec{a} - \vec{b}$ and $5\vec{a} - 3\vec{b}$ respectively. Express $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ in terms of $\vec{a}$ and $\vec{b}$. Are P, Q and R collinear?

2. The position vectors of A, B, P are $\vec{a}$, $\vec{b}$ and $\vec{p}$. Calculate the ratio $AP : PB$ if $\vec{p}$ has the following values.
   (i) $\frac{1}{3}(\vec{a} + 2\vec{b})$ (ii) $\frac{7}{5}\vec{a} - \frac{2}{5}\vec{b}$

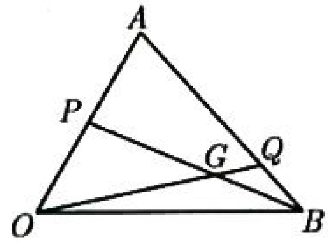
3. The position vectors of A, B and C are $2\vec{p} - \vec{q}$, $k\vec{p} + \vec{q}$ and $12\vec{p} + 4\vec{q}$ respectively. Calculate the value of k if A, B and C are collinear.

4. The position vectors of A, B and C are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Find $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$ for each of the following cases.
   (i) $3\overrightarrow{AC} = \overrightarrow{CB}$ (ii) $\overrightarrow{AB} = 4\overrightarrow{BC}$ (iii) $3\overrightarrow{BC} = 5\overrightarrow{CA}$ (iv) $\overrightarrow{AC} = -2\overrightarrow{CB}$.

5. The position vectors of points P and Q relative to an origin O are $2\vec{a}+5\vec{b}$ and $3\vec{a}-7\vec{b}$ respectively. R is a point on PQ such that $PR : RQ = 1:3$. Find the position vector of R, relative to O, in terms of $\vec{a}$ and $\vec{b}$.

6. In $\triangle ABC$, F is the midpoint of median AD and E is a point on AB such that $AE : EB = 1:3$. If $\overrightarrow{AE} = \vec{a}$ and $\overrightarrow{BD} = \vec{b}$, find $\overrightarrow{BF}$, $\overrightarrow{EF}$ and $\overrightarrow{AC}$. Is $EF \parallel AC$?

7. In the diagram, P is the midpoint of OA and Q lies on AB such that $\overrightarrow{AQ} = 3\overrightarrow{QB}$. Given that $\overrightarrow{OA} = 5\vec{s}$ and $\overrightarrow{OB} = 10\vec{t}$, express the followings in terms of $\vec{s}$ and $\vec{t}$.
(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{BQ}$ (iii) $\overrightarrow{OQ}$ (iv) $\overrightarrow{BP}$
Given that $\overrightarrow{BG} = \lambda\overrightarrow{BP}$ and $\overrightarrow{OG} = \mu\overrightarrow{OQ}$, evaluate λ and μ.



8. $OPQR$ is a parallelogram and PQ is produced to S such that $PQ = 3QS$. T is a point on OP such that $OT = \frac{1}{3}OP$, and D is the point of intersection of OQ and TS. If $\overrightarrow{OT} = \vec{a}$ and $\overrightarrow{OR} = 3\vec{b}$, find the ratio of OD and DQ.

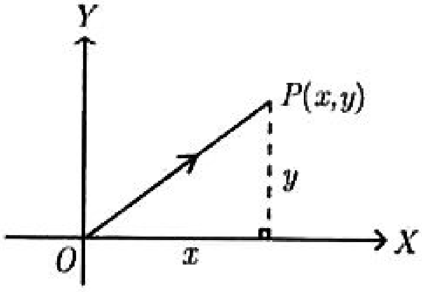
9. $OABC$ is a trapezium with $\overrightarrow{CB} = \frac{1}{2}\overrightarrow{OA}$ and D is a point on OA such that $OD = \frac{3}{4}OA$. If OB and CD intersect at G, find the ratio $CG : GD$.

10. If G is the centroid of a triangle ABC, show that
(i) $\overrightarrow{AB} + \overrightarrow{AC} = 3\overrightarrow{AG}$, (ii) $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$.

9.4 Two-Dimensional Vectors

In a Cartesian coordinate system, a vector can be represented by column matrix, later we will call **column vector**.

For a point $P(x, y)$ in the Cartesian coordinate system, the position vector of the point $P(x, y)$ can be written as $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$. By using Pythagoras theorem, we get



$$\left|\overrightarrow{OP}\right| = OP = \sqrt{x^2 + y^2}.$$

Definition. A vector with magnitude 1 is called a **unit** vector.

For a nonzero vector $\overrightarrow{a}$, the unit vector in the (same) direction of $\overrightarrow{a}$, denoted by $\hat{a}$, is given by $\hat{a} = \frac{1}{|\overrightarrow{a}|}\overrightarrow{a}$.

In Fig. 9.6, the coordinates of I and J are $(1, 0)$ and $(0, 1)$ respectively.

$\overrightarrow{OI} = \hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OJ} = \hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

are known as the **standard unit vectors** in Cartesian coordinate system.

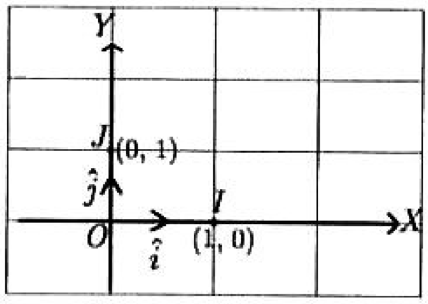
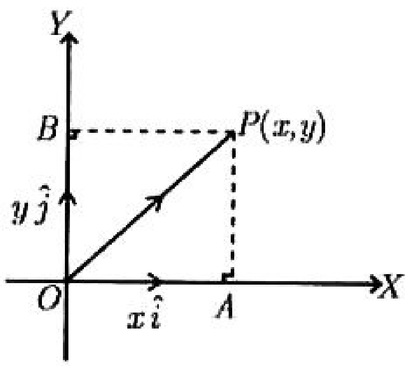


Fig. 9.6

The position vector of the point $P(x, y)$ can be represented by the standard unit vectors, i.e.,

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} = x\,\hat{i} + y\,\hat{j}.$$



So,

$$\begin{pmatrix} x \\ y \end{pmatrix} = x\,\hat{i} + y\,\hat{j}.$$

Also,

$$-\overrightarrow{OP} = -(x\,\hat{i} + y\,\hat{j}) = -x\,\hat{i} - y\,\hat{j}$$

Hence,

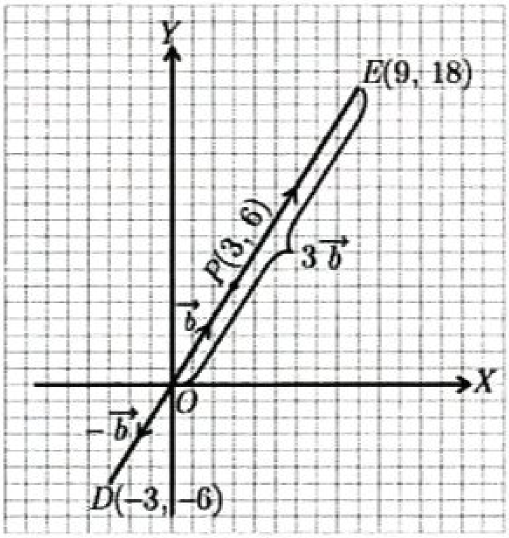
$$-\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}.$$

For any scalar k,

$$k\overrightarrow{OP} = k(x\ \hat{i} + y\ \hat{j}) = kx\ \hat{i} + ky\ \hat{j}.$$

Thus,

$$k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}.$$

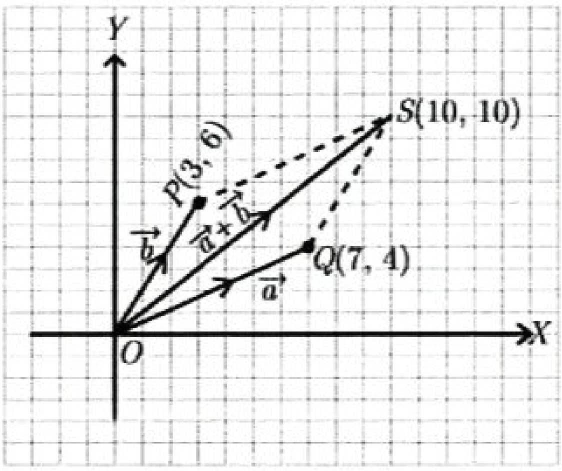


For two points $P(x, y)$ and $Q(u, v)$,

$$\overrightarrow{OP} + \overrightarrow{OQ} = (x\ \hat{i} + y\ \hat{j}) + (u\ \hat{i} + v\ \hat{j}) = (x+u)\hat{i} + (y+v)\hat{j}.$$

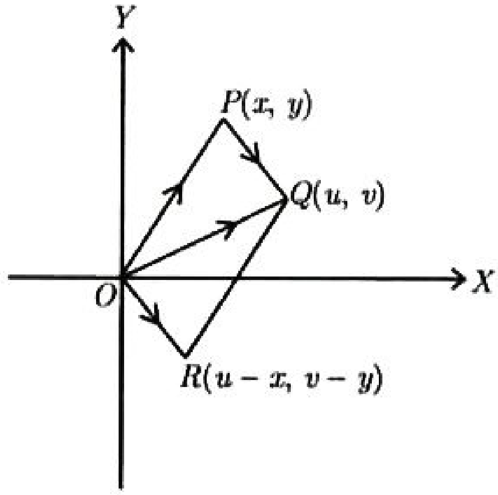
Therefore,

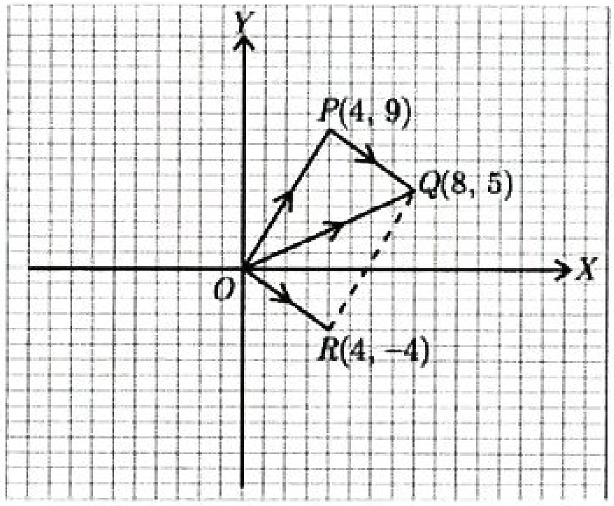
$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x+u \\ y+v \end{pmatrix}.$$



Produce a parallelogram $OPQR$. Then $\overrightarrow{PQ} = \overrightarrow{OR}$. We have known that $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$. Thus $\overrightarrow{PQ}$ can be represented by a position vector $\overrightarrow{OR}$ with terminal point $R(u-x, v-y)$:

$$\overrightarrow{PQ} = \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u-x \\ v-y \end{pmatrix} = \overrightarrow{OR}.$$





Example 12.

Let the coordinates of the points P and Q be (2, 3) and $(q, 2q)$ respectively. Given that $\overrightarrow{PQ}$ is a unit vector, calculate the possible values of q.

Solution

$P(2,3)$ and $Q(q,2q)$ are given.

$\therefore \overrightarrow{OP} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} q \\ 2q \end{pmatrix}$.

$\therefore \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} q \\ 2q \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} q-2 \\ 2q-3 \end{pmatrix}$

Since $\overrightarrow{PQ}$ is a unit vector, $\left|\overrightarrow{PQ}\right| = 1$.

$$\begin{aligned} \sqrt{(q-2)^2 + (2q-3)^2} &= 1 \\ q^2 - 4q + 4 + 4q^2 - 12q + 9 - 1 &= 0 \\ 5q^2 - 16q + 12 &= 0 \\ (5q-6)(q-2) &= 0 \\ q = \frac{6}{5} \quad &\text{or} \quad q = 2 \end{aligned}$$

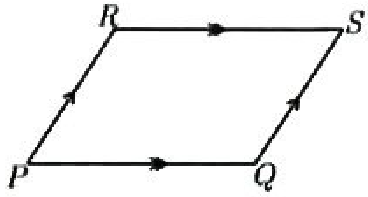
Example 13.

The coordinates of P, Q, and R are $(1,2),(7,3)$ and $(4,7)$ respectively. Find the coordinates of S if $PQSR$ is a parallelogram.

Solution

$P(1,2)$, $Q(7,3)$ and $R(4,7)$ are given.

$\overrightarrow{OP}=\begin{pmatrix}1\\2\end{pmatrix}, \overrightarrow{OQ}=\begin{pmatrix}7\\3\end{pmatrix}, \overrightarrow{OR}=\begin{pmatrix}4\\7\end{pmatrix}.$



Since $PQSR$ is a parallelogram,

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{RS}\\ \overrightarrow{OQ}-\overrightarrow{OP} &= \overrightarrow{OS}-\overrightarrow{OR}\\ \overrightarrow{OS} &= \overrightarrow{OQ}-\overrightarrow{OP}+\overrightarrow{OR}\\ &= \begin{pmatrix}7\\3\end{pmatrix}-\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}4\\7\end{pmatrix}\\ &= \begin{pmatrix}7-1+4\\3-2+7\end{pmatrix}\\ &= \begin{pmatrix}10\\8\end{pmatrix}\end{aligned}$$

Therefore the coordinates of S are (10, 8).

Example 14.

The coordinates of A, B and C are (1, 2), (7, 1) and $(-3,7)$ respectively. If $\overrightarrow{OC}=h\,\overrightarrow{OA}+k\,\overrightarrow{OB}$, where h and k are constants, find the values of h and k.

Solution

$$\overrightarrow{OC} = h\,\overrightarrow{OA}+k\,\overrightarrow{OB}$$

$$\begin{pmatrix}-3\\7\end{pmatrix} = h\begin{pmatrix}1\\2\end{pmatrix}+k\begin{pmatrix}7\\1\end{pmatrix}=\begin{pmatrix}h\\2h\end{pmatrix}+\begin{pmatrix}7k\\k\end{pmatrix}=\begin{pmatrix}h+7k\\2h+k\end{pmatrix}$$

$\therefore h+7k=-3$ and $2h+k=7$.

By solving these equations, $h=4$ and $k=-1$.

Example 15.

The position vectors of the points L and M are $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ respectively. Find the unit vectors parallel to $\overrightarrow{LM}$.

Solution

$\overrightarrow{OL} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ and $\overrightarrow{OM} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

$$\therefore \overrightarrow{LM} = \overrightarrow{OM} - \overrightarrow{OL} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

Let $\hat{a}$ be the unit vector in the direction of $\overrightarrow{LM}$.

$$\therefore \hat{a} = \frac{1}{\left|\overrightarrow{LM}\right|}\overrightarrow{LM} = \frac{1}{\sqrt{3^2 + (-4)^2}}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$$

So the unit vectors parallel to $\overrightarrow{LM}$ are $\hat{a} = \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$ and $-\hat{a} = \begin{pmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$.

Example 16.

The position vectors of A and B are $\begin{pmatrix} 4 \\ 14 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ 2 \end{pmatrix}$ respectively. Given that C lies on AB and has position vector $\begin{pmatrix} 2t \\ t \end{pmatrix}$, find the value of t and the ratio $AC : CB$.

Solution

$\overrightarrow{OA} = \begin{pmatrix} 4 \\ 14 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 12 \\ 2 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 2t \\ t \end{pmatrix}$.

Since C lies on AB, then A, C and B are collinear.
Let $\overrightarrow{AC} = h\,\overrightarrow{CB}$, for some nonzero scalar h.

$$\begin{aligned}
\therefore \overrightarrow{OC} - \overrightarrow{OA} &= h(\overrightarrow{OB} - \overrightarrow{OC}) \\
\begin{pmatrix} 2t \\ t \end{pmatrix} - \begin{pmatrix} 4 \\ 14 \end{pmatrix} &= h\left[\begin{pmatrix} 12 \\ 2 \end{pmatrix} - \begin{pmatrix} 2t \\ t \end{pmatrix}\right] \\
\begin{pmatrix} 2t - 4 \\ t - 14 \end{pmatrix} &= h\begin{pmatrix} 12 - 2t \\ 2 - t \end{pmatrix} = \begin{pmatrix} 12h - 2th \\ 2h - th \end{pmatrix}
\end{aligned}$$

Thus $2t - 4 = 12h - 2th$ and $t - 14 = 2h - th$.

$$
\begin{aligned}
\therefore \frac{2t-4}{t-14} &= \frac{12h-2th}{2h-th} \\
\frac{2t-4}{t-14} &= \frac{12-2t}{2-t} \\
4t - 2t^2 - 8 + 4t &= 12t - 2t^2 - 168 + 28t \\
8t - 8 &= 40t - 168 \\
32t &= 160 \\
t &= 5
\end{aligned}
$$

Substituting $t = 5$ in $t - 14 = 2h - th$

$$
\begin{aligned}
5 - 14 &= 2h - 5h \\
h &= 3 \\
\therefore \overrightarrow{AC} &= 3\overrightarrow{CB} \\
\therefore AC &= 3CB \\
\therefore AC : CB &= 3 : 1
\end{aligned}
$$

Example 17.

The vector $\overrightarrow{OA}$ has the magnitude of 39 units and has the same direction as $5\hat{i} + 12\hat{j}$. The vector $\overrightarrow{OB}$ has the magnitude of 25 units and has the same direction as $-3\hat{i} + 4\hat{j}$. Express $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in terms of $\hat{i}$ and $\hat{j}$ and find the magnitude of $\overrightarrow{AB}$.

Solution

Let $\overrightarrow{p} = 5\hat{i} + 12\hat{j}$ and $\overrightarrow{q} = -3\hat{i} + 4\hat{j}$.

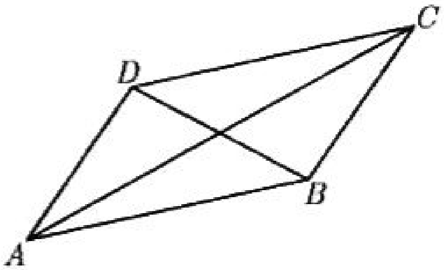
$$
\begin{aligned}
|\overrightarrow{p}| &= \sqrt{5^2 + 12^2} = \sqrt{169} = 13 \\
\hat{p} &= \frac{1}{|\overrightarrow{p}|}\overrightarrow{p} = \frac{1}{13}(5\hat{i} + 12\hat{j})
\end{aligned}
$$

$$
\begin{aligned}
\therefore \overrightarrow{OA} &= 39\hat{p} = 39\left[\frac{1}{13}(5\hat{i} + 12\hat{j})\right] = 15\hat{i} + 36\hat{j} \\
|\overrightarrow{q}| &= \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5 \\
\hat{q} &= \frac{1}{|\overrightarrow{q}|}\overrightarrow{q} = \frac{1}{5}(-3\hat{i} + 4\hat{j}) \\
\therefore \overrightarrow{OB} &= 25\hat{q} = 25\left[\frac{1}{5}(-3\hat{i} + 4\hat{j})\right] = -15\hat{i} + 20\hat{j} \\
\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = -15\hat{i} + 20\hat{j} - (15\hat{i} + 36\hat{j}) = -30\hat{i} - 16\hat{j} \\
\left|\overrightarrow{AB}\right| &= \sqrt{(-30)^2 + (-16)^2} = \sqrt{1156} = 34
\end{aligned}
$$

Example 18.

Prove that the sum of squares of diagonals of a parallelogram is equal to the sum of squares of four sides of the parallelogram.

Solution



Let $ABCD$ be a parallelogram with $\overrightarrow{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\overrightarrow{AD} = \begin{pmatrix} u \\ v \end{pmatrix}$. Then

$\left|\overrightarrow{AB}\right| = \sqrt{x^2+y^2}$ and $\left|\overrightarrow{AD}\right| = \sqrt{u^2+v^2}$.

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x+u \\ y+v \end{pmatrix}$$

Also,

$$\overrightarrow{DB} = \overrightarrow{AB} - \overrightarrow{AD} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x-u \\ y-v \end{pmatrix}.$$

$$\left|\overrightarrow{AC}\right| = \sqrt{(x+u)^2+(y+v)^2} \text{ and } \left|\overrightarrow{DB}\right| = \sqrt{(x-u)^2+(y-v)^2}.$$

$$\begin{aligned} AC^2 + DB^2 &= \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{DB}\right|^2 \\ &= (x+u)^2+(y+v)^2+(x-u)^2+(y-v)^2 \\ &= 2x^2+2u^2+2y^2+2v^2 \\ &= 2(x^2+y^2)+2(u^2+v^2) \\ &= 2\left|\overrightarrow{AB}\right|^2 + 2\left|\overrightarrow{AD}\right|^2 \\ &= 2AB^2 + 2AD^2 \\ &= AB^2+CD^2+AD^2+BC^2 \qquad (\because AB = CD \text{ and } AD = BC) \end{aligned}$$

Exercise 9.4

1. $ABCD$ is a trapezium such that $\overrightarrow{DC} = 2\overrightarrow{AB}$. If $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\overrightarrow{AD} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, find the magnitude of $\overrightarrow{BC}$.

2. The coordinates of points P, Q and R are $(1, 2)$, $(7, 3)$ and $(4, 7)$ respectively. If $PQSR$ is a parallelogram, find the coordinates of S by vector method. If PS and QR meet at T, find the coordinates of T by using vectors.

3. Given that $P = (4, -9)$ and $Q = (-8, 3)$. If R is the midpoint of PQ and S is the point such that $\overrightarrow{RS} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$, find the coordinates of the point S.

4. $ABCD$ is a parallelogram with $\overrightarrow{AB} = 2\hat{i} + 5\hat{j}$ and $\overrightarrow{AC} = 8\hat{i} + 14\hat{j}$. Find $\overrightarrow{AD}$. If D has the coordinates (3, 7), find the coordinates of the point E of intersection of the diagonals.

5. If $\overrightarrow{a} = -\hat{i} + 2\hat{j}$, $\overrightarrow{b} = 3\hat{i} + 5\hat{j}$ and $\overrightarrow{c} = \hat{i} + 6\hat{j}$, calculate the modulus of the following vectors.
(i) $\overrightarrow{a} + 2\overrightarrow{b} + 2\overrightarrow{c}$ (ii) $3\overrightarrow{a} - 3\overrightarrow{b} + 4\overrightarrow{c}$ (iii) $3\overrightarrow{a} + 6\overrightarrow{b} - \overrightarrow{c}$

6. The coordinates of A and B be $(2, 5)$ and $(3, 4)$ respectively. Find the unit vector $\hat{p}$ in the direction of $\overrightarrow{AB}$. If $R(x, y)$ is any point on the line containing AB, and $\overrightarrow{AR} = h\hat{p}$, find the relation between x and y.

7. The vector $\overrightarrow{OP}$ has a magnitude of 26 units and has the same direction as $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$. The vector $\overrightarrow{OQ}$ has a magnitude of 20 units and has the opposite direction as $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Express $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ as column vectors and find the unit vector in the direction of $\overrightarrow{PQ}$.

8. A, B and C are points with position vectors $\hat{i}+3\hat{j}$, $2\hat{i}+5\hat{j}$ and $k\hat{i}-4\hat{j}$ respectively. Find the value of k if A, B and C are collinear.

9. The position vectors of the points A and B are $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ respectively. Given that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$, express the position vector of P in terms of λ. Find the value of λ such that $\overrightarrow{OP}$ is parallel to $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

10. Prove that the bisector of an interior angle of a triangle divides the opposite side internally into a ratio equal to the ratio of the other two sides of the triangle.

Chapter 10

Trigonometry

Trigonometry deals with the measurement of sides and angles of a triangle. It has been widely used in Astronomy, Surveying, Geography, Physics, Navigation etc.To study trigonometry, the student should already be acquainted with the theorems on similar triangles. Only then will they find it easy to understand the definition of trigonometric ratio. In every branch of higher mathematics, whether pure or applied, a knowledge of trigonometry is of great value. In this chapter, first we introduce circular measure of any angle. Also, compound angle formulae and its related expansion formulae are expressed. Finally, we express the derivation of Law of Sines and Law of Cosines.

10.1 Trigonometric Ratios of Any Angle

Consider a circle of unit radius with its centre at the origin of the XY-plane.

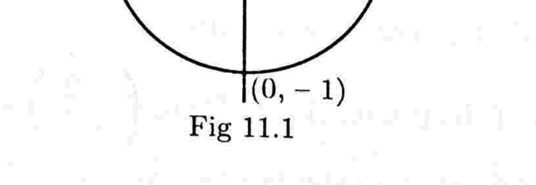


Fig 11.1

Suppose a point P is started at $A(1,0)$. OP rotates through an angle θ in degree from the positive X-axis, counter clockwise if $\theta > 0°$ and clockwise if $\theta < 0°$.

We can locate the exact position of P for any specific value of θ. If $\theta > 360°$, we shall continue around the circle passing through $(1,0)$ until we reach required angle θ. Now to every value of θ there is associated a point called the **terminal point**

$P(x, y)$, which is the point of intersection of the terminal side and the circumference of the unit circle, we define the six trigonometric ratios for θ as follows:

$$\cos\theta = x \qquad \sec\theta = \frac{1}{x}, x \neq 0$$

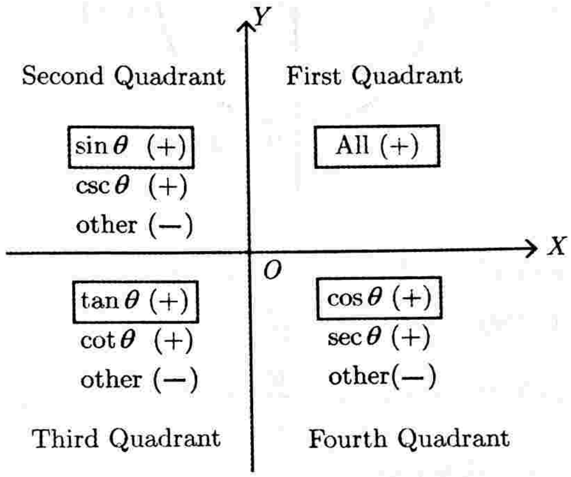
$$\sin\theta = y \qquad \csc\theta = \frac{1}{y}, y \neq 0$$

$$\tan\theta = \frac{y}{x}, x \neq 0 \qquad \cot\theta = \frac{x}{y}, y \neq 0$$

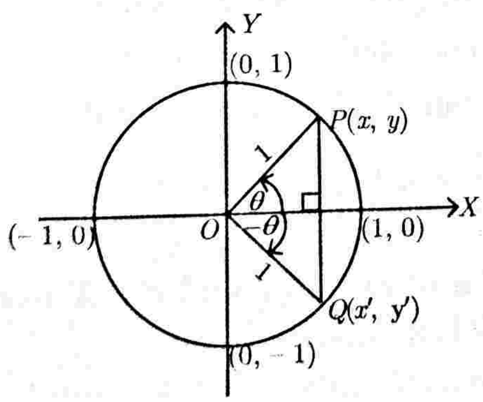
According to the above definition, we can see from Figure 11.1, that

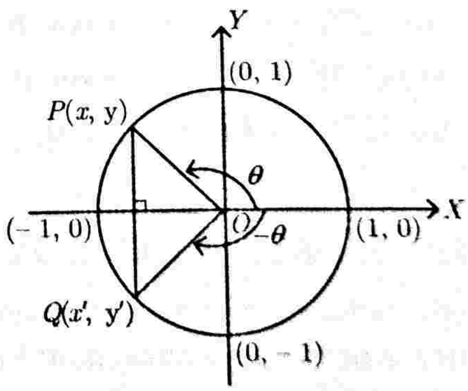
$$-1 \leq \cos\ \theta \leq 1 \quad \text{and} \quad -1 \leq \sin\ \theta \leq 1.$$

The signs of $x = \cos\theta$ and $y = \sin\theta$ depend upon the quadrant in which $P(x, y)$ lies.



10.2 Negative Angles



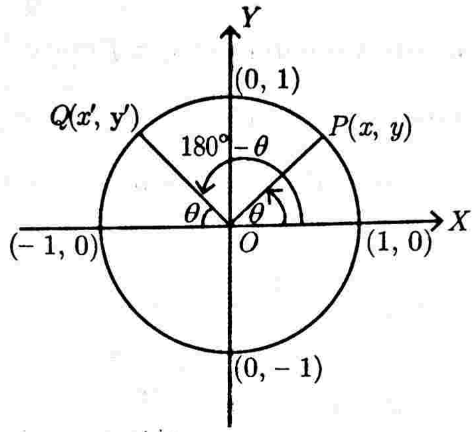


Consider a unit circle with a point $P(x, y)$ and OP rotates through an angle θ from the positive X-axis. Under the reflection in the positive X-axis, the point P is mapped onto the point $Q(x', y')$.

From the figure,

$$\cos(-\theta) = x' = x = \cos\theta,$$
$$\sin(-\theta) = y' = -y = -\sin\theta,$$
$$\tan(-\theta) = \frac{y'}{x'} = \frac{-y}{x} = -\tan\theta.$$

Consider a unit circle with a point $P(x, y)$ and OP rotates through an angle θ from the X-axis. Under the reflection in the Y-axis, the point P is mapped onto the point $Q(x', y')$.



From the figure,

$$\cos(180° - \theta) = x' = -x = -\cos\theta.$$
$$\sin(180° - \theta) = y' = y = \sin\theta,$$
$$\tan(180° - \theta) = \frac{y'}{x'} = \frac{y}{-x} = -\tan\theta.$$

Similarly, we can derive the followings.

$$\sin(90° + \theta) = \cos\theta, \qquad \sin(180° + \theta) = -\sin\theta,$$
$$\cos(90° + \theta) = -\sin\theta, \qquad \cos(180° + \theta) = -\cos\theta,$$
$$\tan(90° + \theta) = -\cot\theta. \qquad \tan(180° + \theta) = \tan\theta.$$

Also,

$$\sin(270° - \theta) = -\cos\theta, \qquad \sin(360° - \theta) = -\sin\theta,$$
$$\cos(270° - \theta) = -\sin\theta, \qquad \cos(360° - \theta) = \cos\theta,$$
$$\tan(270° - \theta) = \cot\theta. \qquad \tan(360° - \theta) = -tan\,\theta.$$

10.3 The Basic Acute Angle

The acute angle between the terminal side of a given angle and the X-axis is called the **basic acute angle**. The basic acute angle is a positive angle.

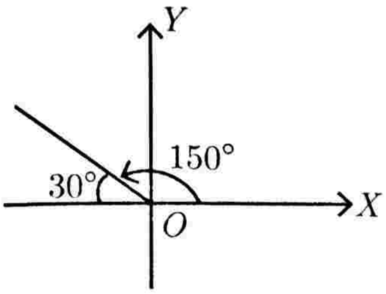
Example 1.

Find the basic acute angle of

(a) 150° (b) 210° (c) 300° (d) −140° (e) −35°

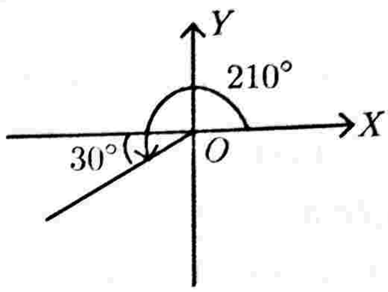
Solution

(a)



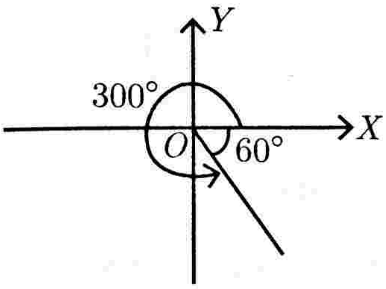
The basic acute angle = 30°

(b)



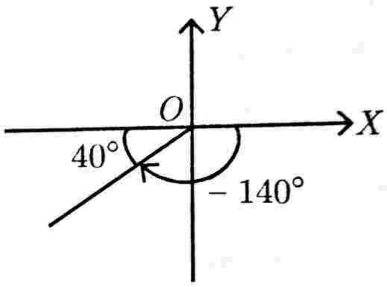
The basic acute angle = 30°

(c)



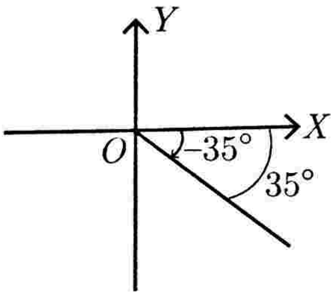
The basic acute angle = 60°

(d)



The basic acute angle = 40°

(e)



The basic acute angle = 35°

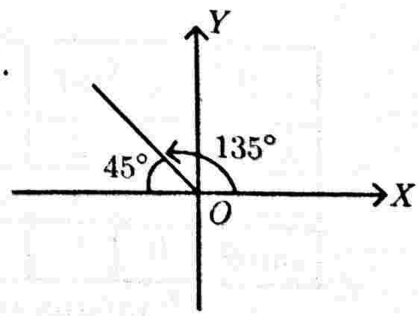
Example 2.

Using the basic acute angle, find the six trigonometric ratios for the obtuse angle 135°.

Solution

135° is in the second quadrant, with basic acute angle 45°.

The cosine ratio is negative and the sine ratio is positive.



So

$$\cos 135° = -\cos 45° = -\frac{\sqrt{2}}{2}, \qquad \sec 135° = -\sqrt{2},$$
$$\sin 135° = \sin 45° = \frac{\sqrt{2}}{2}, \qquad \csc 135° = \sqrt{2},$$
$$\tan 135° = -\tan 45° = -1, \qquad \cot 135° = -1$$

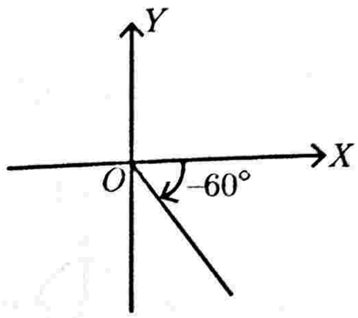
Example 3.

Find the value of $\cos(-60°)$.

Solution

$-60°$ is in the fourth quadrant, with basic acute angle $60°$.
Cosine ratio is positive.

$\therefore \cos(-60°) = \cos 60° = \frac{1}{2}$



Alternative Method

$\cos(-60°) = \cos 60° = \frac{1}{2}$ (using an identity for trigonometric ratios of negative angles)

Note: The trigonometric ratios of an angle θ can be formed by following steps:

(1) Determine the quadrant in which angle θ is in.

(2) Find the basic acute angle.

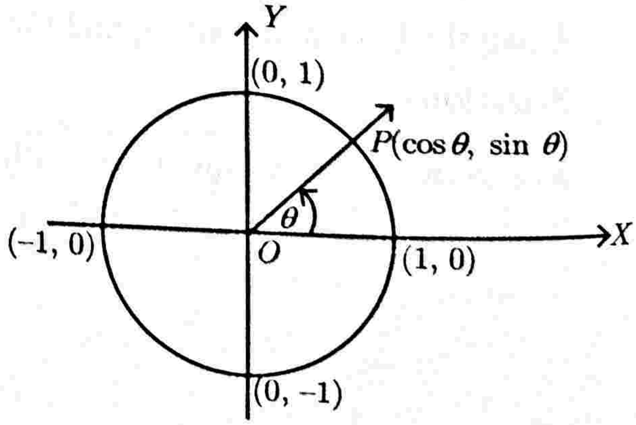
(3) The trigonometric ratio of angle θ is equal to positive or negative value of the trigonometric ratio of the basic acute angle. The sign is determined by the quadrant in which θ lies.

10.4 Trigonometric Ratios of $0°, 90°, 180°, 270°, 360°$

Consider a circle of unit radius with its centre at the origin of the XY-plane.

θ	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin\theta$	0	1	0	−1	0
$\cos\theta$	1	0	−1	0	1
$\tan\theta$	0	*	0	*	0

(* means undefined)



Example 4.

Find the values of $x, 0° \leq x \leq 360°$ for the following equations.

(a) $\sin x = -\frac{1}{2}$ (b) $\cos x = \frac{1}{2}$ (c) $\tan x = -1$

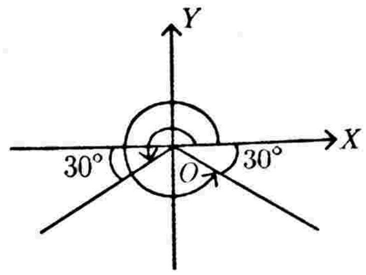
Solution

(a) Since $\sin x = -\frac{1}{2}$,

x lies in the third or the fourth quadrant.

Since $\sin 30° = \frac{1}{2}$,

the basic acute angle is $30°$.



Hence $x = 180° + 30°$ or $x = 360° - 30°$

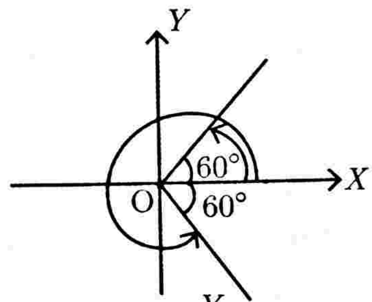
$x = 210°$ or $x = 330°$

(b) Since $\cos x = \frac{1}{2}$,

x lies in the first or the fourth quadrant.

Since $\cos 60° = \frac{1}{2}$,

the basic acute angle is $60°$.



Hence $x = 60°$ or $x = 360° - 60°$

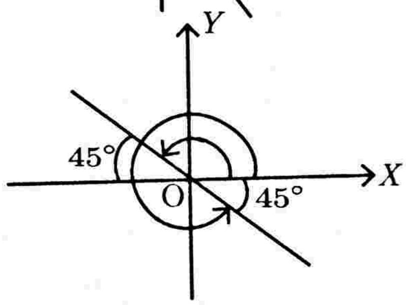
$x = 60°$ or $x = 300°$

(c) Since $\tan x = -1$,

x lies in the second or the fourth quadrant.

Since $\tan 45° = 1$,

the basic acute angle is $45°$.



Hence $x = 180° - 45°$ or $x = 360° - 45°$

$x = 135°$ or $x = 315°$

Example 5.

Solve the equation $\cot^2 x \cos x = 3 \cos x$ for $0° \leq x \leq 360°$.

Solution

$$\begin{aligned} \cot^2 x \cos x &= 3 \cos x \\ \cot^2 x \cos x - 3 \cos x &= 0 \\ \cos x(\cot^2 x - 3) &= 0 \\ \cos x = 0 \quad &\text{or} \quad \cot^2 x - 3 = 0 \\ \cos x = 0 \quad &\text{or} \quad \cot^2 x = 3 \\ \cos x = 0 \quad &\text{or} \quad \cot x = \pm\sqrt{3} \end{aligned}$$

If $\cos x = 0$, then $x = 90°$ or $270°$.

If $\cot x = \sqrt{3}$, then $x = 30°$ or $210°$.
If $\cot x = -\sqrt{3}$, then $x = 150°$ or $330°$.
$\therefore x = 30°$ or $90°$ or $150°$ or $210°$ or $270°$ or $330°$

Exercise 10.1

1. Using the basic acute angle, find the six trigonometric ratios of
   (i) $x = 120°$ (ii) $x = 210°$ (iii) $x = -30°$ (iv) $x = 300°$.

2. Find the values of $\theta, 0° \leq \theta \leq 360°$ for the following equations.
   (i) $\sin\theta = -\frac{\sqrt{3}}{2}$ (ii) $\tan\theta = -\sqrt{3}$ (iii) $\cot\theta = \sqrt{3}$
   (iv) $\sin(2\theta - 15°) = \frac{\sqrt{3}}{2}$ (v) $\cot(2\theta + 30°) = -1$ (vi) $\sec 3\theta = \frac{\sqrt{2}}{2}$

3. Find the exact values of the followings.
   (i) $\csc(-150°)$ (ii) $\tan 210°$ (iii) $\sin(-135°)$
   (iv) $\cot 120°$ (v) $\sec(-30°)$ (vi) $\cos 225°$

4. Find the exact values of the followings.
   (i) $\tan 750°$ (ii) $\cos(-780°)$ (iii) $\csc(-765°)$
   (iv) $\cot 240°$ (v) $\sec 330°$ (vi) $\sin(-660°)$

5. Solve the following equations for $0° \leq x \leq 360°$.
   (i) $2\sin x\cos x = \sin x$ (ii) $\cos^2 x - cos x = 2$
   (iii) $\cos x = 2\sin^2 x - 1$ (iv) $2\cos^2 x = \sin x + 1$

6. If $\alpha + \beta + \gamma = 180°$, prove that
   (i) $\cos(\alpha + \beta) = \sin(270° - \gamma)$ (ii) $\cos\frac{\alpha+\beta}{2} = \sin(180° - \frac{\gamma}{2})$
   (iii) $\cot\frac{\gamma}{2} = \tan(180° + \frac{\alpha+\beta}{2})$.

7. Prove that $\tan\theta + \tan(180° - \theta) + \cot(90° + \theta) = \tan(360° - \theta)$.

8. Prove that

$$\sin(90° + \theta)\ \cos(180° - \theta)\ \cot(270° + \theta)$$
$$= \sin(90° - \theta)\ \sin(270° - \theta)\ \cot(90° + \theta).$$

10.5 Further Trigonometric Identities

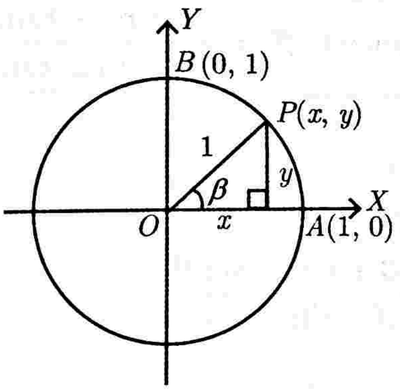
(a) Compound Angle Formulae

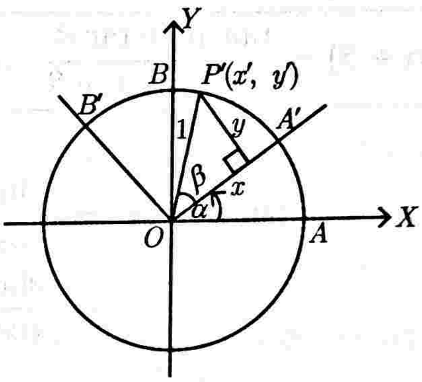
Consider a circle, with unit radius, centred at the origin of the XY- plane. Then the points $A(1,0)$ and $B(0,1)$ lie on the circle. Let $P(x,y) = (\cos\beta, \sin\beta)$.

The point P is reached from O by moving a distance x along OA and a distance y parallel to OB, that is

$$\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Consider now a rotation of the plane about O, through an angle α in the positive (counterclockwise) direction, in which $\angle A'OB' = 90°$, A moves to $A'(\cos\alpha, \sin\alpha)$ and B moves to $B'(-\sin\alpha, \cos\alpha)$, while P moves to P' .





Then P' is reached from O by moving a distance x along OA', and a distance y parallel to OB' , so that the position vector of $P'(x', y')$ is given by

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= x\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} + y\begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix} \\ &= \begin{pmatrix} x\cos\alpha \\ x\sin\alpha \end{pmatrix} + \begin{pmatrix} -y\sin\alpha \\ y\cos\alpha \end{pmatrix} \\ &= \begin{pmatrix} x\cos\alpha - y\sin\alpha \\ x\sin\alpha + y\cos\alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha \\ \cos\beta\sin\alpha + \sin\beta\cos\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{pmatrix} \end{aligned}$$

Since $\angle AOP' = \alpha + \beta$, P' represents the terminal point of the angle $\alpha + \beta$ and has coordinates $(\cos(\alpha+\beta), \ \sin(\alpha+\beta))$. Corresponding these two forms for the coordinates of P', we find that

$$\boxed{\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}}$$

Consequently,

$$\begin{aligned}\cos(\alpha-\beta) &= \cos(\alpha+(-\beta))\\ &= \cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta)\\ &= \cos\alpha\cos\beta+\sin\alpha\sin\beta,\\ \sin(\alpha-\beta) &= \sin(\alpha+(-\beta))\\ &= \sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)\\ &= \sin\alpha\cos\beta-\cos\alpha\sin\beta.\end{aligned}$$

$$\boxed{\begin{aligned}\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta\\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta\end{aligned}}$$

We can also prove that

$$\boxed{\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \quad \text{and} \quad \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}$$

Indeed,

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}\end{aligned}$$

$$\therefore \tan(\alpha+\beta)=\frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

(b) Double Angle Formulae

From the formulae for compound angle, more identities can be derived.
If $\alpha=\beta$, then by using compound angle formulae,

$$\sin(\alpha+\alpha)=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha, \quad \cos(\alpha+\alpha)=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha$$

and $\tan(\alpha+\alpha)=\dfrac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}$.

Therefore,

$\sin 2\alpha = 2\sin\alpha\cos\alpha$	$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$	$\tan 2\alpha=\dfrac{2\tan\alpha}{1-\tan^2\alpha}$
	$\cos 2\alpha = 1-2\sin^2\alpha$	
	$\cos 2\alpha = 2\cos^2\alpha-1$	

(c) Half Angle Formulae

Using $\cos 2\alpha=1-2\sin^2\alpha$ and $\cos 2\alpha=2\cos^2\alpha-1$,

we have $\sin^2\alpha=\dfrac{1-\cos 2\alpha}{2}$ and $\cos^2\alpha=\dfrac{1+\cos 2\alpha}{2}$.

Then $\sin\alpha = \pm\sqrt{\dfrac{1-\cos 2\alpha}{2}}$ and $\cos\alpha = \pm\sqrt{\dfrac{1+\cos 2\alpha}{2}}$.

It follows that $\boxed{\sin\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}} \quad \text{and} \quad \cos\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}}$

Note : The signs of $\sin\frac{\alpha}{2}$ and $\cos\frac{\alpha}{2}$ depend on the quadrant in which $\frac{\alpha}{2}$ lies.

Since $2\sin^2\frac{\alpha}{2} = 1-\cos\alpha$, $2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \sin\alpha$ and $2\cos^2\frac{\alpha}{2} = 1+\cos\alpha$,

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} \quad \text{and} \quad \tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}.$$

So $\boxed{\tan\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{\sin\alpha} = \dfrac{\sin\alpha}{1+\cos\alpha}}$

Example 6.

Prove that $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$.

Solution

$$\begin{aligned}
\sin(x+y)\sin(x-y) &= (\sin x\cos y + \cos x\sin y)(\sin x\cos y - \cos x\sin y)\\
&= \sin^2 x\cos^2 y - \cos^2 x\sin^2 y\\
&= \sin^2 x(1-\sin^2 y) - (1-\sin^2 x)\sin^2 y\\
&= \sin^2 x - \sin^2 x\sin^2 y - \sin^2 y + \sin^2 x\sin^2 y\\
&= \sin^2 x - \sin^2 y
\end{aligned}$$

Example 7.

Prove that $\sin 75° = \cos 15° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

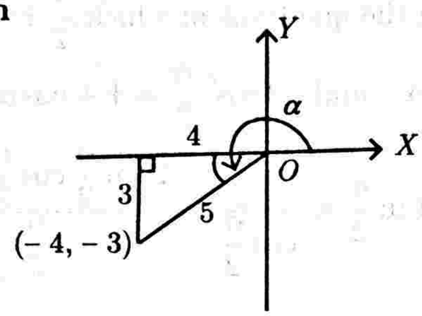
Solution

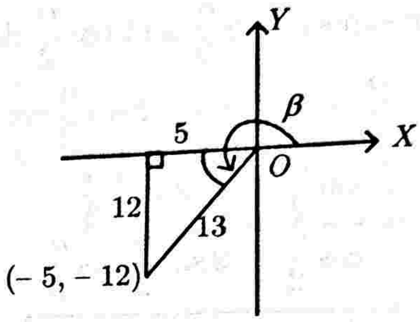
$$\begin{aligned}
\sin 75° &= \sin(45°+30°)\\
&= \sin 45°\cos 30° + \cos 45°\sin 30°\\
&= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}\\
\cos 15° &= \cos(90°-75°) = \sin 75° = \frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}$$

Example 8.

If $\sin\alpha = -\frac{3}{5}$ for $0° \le \alpha < 360°$, $\cos\beta = -\frac{5}{13}$ for $0° \le \beta < 360°$ and that angles α and β are in the same quadrant, find the values of $\sin(\alpha+\beta)$ and $\cos 2\alpha$.

Solution





Angles α and β are in the same quadrant, with both $\sin\alpha$ and $\cos\beta$ being negative. So they lie in the third quadrant.

From the figures, $\cos\alpha = -\frac{4}{5}$ and $\sin\beta = -\frac{12}{13}$.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= (-\frac{3}{5})(-\frac{5}{13}) + (-\frac{4}{5})(-\frac{12}{13}) = \frac{15}{65} + \frac{48}{65} = \frac{63}{65}$$

and $\cos 2\alpha = 1 - 2\sin^2\alpha = 1 - 2(-\frac{3}{5})^2 = 1 - \frac{18}{25} = \frac{7}{25}$

Alternative Solution

Angles α and β are in the same quadrant, with both $\sin\alpha$ and $\cos\beta$ being negative. So they lie in the third quadrant.

$\cos^2\alpha = 1 - \sin^2\alpha = 1 - (-\frac{3}{5})^2 = 1 - \frac{9}{25} = \frac{16}{25}$. $\therefore \cos\alpha = -\frac{4}{5}$

$\sin^2\beta = 1 - \cos^2\beta = 1 - (-\frac{5}{13})^2 = 1 - \frac{25}{169} = \frac{144}{169}$ $\therefore \sin\beta = -\frac{12}{13}$

$$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= (-\frac{3}{5})(-\frac{5}{13}) + (-\frac{4}{5})(-\frac{12}{13}) = \frac{15+48}{65} = \frac{63}{65}$$

and $\cos 2\alpha = 1 - 2\sin^2\alpha = 1 - 2(-\frac{3}{5})^2 = 1 - \frac{18}{25} = \frac{7}{25}$

Example 9.

If $\sin x = a$, where x is an acute angle, express the following in terms of a.
(i) $\sin 2x$ (ii) $\cos 2x$ (iii) $\sin 4x$ (iv) $\cos 4x$

Solution

Since $\sin x = a$, we have $\cos^2 x = 1 - \sin^2 x = 1 - a^2$

$\therefore \cos x = \sqrt{1-a^2}$ ($\because x$ is an acute angle)

(i) $\sin 2x = 2\sin x\cos x = 2a\sqrt{1-a^2}$

(ii) $\cos 2x = 1 - 2\sin^2 x = 1 - 2a^2$

(iii) $\sin 4x = 2\sin 2x\cos 2x = 2(2a\sqrt{1-a^2})(1-2a^2) = 4a(1-2a^2)\sqrt{1-a^2}$

(iv) $\cos 4x = 1 - 2\sin^2 2x = 1 - 2(2a\sqrt{1-a^2})^2 = 1 - 8a^2 + 8a^4$

Example 10.

Express $\cos 3x$ in terms of $\cos x$ and hence find the values of $\sin 18°$ and $\cos 36°$.

Solution

$$\begin{aligned}\cos 3x &= \cos(2x + x)\\ &= \cos 2x\cos x - \sin 2x\sin x\\ &= (2\cos^2 x - 1)\cos x - 2\sin^2 x\cos x\\ &= 2\cos^3 x - \cos x - 2(1-\cos^2 x)\cos x = 4\cos^3 x - 3\cos x\end{aligned}$$

If $x = 18°$, then $5x = 90°$. So $2x = 90° - 3x$.

$$\begin{aligned}\therefore \sin 2x &= \sin(90° - 3x) = \cos 3x\\ 2\sin x\cos x &= 4\cos^3 x - 3\cos x\\ 2\sin x &= 4\cos^2 x - 3 \quad (\because \cos x = \cos 18° \neq 0)\\ 2\sin x &= 4(1-\sin^2 x) - 3\\ 4\sin^2 x + 2\sin x - 1 &= 0\\ \sin x &= \frac{-2\pm\sqrt{4+16}}{8} = \frac{-1\pm\sqrt{5}}{4}\end{aligned}$$

Since $18°$ is an acute angle,

$$\begin{aligned}\sin 18° &= \frac{-1+\sqrt{5}}{4}\\ \cos 36° &= 1 - 2\sin^2 18° = 1 - 2(\frac{6-2\sqrt{5}}{16}) = \frac{\sqrt{5}+1}{4}\end{aligned}$$

Example 11.

(a) Express $\cos x + \sin x$ as single trigonometric ratio and hence solve the equation:

$\cos x + \sin x = 1$, for $0° \le x \le 360°$.

(b) Express $\sqrt{3}\cos x - \sin x$ as single trigonometric ratio and hence solve:

$\sqrt{3}\cos x - \sin x = -1, 0° \le x \le 360°$.

Solution

(a) $\cos x + \sin x = \sqrt{2}(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x)$

$= \sqrt{2}(\sin 45° \cos x + \cos 45° \sin x) = \sqrt{2}\sin(45° + x)$

If $\cos x + \sin x = 1, \quad 0° \le x \le 360°,$

then $\sqrt{2}\sin(45° + x) = 1, \quad 45° \le 45° + x \le 405°.$

$\therefore \sin(45° + x) = \frac{1}{\sqrt{2}}$

$\sin(45° + x) = \sin 45° \text{ or } \sin 135° \text{ or } \sin 405°$

$\therefore 45° + x = 45° \quad \text{or} \quad 45° + x = 135° \quad \text{or} \quad 45° + x = 405°$

$\therefore x = 0° \quad \text{or} \quad x = 90° \quad \text{or} \quad x = 360°$

(b) $\sqrt{3}\cos x - \sin x = 2(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x)$

$= 2(\cos 30° \cos x - \cos 60° \sin x)$

$= 2(\cos 30° \cos x - \sin 30° \sin x) = 2\cos(30° + x)$

If $\sqrt{3}\cos x - \sin x = -1, \quad 0° \le x \le 360°$

then $2\cos(30° + x) = -1, \quad 30° \le 30° + x \le 390°$

$\cos(30° + x) = -\frac{1}{2} = \cos 120° \text{ or } \cos 240°$

$\therefore 30° + x = 120° \quad \text{or} \quad 30° + x = 240°$

$\therefore x = 90° \quad \text{or} \quad x = 210°$

Example 12.

Express $a\sin x - b\cos x$ as a single trigonometric ratio, where a and b are positive.

Solution

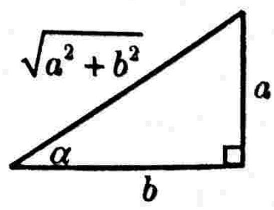
$a\sin x - b\cos x$

$= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x - \frac{b}{\sqrt{a^2+b^2}}cosx\right)$

$= \sqrt{a^2+b^2}(\sin\alpha\sin x - \cos\alpha\cos x)$

$= -\sqrt{a^2+b^2}(\cos x\cos\alpha - \sin x\sin\alpha)$

$= -\sqrt{a^2+b^2}\cos(x+\alpha)$



Exercise 10.2

1. Prove that $\sin(x+y)\sin(x-y) = \cos^2 y - \cos^2 x$.
2. Prove that $\cot(\alpha+\beta) = \dfrac{\cot\alpha\ \cot\beta - 1}{\cot\beta + \cot\alpha}$.
3. Prove that $\cos 75° = \sin 15° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ and $\tan 75° = \cot 15° = 2+\sqrt{3}$.
4. Prove that $\cos(45°+A) + \sin(A-45°) = 0$.
5. Prove that $2\sin(45°+\alpha)\cos(45°+\beta) = \cos(\alpha+\beta) + \sin(\alpha-\beta)$.
6. Express (i) $\sin 3x$ in terms of $\sin x$ and (ii) $\tan 3x$ in terms of $\tan x$.
7. Prove that $\cos(\alpha+45°) = \dfrac{\sqrt{2}}{2}(\cos\alpha - \sin\alpha)$.
8. Prove that $2\sin(30°-x) = \cos x - \sqrt{3}\sin x$.
9. Prove that $\cos^6 x + \sin^6 x = 1 - \dfrac{3}{4}\sin^2 2x$.
10. Given that $\sin\alpha = \dfrac{15}{17}, 0° \le \alpha < 360°, \cos\beta = -\dfrac{3}{5}, 0° \le \beta < 360°$ and that α and β are in the same quadrant, find without using table, the values of
    (i) $\sin 2\alpha$ (ii) $\cos\dfrac{\alpha}{2}$ (iii) $\cos 2\beta$ (iv) $\sin^2\dfrac{\beta}{2}$.
11. Given that α is acute and $\cos\alpha = x$, find, in terms of x, the values of
    (i) $\tan^2\alpha$ (ii) $\sin 2\alpha$ (iii) $\cos 4\alpha$ (iv) $\sin\dfrac{\alpha}{2}$.
12. Express the following as single trigonometric ratios.
    (i) $\sin x - \cos x$ (ii) $5\cos x + 12\sin x$
    (iii) $a\cos x - b\sin x$, where a and b are positive.
13. Find all solutions of (i) $\sin x - \cos x = -1$ for $0° \le x < 360°$.
    (ii) $\sqrt{3}\sin x + \cos x = 0$ for $0° \le x < 360°$.

(d) Product-to-Sum Formulae

From the compound angle formulae

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta) \quad (1)$$
$$\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta) \quad (2)$$

By adding and subtracting equations (1) and (2), we get

$$\boxed{\begin{aligned}2\sin\alpha\cos\beta &= \sin(\alpha+\beta)+\sin(\alpha-\beta).\\ 2\cos\alpha\sin\beta &= \sin(\alpha+\beta)-\sin(\alpha-\beta).\end{aligned}}$$

Again,

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta) \quad (3)$$
$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta) \quad (4)$$

By adding and subtracting, equations (3) and (4), we get

$$\boxed{\begin{aligned}2\cos\alpha\cos\beta &= \cos(\alpha+\beta)+\cos(\alpha-\beta).\\ 2\sin\alpha\sin\beta &= \cos(\alpha-\beta)-\cos(\alpha+\beta).\end{aligned}}$$

(e) Sum-to-Product Formulae

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta \quad (5)$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta-\cos\alpha\sin\beta \quad (6)$$

By adding and subtracting, equations (5) and (6), we get

$$\boxed{\begin{aligned}\sin(\alpha+\beta)+\sin(\alpha-\beta) &= 2\sin\alpha\cos\beta.\\ \sin(\alpha+\beta)-\sin(\alpha-\beta) &= 2\cos\alpha\sin\beta.\end{aligned}}$$

Let $A=\alpha+\beta$ and $B=\alpha-\beta$.

Then $2\alpha = A+B$ and $2\beta = A-B$.

Therefore, $\alpha = \dfrac{A+B}{2}$ and $\beta = \dfrac{A-B}{2}$.

Substituting these values in the above two equations, we get

$$\boxed{\begin{aligned}\sin A+\sin B &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\\ \sin A-\sin B &= 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}.\end{aligned}}$$

Also, we can derive

$$\boxed{\begin{aligned}\cos A+\cos B &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}\\ \cos A-\cos B &= -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.\end{aligned}}$$

Example 13.

Express the following in the form of a sum.

(a) $2\cos(30^\circ+\theta)\cos(30^\circ-\theta)$ (b) $\sin(2\alpha-\beta)\cos(2\alpha+\beta)$

Solution

(a) $2\cos(30° + \theta)\cos(30° - \theta)$

$= \cos(30° + \theta + 30° - \theta) + \cos(30° + \theta - 30° + \theta)$

$= \cos 60° + \cos 2\theta = \frac{1}{2} + \cos 2\theta$

(b) $\sin(2\alpha - \beta)\cos(2\alpha + \beta)$

$= \frac{1}{2}(\sin(2\alpha - \beta + 2\alpha + \beta) + \sin(2\alpha - \beta - 2\alpha - \beta))$

$= \frac{1}{2}(\sin 4\alpha - \sin 2\beta)$

Example 14.

Express the following in the form of a product.

(a) $\sin(30° + \alpha) + \sin(30° - \alpha)$ (b) $\cos(\alpha + 2\beta) - \cos(\alpha - 2\beta)$

Solution

(a) $\sin(30° + \alpha) + \sin(30° - \alpha)$

$= 2\ \sin \frac{(30° + \alpha + 30° - \alpha)}{2} \cos \frac{(30° + \alpha - 30° + \alpha)}{2}$

$= 2\sin 30° \cos\alpha$

$= 2 \times \frac{1}{2}\cos\alpha$

$= \cos\alpha$

(b) $\cos(\alpha + 2\beta) - \cos(\alpha - 2\beta) = -2\ \sin\frac{\alpha + \alpha}{2}\ \sin\frac{2\beta + 2\beta}{2} = -2\sin\alpha\ \sin 2\beta$

Example 15.

Find the value of $\sin 106° - \sin 46° + \cos 104°$.

Solution

$$\begin{aligned}\sin 106° - \sin 46° + \cos 104° &= 2\cos\frac{152°}{2}\sin\frac{60°}{2} + \cos 104° \\ &= 2\cos 76° \sin 30° + \cos(180° - 76°) \\ &= 2 \times \frac{1}{2}\cos 76° - \cos 76° \\ &= 0\end{aligned}$$

Example 16.

Prove that $\sin^2 7x - \sin^2 2x = \sin 9x\ \sin 5x$.

Solution

$$\begin{aligned}
\sin^2 7x - \sin^2 2x &= (\sin 7x + \sin 2x)(\sin 7x - \sin 2x) \\
&= (2\ \sin\frac{7x+2x}{2}\ \cos\frac{7x-2x}{2})(2\ \cos\frac{7x+2x}{2}\ \sin\frac{7x-2x}{2}) \\
&= (2\sin\frac{9x}{2}\cos\frac{5x}{2})(2\cos\frac{9x}{2}\sin\frac{5x}{2}) \\
&= (2\sin\frac{9x}{2}\cos\frac{9x}{2})(2\sin\frac{5x}{2}\cos\frac{5x}{2}) \\
&= \sin 9x\ \sin 5x
\end{aligned}$$

Alternative solution

$$\begin{aligned}
\sin 9x \sin 5x &= -\frac{1}{2}(\cos(9x+5x) - \cos(9x-5x)) \\
&= -\frac{1}{2}(\cos 14x - \cos 4x) \\
&= -\frac{1}{2}(\cos(2\times 7x) - \cos(2\times 2x)) \\
&= -\frac{1}{2}(1 - 2\sin^2 7x - 1 + 2\sin^2 2x) \\
&= -\frac{1}{2}(-2\sin^2 7x + 2\sin^2 2x) \\
&= \sin^2 7x - \sin^2 2x
\end{aligned}$$

Example 17.

If $\alpha+\beta+\gamma = 180°$, prove that $\sin 2\alpha - \sin 2\beta - \sin 2\gamma = -4\sin\alpha\cos\beta\cos\gamma$.

Solution

$$\begin{aligned}
\sin 2\alpha - \sin 2\beta - \sin 2\gamma &= 2\ \cos(\alpha+\beta)\ \sin(\alpha-\beta) - 2\sin\gamma\ \cos\gamma \\
&= 2\ \cos(180° - \gamma)\ \sin(\alpha-\beta) - 2\sin\gamma\cos\gamma \\
&= -2\cos\gamma\ \sin(\alpha-\beta) - 2\sin\gamma\ \cos\gamma \\
&= -2(\ \sin(\alpha-\beta) + \sin\gamma\ )\ \cos\gamma \\
&= -2(2\sin\frac{\alpha-\beta+\gamma}{2}\cos\frac{\alpha-\beta-\gamma}{2})\cos\gamma \\
&= -2\left(2\sin\frac{\alpha+\gamma-\beta}{2}\cos\frac{\alpha-(\beta+\gamma)}{2}\right)\cos\gamma \\
&= -4\sin\frac{180° - 2\beta}{2}\cos\frac{2\alpha - 180°}{2}\cos\gamma \\
&= -4\sin(90° - \beta)\cos(\alpha - 90°)\cos\gamma
\end{aligned}$$

$$= -4\sin(90^\circ - \beta)\cos(90^\circ - \alpha)\cos\gamma$$
$$= -4\cos\beta\sin\alpha\cos\gamma$$

Example 18.

If $\alpha + \beta + \gamma = 180^\circ$, show that

$$\cos\frac{\alpha}{2} + \cos\frac{\beta}{2} + \cos\frac{\gamma}{2} = 4\cos\frac{\beta+\gamma}{4}\cos\frac{\alpha+\gamma}{4}\cos\frac{\alpha+\beta}{4}.$$

Solution

$$4\cos\frac{\beta+\gamma}{4}\cos\frac{\alpha+\gamma}{4}\cos\frac{\alpha+\beta}{4}$$
$$= 2\cos\frac{180^\circ-\alpha}{4}\left(2\cos\frac{\alpha+\gamma}{4}\cos\frac{\alpha+\beta}{4}\right)$$
$$= 2\cos\frac{180^\circ-\alpha}{4}\left(\cos\frac{\alpha+\beta+\gamma+\alpha}{4} + \cos\frac{\gamma-\beta}{4}\right)$$
$$= 2\cos\frac{180^\circ-\alpha}{4}\left(\cos\frac{180^\circ+\alpha}{4} + \cos\frac{\gamma-\beta}{4}\right)$$
$$= 2\cos\frac{180^\circ-\alpha}{4}\cos\frac{180^\circ+\alpha}{4} + 2\cos\frac{180^\circ-\alpha}{4}\cos\frac{\gamma-\beta}{4}$$
$$= \cos\frac{(180^\circ+\alpha+180^\circ-\alpha)}{4} + \cos\frac{(180^\circ+\alpha-180^\circ+\alpha)}{4} + 2\cos\frac{180^\circ-\alpha}{4}\cos\frac{\gamma-\beta}{4}$$
$$= \cos\frac{360^\circ}{4} + \cos\frac{2\alpha}{4} + 2\cos\frac{\beta+\gamma}{4}\cos\frac{\gamma-\beta}{4}$$
$$= \cos 90^\circ + \cos\frac{\alpha}{2} + \cos\frac{2\gamma}{4} + \cos\frac{2\beta}{4}$$
$$= \cos\frac{\alpha}{2} + \cos\frac{\gamma}{2} + \cos\frac{\beta}{2}$$

Exercise 10.3

1. Express the following in the form of a sum.
   (a) $2\sin 4\theta\cos 2\theta$ (b) $2\cos 4\theta\sin 3\theta$ (c) $2\cos 6\theta\cos 4\theta$ (d) $2\sin 4\theta\sin 10\theta$
2. Express the following in the form of a product.
   (a) $\sin 6\theta + \sin 2\theta$ (b) $\sin 7\theta - \sin 3\theta$ (c) $\cos 5\theta + \cos 3\theta$ (d) $\cos 11\theta - \cos 5\theta$
3. Find the value of $\cos 40^\circ + \cos 80^\circ + \cos 160^\circ$.
4. Prove that $\sin^2 9x - \sin^2 3x = \sin 12x\cos 6x$.
5. If $\alpha + \beta + \gamma = 180^\circ$, prove that
   $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$.

6. If $\alpha + \beta + \gamma = 180°$, prove that
$\tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} + \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = 1.$

7. If $\alpha + \beta + \gamma = 180°$, prove that
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 1.$

8. If $\alpha + \beta + \gamma = 180°$, prove that
$\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} = 1 + 4\sin\frac{180° - \alpha}{4}\sin\frac{180° - \beta}{4}\sin\frac{180° - \gamma}{4}.$

10.6 The Law of Cosines and The Law of Sines

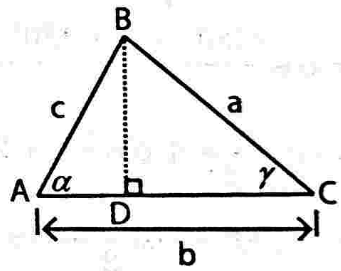
There are some important relationships between the parts of a triangle. We will study two of these, namely, the law of cosines and the law of sines.

The Law of Cosines

If $\triangle ABC$ is an arbitrary triangle with angles α, β, γ and corresponding opposite sides a, b, c respectively, then

$$\boxed{\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ b^2 &= c^2 + a^2 - 2ca\cos\beta \\ c^2 &= a^2 + b^2 - 2ab\cos\gamma \end{aligned}}$$

Proof



Case(i) $\angle A$ is acute angle.

In right $\triangle ADB$, $\frac{AD}{AB} = \cos\alpha$

$AD = AB\cos\alpha = c\cos\alpha$

In right $\triangle ADB$, $BD^2 = AB^2 - AD^2$

In right $\triangle BDC$, $BD^2 = BC^2 - DC^2$

Therefore,

$$BC^2 - DC^2 = AB^2 - AD^2$$

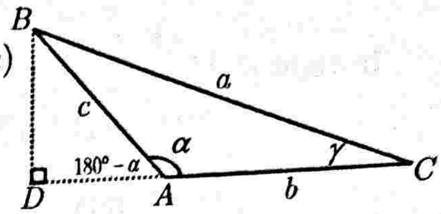
$$\begin{aligned} \therefore\ BC^2 &= DC^2 + AB^2 - AD^2 \\ &= (AC - AD)^2 + AB^2 - AD^2 \\ &= AC^2 - 2AC\,.\,AD + AD^2 + AB^2 - AD^2 \\ &= AC^2 + AB^2 - 2AC\,.\,AD \\ a^2 &= b^2 + c^2 - 2bc\cos\alpha \end{aligned}$$

Similarly, we can prove that

$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \text{and} \quad c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Case(ii) $\angle A$ is obtuse angle.

$$\begin{aligned}
\text{In right } \triangle ADB, \frac{AD}{AB} &= \cos(180^\circ - \alpha)\\
AD &= AB\cos(180^\circ - \alpha)\\
&= -c\cos\alpha\\
\text{In right } \triangle ADB, BD^2 &= AB^2 - AD^2\\
\text{In right } \triangle BDC, BD^2 &= BC^2 - DC^2
\end{aligned}$$



Therefore,

$$\begin{aligned}
BC^2 - DC^2 &= AB^2 - AD^2\\
BC^2 &= DC^2 + AB^2 - AD^2\\
&= (AC + AD)^2 + AB^2 - AD^2\\
&= AC^2 + 2AC\,.\,AD + AD^2 + AB^2 - AD^2\\
&= AC^2 + AB^2 + 2AC\,.\,AD\\
a^2 &= b^2 + c^2 + 2b(-c\cos\alpha)\\
a^2 &= b^2 + c^2 - 2bc\cos\alpha
\end{aligned}$$

Similarly, we can prove that

$b^2 = c^2 + a^2 - 2ca\cos\beta$ and $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

The Law of Cosines is used when we are given either
(1) two sides of a triangle and the included angle or
(2) three sides of a triangle.

The Law of Sines

If $\triangle ABC$ is an arbitrary triangle with angles α, β, γ and corresponding opposite sides a, b, c respectively, then

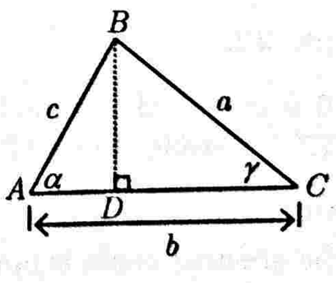
$$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.}$$

Equivalently, $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}.$

Proof

Case(i) $\angle A$ is acute angle.

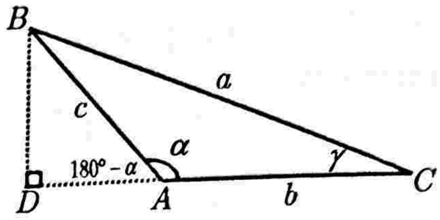
$$\begin{aligned}
\text{In right } \triangle ADB, \frac{BD}{AB} &= \sin\alpha\\
BD &= AB\sin\alpha = c\sin\alpha\\
\text{In right } \triangle BDC, \frac{BD}{BC} &= \sin\gamma\\
BD &= BC\sin\gamma = a\sin\gamma\\
\text{Therefore, } a\sin\gamma &= c\sin\alpha\\
\frac{a}{\sin\alpha} &= \frac{c}{\sin\gamma}.
\end{aligned}$$



Similarly, we can prove that $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$.

Case(ii) $\angle A$ is obtuse angle.

$$\begin{aligned}
\text{In right } \triangle ADB, \frac{BD}{AB} &= \sin(180^\circ - \alpha)\\
BD &= AB\sin(180^\circ - \alpha)\\
&= c\sin\alpha\\
\text{In right } \triangle BDC, \frac{BD}{BC} &= \sin\gamma\\
BD &= BC\sin\gamma = a\sin\gamma\\
\text{Therefore, } a\sin\gamma &= c\sin\alpha\\
\frac{a}{\sin\alpha} &= \frac{c}{\sin\gamma}.
\end{aligned}$$



Similarly, we can prove that $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$.

Hence, $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$.

The Law of Sines is used to solve the triangle when we are given either (1) two angles and one side or

(2) two sides and an angle opposite to one side.

Example 19.

If $a = 7, b = 5$ and $c = 8$, find the values of $\angle A$ and $\angle B$.
Given that $\cos 38^\circ 44' = 0.7857$, $\sin 38^\circ 44' = 0.6257$.

Solution

By the law of cosines,

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{5^2 + 8^2 - 7^2}{2 \times 5 \times 8} = \frac{40}{80} = \frac{1}{2} = \cos 60^\circ$$

$\therefore \angle A = 60^\circ$.

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac} = \frac{8^2 + 7^2 - 5^2}{2 \times 8 \times 7} = \frac{88}{2 \times 8 \times 7} = \frac{11}{14} = \cos 38^\circ 44'$$

$\therefore \angle B = 38^\circ 44'$.

Example 20.

If $a = 6, b = 13$ and $c = 11$, find the greatest angle of the triangle ABC. Given that $\cos 84^\circ 47' = 0.0909$, $\sin 84^\circ 47' = 0.9959$.

Solution

Since the greatest angle is opposite to the greatest side, the required angle is B.

By the law of cosines,

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$\therefore \cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{11^2+6^2-13^2}{2\times 11\times 6} = -\frac{12}{2\times 11\times 6} = -\frac{1}{11}$$

$$\therefore \cos B = -0.0909 = -\cos 84°47' = \cos(180° - 84°47') = \cos 95°13'$$

$$\therefore \angle B = 95°13'.$$

Example 21.

Solve the triangle ABC with $\angle A = 78°32', \angle C = 71°28'$ and $b = 4$.

($\sin 78°32' = 0.9800$, $\sin 71°28' = 0.9481$, $\cos 78°32' = 0.2000, \cos 71°28' = 0.3179$)

Solution

$\angle B = 180° - (\angle A + \angle C) = 180° - (78°32' + 71°28') = 30°$.

By the law of sines,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\therefore a = \frac{b\sin A}{\sin B} = \frac{4\sin 78°32'}{\sin 30°} = \frac{4\times 0.9800}{\frac{1}{2}} = 8\times 0.9800 = 7.8400$$

Again, $\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$

$$\therefore c = \frac{b\sin C}{\sin B} = \frac{4\sin 71°28'}{\sin 30°} = \frac{4\times 0.9481}{\frac{1}{2}} = 8\times 0.9481 = 7.5848.$$

Example 22.

$\triangle ABC$ is an arbitrary triangle with angles α, β, γ and corresponding opposite sides a, b, c respectively, then prove that

$$a = b\cos\gamma + c\cos\beta.$$

Solution

Since $\alpha + \beta + \gamma = 180°$, we have $\alpha = 180° - (\beta + \gamma)$.

$$\therefore \sin\alpha = \sin(180° - (\beta+\gamma))$$

$$\therefore \sin\alpha = \sin(\beta+\gamma)$$

$$\therefore \sin\alpha = \sin\beta\cos\gamma + \cos\beta\sin\gamma$$

$$\therefore 1 = \frac{\sin\beta}{\sin\alpha}\cos\gamma + \frac{\sin\gamma}{\sin\alpha}\cos\beta \cdots\cdots (1)$$

By the law of sines,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Thus $\dfrac{\sin\beta}{\sin\alpha} = \dfrac{b}{a}$ and $\dfrac{\sin\gamma}{\sin\alpha} = \dfrac{c}{a}$.

Substituting these values in Equation (1), we get

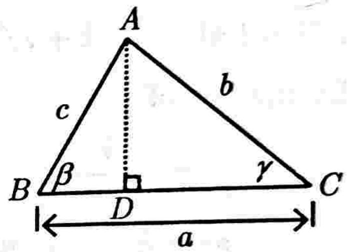
$$\begin{aligned} 1 &= \frac{b}{a}\cos\gamma + \frac{c}{a}\cos\beta \\ \therefore a &= b\cos\gamma + c\cos\beta. \end{aligned}$$

Alternative solution

There are two cases.

Case(i) $\angle B$ is acute angle.

Draw $AD \perp BC$.



In right $\triangle ADB$,

$$\begin{aligned} \frac{BD}{AB} &= \cos\beta \\ BD &= AB\cos\beta = c\cos\beta \end{aligned}$$

In right $\triangle ADC$,

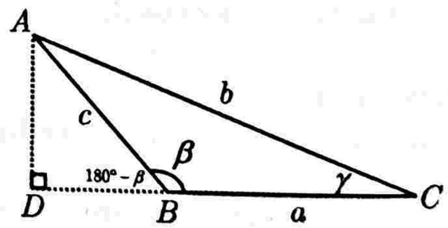
$$\begin{aligned} \frac{DC}{AC} &= \cos\gamma \\ DC &= AC\cos\gamma = b\cos\gamma \end{aligned}$$

$\therefore a = BC = BD + DC = c\cos\beta + b\cos\gamma$.

Case(ii) $\angle B$ is obtuse angle.

Draw $AD \perp CB$ produced.

In right $\triangle ADC$,



$$\begin{aligned} \frac{DC}{AC} &= \cos\gamma \\ DC &= AC\cos\gamma = b\cos\gamma \end{aligned}$$

In right $\triangle ADB$,

$$\begin{aligned} \frac{DB}{AB} &= \cos(180^\circ - \beta) \\ DB &= AB\cos(180^\circ - \beta) = c\cos(180^\circ - \beta) \\ \therefore a = BC = DC - DB &= b\cos\gamma - c\cos(180^\circ - \beta) = b\cos\gamma + c\cos\beta. \end{aligned}$$

Example 23.

Prove that $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$.

Solution

Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$.($k$= constant)

Then $a = k\sin A$, $b = k\sin B$ and $c = k\sin C$.

$\therefore (b+c)\cos A + (c+a)\cos B + (a+b)\cos C$

$= (k\sin B + k\sin C)\cos A + (k\sin C + k\sin A)\cos B + (k\sin A + k\sin B)\cos C$

$= k(\sin B\cos A + \sin C\cos A + \sin C\cos B + \sin A\cos B + \sin A\cos C + \sin B\cos C)$

$= k(\sin(A+B) + \sin(A+C) + \sin(B+C))$

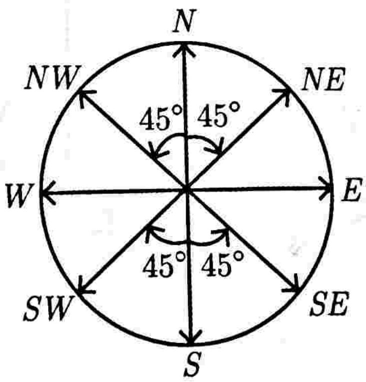
$= k(\sin(180° - C) + \sin(180° - B) + \sin(180° - A))$

$= k\sin C + k\sin B + k\sin A$

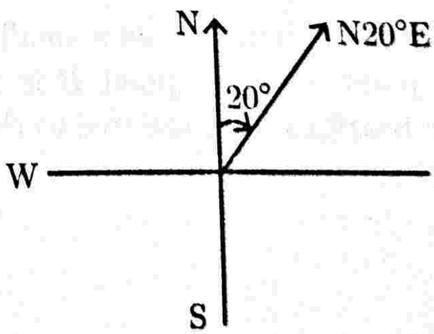
$= c+b+a$

10.7 Bearings

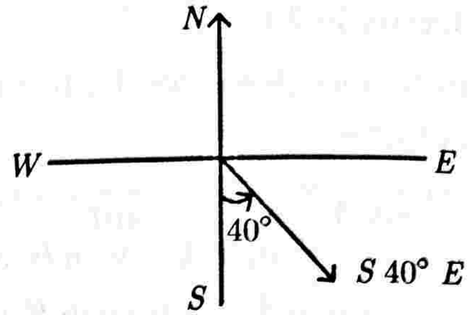
The four cardinal directions are North (N), South(S), East(E) and West(W). In the figure shown below, Northeast(NE), Northwest(NW), Southeast(SE) and Southwest(SW) are the inter-cardinal directions.



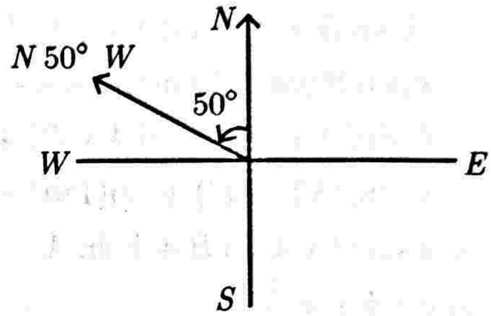
A bearing of $N\,20°\,E$ means an angle of 20° measured from the N towards E.



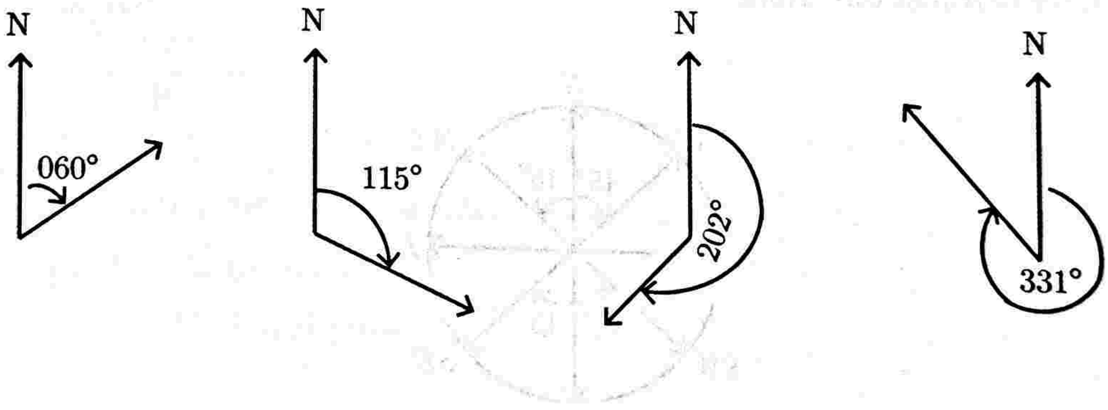
A bearing of $S\,40^\circ\,E$ means an angle of 40° measured from the S towards E.



A bearing of $N\,50^\circ\,W$ means an angle of 50° measured from the N towards W.



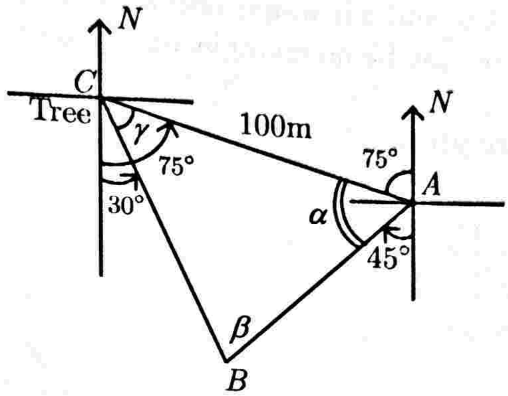
Bearing quoted in the above way is always measured from N and S, but not from E and W. There is a second way of stating a bearing in which the angle is measured from N in the clockwise direction, with N being reckoned as 0°. Three figures are always stated, for example 005° is written instead of 5°, 035° for 35° etc. East will be 90°, South will be 180° and West will be 270° are shown in the figure.



Example 24.

A bridge is built across a small lake from a point A to a point B. The bearing from the point A to the point B is $S45^\circ W$. There is a tree 100 metres from the point A. The bearings from the tree to the points A and B are $S75^\circ\,E$ and $S30^\circ\,E$ respectively. Find the distance from the point A to the point B.

Solution



Let AB be the distance from the point A to the point B.

$\gamma = 75° - 30° = 45°$

$$\alpha = 180° - (45° + 75°) = 60°$$

$$\beta = 180° - (45° + 60°) = 75°$$

By the law of sines,

$$\frac{AB}{\sin\gamma} = \frac{AC}{\sin\beta}$$

$$\therefore\ AB = \frac{AC\sin\gamma}{\sin\beta}$$

$$= \frac{100 \times \sin 45°}{\sin 75°}$$

$$= \frac{\frac{100 \times \sqrt{2}}{2}}{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

$$= \frac{200\sqrt{2}}{\sqrt{6}+\sqrt{2}} \times \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}$$

$$= \frac{200\sqrt{2}(\sqrt{6}-\sqrt{2})}{4}$$

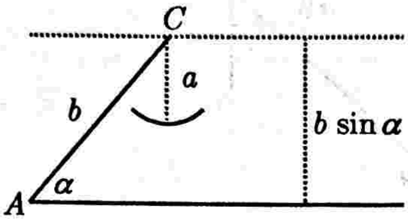
$$= 100(\sqrt{3}-1) \text{ metres.}$$

Ambiguous Case

The case in which two sides and an angle opposite to one are given is called the ambigious case because there may be no triangle, one triangle or two triangles satisfying the given conditions.

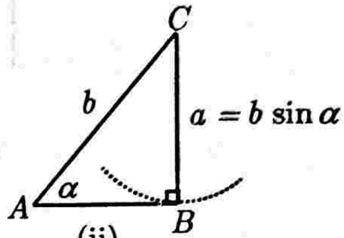
Suppose that a, b, and α are given.

Case (i) $\alpha < 90°$



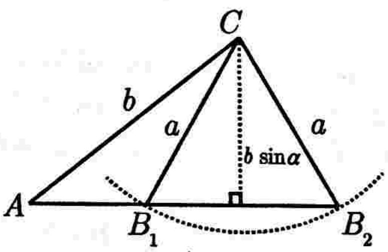
(i)

No solution : $a < b\sin\alpha$



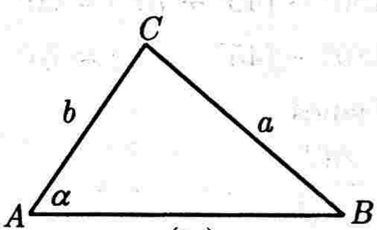
(ii)

One solution : $a = b\sin\alpha$



(iii)

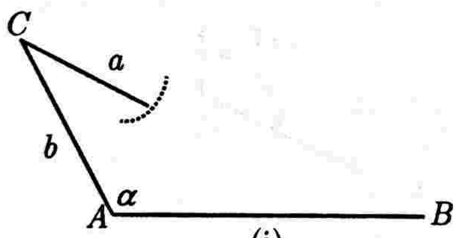
Two solutions : $b\sin\alpha < a < b$



(iv)

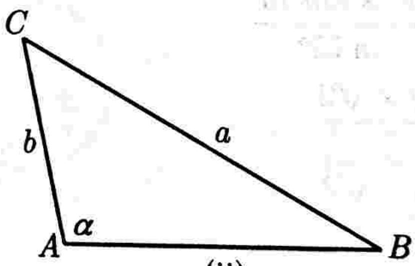
One solution : $a > b$

Case (ii) $\alpha \geq 90°$



(i)

No solution : $a \leq b$



(ii)

One solution : $a > b$

Example 25.

Find the number of solutions for each triangle.

(a) $a = 6$, $b = 12$ and $\alpha = 30°$

(b) $a = 10$, $b = 12$ and $\alpha = 45°$

(c) $a = 4$, $b = 12$ and $\alpha = 60°$

Solution

(a) $b\sin\alpha = 12\sin 30° = 12 \times \dfrac{1}{2} = 6$

$\therefore a = b\sin\alpha$

$\therefore$ One solution.

(b) $b\sin\alpha = 12\sin 45° = 12 \times \dfrac{\sqrt{2}}{2} = 6\sqrt{2}$

$\therefore b\sin\alpha < a < b$

$\therefore$ Two solutions.

(c) $b\sin\alpha = 12\sin 60° = 12 \times \dfrac{\sqrt{3}}{2} = 6\sqrt{3}$

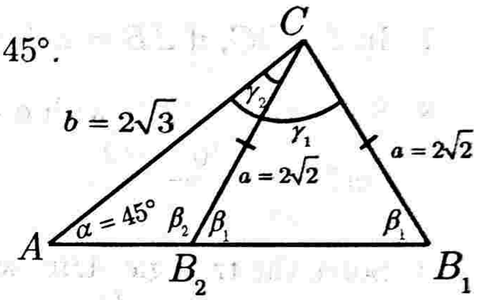
$\therefore a < b\sin\alpha$

$\therefore$ No solution.

When two sides and an angle opposite to one of them are given, it is important to sketch and label the triangle first. Then determine the number of possible solutions.

Example 26.

Solve the $\triangle ABC$ with $a = 2\sqrt{2}, b = 2\sqrt{3}$ and $\alpha = 45°$.



Solution

$b\sin\alpha = 2\sqrt{3}\sin 45° = 2\sqrt{3} \times \dfrac{\sqrt{2}}{2} = \sqrt{6}.$

$\therefore b\sin\alpha < a < b. \qquad (a = 2\sqrt{2} = \sqrt{8})$

There are two solutions.

In $\triangle AB_1C$,

$$\frac{\sin\beta_1}{b} = \frac{\sin\alpha}{a}$$

$$\therefore \sin\beta_1 = \frac{b\sin\alpha}{a} = \frac{2\sqrt{3}\sin 45°}{2\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{3}}{2} = \sin 60°$$

$\therefore \beta_1 = 60°$

$\gamma_1 = 180° - (\alpha + \beta_1) = 180° - (45° + 60°) = 75°$

Again, $\dfrac{AB_1}{\sin 75°} = \dfrac{2\sqrt{2}}{\sin 45°}$

$$\therefore AB_1 = \frac{2\sqrt{2}\sin 75°}{\sin 45°} = \frac{2\sqrt{2}}{\frac{\sqrt{2}}{2}} \times \frac{(\sqrt{6}+\sqrt{2})}{4} = \sqrt{6}+\sqrt{2}$$

In $\triangle AB_2C$,

$\beta_2 = 180° - \beta_1 = 180° - 60° = 120°$

$\gamma_2 = 180° - (\alpha + \beta_2) = 180° - (45° + 120°) = 15°$

$$\frac{AB_2}{\sin 15°} = \frac{2\sqrt{2}}{\sin 45°}$$

$$\therefore AB_2 = \frac{2\sqrt{2}\sin 15°}{\sin 45°} = \frac{2\sqrt{2}}{\frac{\sqrt{2}}{2}} \times \frac{(\sqrt{6}-\sqrt{2})}{4} = \sqrt{6}-\sqrt{2}$$

Exercise 10.4

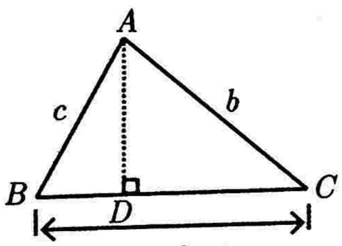
1. Find a if $b = 4, c = 11$ and $\alpha = 60°$.
2. Find b if $a = 20, c = 8$ and $\beta = 45°$.
3. Find c if $\gamma = 30°, \alpha = 45°$ and $a = 100$.
4. Find a and c if $\alpha = 30°, \beta = 120°$ and $b = 45$.
5. Find γ if $a = 12, b = 5$ and $c = 13$.
6. In $\triangle ABC$, if $\angle A = 60°$, $a = \sqrt{3}$ and $c = 2$, find b.
7. In $\triangle ABC$, if $\angle B = \angle A + 15°, \angle C = \angle B + 15°$ and $BC = 6$, find AC.
8. Solve the $\triangle ABC$ with $a = 2, c = \sqrt{6}$ and $\gamma = 60°$, having given that $\sin 75° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.
9. Solve the triangle ABC with $a = 2\sqrt{6}, c = 6 - 2\sqrt{3}, \angle B = 75°$, having given that $\cos 75° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$.
10. Solve the triangle ABC with $\angle B = 60°$, $\angle C = 15°$ and $b = \sqrt{8}$, having given that $\sin 105° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$, and $\sin 15° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$.
11. Solve the $\triangle ABC$ with $\angle A = 105°, \angle C = 30°, b = 4$, having given that $\sin 105° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.
12. The sides of a triangle are $2, 2\frac{2}{3}, 3\frac{1}{3}$, find the greatest angle.
13. Given $\angle A = 45°, \angle B = 60°$, show that $AB : BC = \sqrt{3}+1 : 2$, having given that $\sin 75° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.
14. If $\angle C = 60°, b = 2\sqrt{3}, c = 3\sqrt{2}$, find $\angle A$.
15. Solve the $\triangle ABC$ with $a = \dfrac{1}{\sqrt{6}-\sqrt{2}}, b = \dfrac{1}{\sqrt{6}+\sqrt{2}}, \angle C = 60°$.
16. Prove that $(b - c)\cos\dfrac{A}{2} = a\sin\dfrac{B-C}{2}$.
17. Prove that $a\sin(B - C) + b\sin(C - A) = c\sin(B - A)$.

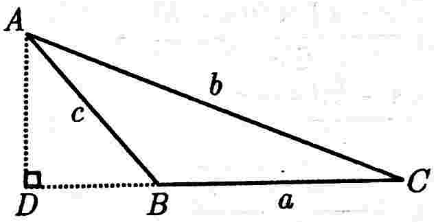
18. A and B are two points on one bank of a straight river, distant from one another at 936 m. C is on the other bank and the measures of the angle CAB, angle CBA respectively $75°$ and $45°$. Find the width of the river, having given that $\sin 75° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

19. In $\triangle ABC$, $AB = x+1$, $BC = x+3$ and $AC = x-1$ where $x > 3$. Prove that $\cos A = \dfrac{7-x}{2(1-x)}$. Find also the integral values of x for which $\angle A$ is acute.

20. If α, β, γ are the angles of a triangle and $\tan\ \alpha = 2$ and $\tan\ \beta = 3$, prove that $\tan\ \gamma = 1$. If a, b, c are the corresponding sides of the triangle, prove that $\dfrac{a}{2\sqrt{2}} = \dfrac{b}{3} = \dfrac{c}{\sqrt{5}}$.

10.8 The Area of a Triangle





Consider $\triangle ABC$. Draw $AD \perp BC$. Then the area of $\triangle ABC$ is

$$\alpha(\triangle ABC) = \frac{1}{2} \times BC \times AD = \frac{1}{2} \times BC \times AB \sin B = \frac{1}{2}\, ac\ \sin B.$$

Similarly, we can prove that

$$\alpha(\triangle ABC) = \frac{1}{2}\, ab\ \sin C,$$

and $$\alpha(\triangle ABC) = \frac{1}{2}\, bc\ \sin A.$$

Generally,

$$\textit{Area of a triangle} = \frac{1}{2}(\textit{product of lengths of two sides}) \times (\textit{sine of included angle})$$

Theorem 10.1. (Heron's Theorem)

Let the lengths of the sides of a triangle be a, b and c. Then the area of the triangle is given by

$$\textit{Area} = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

where s is the *semi-perimeter*, that is, $s = \dfrac{a+b+c}{2}$.

Proof

Let ABC be a triangle with angles α, β, γ and corresponding opposite sides a, b, c respectively. Then the area of $\triangle ABC$ is

$$\alpha(\triangle ABC) = \frac{1}{2}bc\sin\alpha = \frac{1}{2}bc(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}) = bc\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

Since $2\sin^2\frac{\alpha}{2} = 1-\cos\alpha$, we have

$$2\sin^2\frac{\alpha}{2} = 1 - \frac{b^2+c^2-a^2}{2bc} \quad (\because a^2 = b^2+c^2-2bc\cos\alpha)$$

$$= \frac{a^2-(b^2+c^2-2bc)}{2bc}$$

$$= \frac{a^2-(b-c)^2}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc}$$

Since $a+b+c = 2s, \quad a+b-c = a+b+c-2c = 2s-2c = 2(s-c)$

and $a-b+c = a+b+c-2b = 2s-2b = 2(s-b)$

$$\therefore\ 2\sin^2\frac{\alpha}{2} = \frac{2(s-c)\times 2(s-b)}{2bc} = \frac{2(s-b)(s-c)}{bc}$$

$$\therefore\ \sin^2\frac{\alpha}{2} = \frac{(s-b)(s-c)}{bc}$$

$$\therefore\ \sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

Since $2\cos^2\frac{\alpha}{2} = 1+\cos\alpha$, then $2\cos^2\frac{\alpha}{2} = 1 + \frac{b^2+c^2-a^2}{2bc}$

$$= \frac{2bc+b^2+c^2-a^2}{2bc}$$

$$= \frac{(b+c)^2-a^2}{2bc}$$

$$= \frac{(b+c-a)(b+c+a)}{2bc}$$

$$= \frac{(a+b+c-2a)(a+b+c)}{2bc}$$

$$= \frac{(2s-2a)2s}{2bc} = \frac{2s(s-a)}{bc}$$

$$\therefore\ \cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

So, the area of $\triangle ABC = bc\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$$= bc\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Example 27.

Prove that the area of an equilateral triangle with sides of length a is $\frac{a^2\sqrt{3}}{4}$.

Solution

The semi-perimeter is $s = \frac{a+a+a}{2} = \frac{3a}{2}$.

$$\begin{aligned}\text{Area of equilateral triangle} &= \sqrt{s(s-a)(s-b)(s-c)}\\ &= \sqrt{\frac{3a}{2}(\frac{3a}{2}-a)(\frac{3a}{2}-a)(\frac{3a}{2}-a)}\\ &= \sqrt{\frac{3a}{2}\times\frac{a}{2}\times\frac{a}{2}\times\frac{a}{2}} = \frac{a^2}{4}\sqrt{3}\text{ sq.units.}\end{aligned}$$

Example 28.

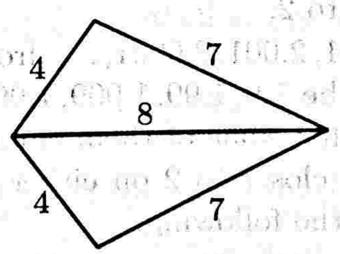
Find the area of a triangle in which two sides are 30 cm and 12 cm, and the included angle is 30°.

Solution

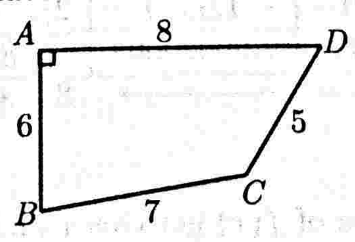
The area of a triangle $= \frac{1}{2}\times 30\times 12\times\sin 30^\circ = \frac{1}{2}\times 30\times 12\times\frac{1}{2} = 90\text{ cm}^2$.

Exercise 10.5

1. In $\triangle ABC, a = 10, b = 20$ and $\angle C = 45^\circ$, find the area of $\triangle ABC$.
2. In $\triangle PQR, p = 12, q = 6$ and $\angle R = 60^\circ$, find the area of $\triangle PQR$.
3. Find the area of the triangle if the sides have lengths:
   (a) 5, 6, 7 (b) 5, 12, 15 (c) 10, 17, 21.
4. The sides of an isosceles triangle measure 7, 7 and 10. Find the length of the altitude to one of the congruent sides.
5. Find the area of the kite.



6. Find the area of the quadrilateral $ABCD$.



Chapter 11

Methods of Differentiation

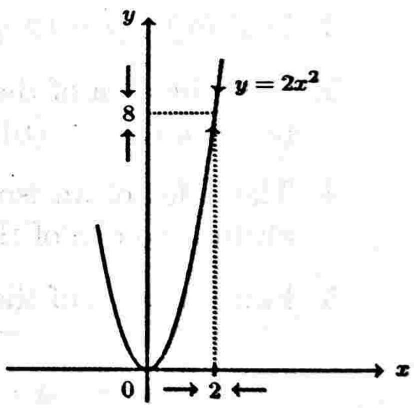
In this chapter we shall consider real-valued functions whose domains are arbitrary subsets of $\mathbb{R}$. We first discuss the fundamental notion of limit of a function and the concept of derivatives. We next introduce sum rule, difference rule, product rule, quotient rule and chain rule that will be used to calculate derivatives more efficiently. Finally, we shall explain the method of differentiation for implicit functions.

11.1 Limit of Functions

The basic idea of limit is to describe the behaviour of function when the independent variable approaches a given value. For example, let us consider the behaviour of the function

$$f(x) = 2x^2$$

for x-values closer and closer to 2.



Here x would be $2.1, 2.01, 2.001, 2.0001, \ldots$ from the right of 2 as well as x would be $1.9, 1.99, 1.999, 1.9999, \ldots$ from the left of 2. It is seen that none of them equal to 2, but x are selected closer and closer to 2 on either left or right side of 2. Then we find the following:

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$f(x)$	7.22	7.9202	7.992002	7.99920002		8.00080002	8.008002	8.0802	8.82

Left side → 8 ← Right side

It turns out that the values of $f(x)$ get closer and closer to 8 from either side. We write

$$\lim_{x \to 2} 2x^2 = 8 \quad \text{or} \quad 2x^2 \to 8 \quad \text{as} \quad x \to 2.$$

If the values of $f(x)$ get closer and closer to a by taking the values of x sufficiently close to c (but not equal to c), then we write

$$\lim_{x \to c} f(x) = a$$

which is read as " the limit of $f(x)$ is a as x approaches c or a is a limit of f". We can also write

$$f(x) \to a \quad \text{as} \quad x \to c.$$

Notice the phrase "**the value of x sufficiently close to c (but not equal to c)**" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches c, we never consider $x = c$.

Limit of a Constant Function

For the constant function $f(x) = a$,

$$\lim_{x \to c} f(x) = \lim_{x \to c} a = a, \qquad \text{where } c \text{ is a real number.}$$

Limit of Identity Function

For the identity function $f(x) = x$,

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c, \qquad \text{where } c \text{ is a real number.}$$

It is easily seen that

$$\lim_{x \to 2} 5 = 5, \qquad \lim_{x \to 0}(-2) = -2, \qquad \lim_{x \to 1} x = 1, \qquad \lim_{x \to -2} x = -2.$$

The Limit Laws

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist and c and k are real number, then

1. Sum Rule: $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$.

2. Constant Multiple Rule: $\lim_{x \to c} [k \cdot f(x)] = k \lim_{x \to c} f(x)$

3. Product Rule: $\lim_{x \to c} [f(x) \cdot g(x)] = [\lim_{x \to c} f(x)] [\lim_{x \to c} g(x)]$

4. Quotient Rule: $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}, \quad \lim_{x \to c} g(x) \neq 0.$

5. Power Rule: $\lim_{x \to c}[f(x)]^n = [\lim_{x \to c} f(x)]^n$, n is a positive integer

6. Root Rule: $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$.

(If n is even, we assume that $\lim_{x \to c} f(x) \geq 0$.)

Example 1.

Find (a) $\lim_{x\to 3}(x+6)$ (b) $\lim_{x\to -1}(2-x)$ (c) $\lim_{x\to 3}(6x)$ (d) $\lim_{x\to -1} x(x-1)$.

Solution

(a) $\lim_{x\to 3}(x+6) = \lim_{x\to 3} x + \lim_{x\to 3} 6 = 3+6 = 9$

(b) $\lim_{x\to -1}(2-x) = \lim_{x\to -1} 2 - \lim_{x\to -1} x = 2-(-1) = 3$

(c) $\lim_{x\to 3}(6x) = [\lim_{x\to 3} 6][\lim_{x\to 3} x] = 6\cdot 3 = 18$

(d) $\lim_{x\to -1} x(x-1) = [\lim_{x\to -1} x][\lim_{x\to -1}(x-1)] = (-1)(-2) = 2$

Example 2.

Find $\lim_{x\to 2}\dfrac{4x^3+3x-1}{x+1}$.

Solution

$$\lim_{x\to 2}\frac{4x^3+3x-1}{x+1} = \frac{\lim_{x\to 2}(4x^3+3x-3)}{\lim_{x\to 2}(x+1)} = \frac{35}{3}.$$

Example 3.

Find (a) $\lim_{x\to 2}(2x-3)^3$ (b) $\lim_{x\to 0}\sqrt{7x^2+6}$ (c) $\lim_{x\to -1}(x^3-2x+1)^{4/3}$.

Solution

(a) $\lim_{x\to 2}(2x-3)^3 = [\lim_{x\to 2}(2x-3)]^3 = 1^3 = 1$

(b) $\lim_{x\to 0}\sqrt{7x^2+6} = \sqrt{\lim_{x\to 0}(7x^2+6)} = \sqrt{6}$

(c) $\lim_{x\to -1}(x^3-2x+1)^{4/3} = \sqrt[3]{\lim_{x\to -1}(x^3-2x+1)^4}$

$= \sqrt[3]{[\lim_{x\to -1}(x^3-2x+1)]^4} = \sqrt[3]{2^4} = 2\sqrt[3]{2}$

Limit of a Polynomial Function

If P is a polynomial function, then

$$\lim_{x\to c} P(x) = P(c), \quad \text{where } c \text{ is any number.}$$

Example 4.

Find $\lim_{x\to 2}(5x^3-3x^2+2x-1)$.

Solution

$$\lim_{x\to 2}(5x^3-3x^2+2x-1) = 5\cdot 2^3 - 3\cdot 2^2 + 2\cdot 2 - 1 = 40-12+4-1 = 31$$

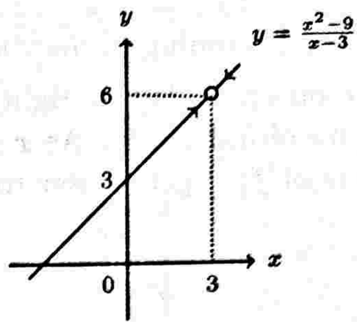
When we consider the rational function's behaviour near a particular point x_0 that leads to division by zero, the quotient rule in the Limit Laws cannot be used.

Let us consider the function $f(x) = \dfrac{x^2 - 9}{x - 3}$.

This function is defined for all real numbers x except $x = 3$.

That is,

$$f(x) = \frac{x^2 - 9}{x - 3}, \quad x \neq 3.$$



We can simplify that

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3 \quad \text{for} \quad x \neq 3.$$

The graph of f is the line $y = x + 3$ with the point $(3, 6)$ removed which is shown as a "hole". Even though $f(3)$ is not defined, we can find the value of $f(x)$ as close as 6 by choosing x close enough to 3.

Thus
$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3}(x + 3) = 6.$$

Example 5.

Find (a) $\displaystyle\lim_{x \to 1} \frac{x^3 - x^2 - 4x + 4}{x - 1}$ (b) $\displaystyle\lim_{x \to 0} \frac{x^2 - 3x}{x}$ (c) $\displaystyle\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}$.

Solution

(a)
$$\begin{aligned}\lim_{x \to 1} \frac{x^3 - x^2 - 4x + 4}{x - 1} &= \lim_{x \to 1} \frac{x^2(x - 1) - 4(x - 1)}{x - 1} \\ &= \lim_{x \to 1} \frac{(x - 1)(x^2 - 4)}{x - 1} \\ &= \lim_{x \to 1}(x^2 - 4) = -3.\end{aligned}$$

(b)
$$\lim_{x \to 0} \frac{x^2 - 3x}{x} = \lim_{x \to 0} \frac{x(x - 3)}{x} = \lim_{x \to 0} (x - 3) = -3.$$

(c)
$$\begin{aligned}\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} &= \lim_{x \to 0} \left[\frac{\sqrt{x + 1} - 1}{x} \times \frac{\sqrt{x + 1} + 1}{\sqrt{x + 1} + 1}\right] \\ &= \lim_{x \to 0} \frac{1}{(\sqrt{x + 1} + 1)} = \frac{1}{2}.\end{aligned}$$

Exercise 11.1

1. Find the limits of the following:

(a) $\displaystyle\lim_{x \to 2} (x^3 - 3x)$ (b) $\displaystyle\lim_{x \to 4} (\sqrt{x^2 - x} - x)$ (c) $\displaystyle\lim_{x \to 5} \frac{x^2 - 2x}{x - 2}$

(d) $\displaystyle\lim_{x \to -3} (2x^2 - 5x + 1)$ (e) $\displaystyle\lim_{x \to 1} \frac{x^2 - x}{x - 1}$ (f) $\displaystyle\lim_{x \to 0} \sqrt{4x + 9}$

(g) $\displaystyle\lim_{x \to 0} \frac{1}{x}\left(\frac{4}{x + 4} - 1\right)$ (h) $\displaystyle\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1}$ (i) $\displaystyle\lim_{x \to 9} \frac{9x - x^2}{3 - \sqrt{x}}$

One-Sided Limits of a Function

At the beginning of the chapter, we describe $\lim_{x\to c} f(x) = a$ by saying that x gets closer to c on either left or right side of c. If we only approaches c from one side, we have **a one-sided limit**. As x gets closer to c, but remains less than c, the corresponding value of $f(x)$ gets closer to L. The notation

$$\lim_{x\to c^-} f(x) = L$$

is called the **left-hand limit**.

As x gets closer to c, but remains greater than c, the corresponding value of $f(x)$ gets closer to R. The notation

$$\lim_{x\to c^+} f(x) = R$$

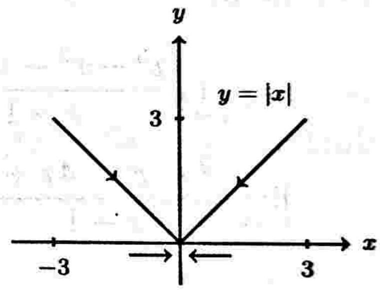
is called the **right-hand limit**.

Example 6.

Find $\lim_{x\to 0^+} |x|$ and $\lim_{x\to 0^-} |x|$.

Solution

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$



If $x > 0$, then $|x| = x$ and x is getting close to 0,

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0.$$

If $x < 0$, then $|x| = -x$ and x is getting close to 0,

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0.$$

Note: In this example we see that $\lim_{x\to 0^+} |x| = \lim_{x\to 0^-} |x|$. Thus $|x|$ has a limit 0 as x approaches 0.

A function $f(x)$ has a limit as x approaches c if and only if left-hand and right-hand limits exist and have the same value:

$$\lim_{x\to c} f(x) = a \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = a.$$

However, some functions show different behaviours on the two sides of an x-value. We often see the following case

$$\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x).$$

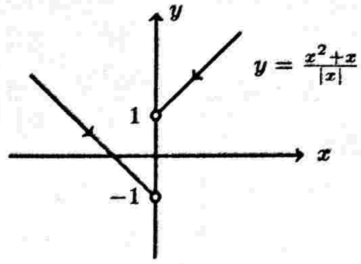
In that case we say that $f(x)$ has **no limit** when x closes to c.

Example 7.

Find the limit of $f(x) = \dfrac{x^2+x}{|x|}$ when $x \to 0$ if the limit of $f(x)$ exists.

Solution

$$f(x) = \frac{x^2+x}{|x|}, \qquad x \neq 0.$$



When $x > 0$, $\quad f(x) = \dfrac{x^2+x}{x} = x+1.$

When $x < 0$, $\quad f(x) = \dfrac{x^2+x}{-x} = -x-1.$

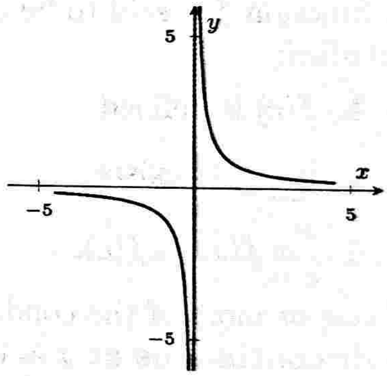
We observe that when $x > 0$, $\displaystyle\lim_{x\to 0+} \frac{x^2+x}{|x|} = \lim_{x\to 0+}(x+1) = 1$ and

when $x < 0$, $\displaystyle\lim_{x\to 0-} \frac{x^2+x}{|x|} = \lim_{x\to 0-}(-x-1) = -1.$

Hence
$$\lim_{x\to 0+} \frac{x^2+x}{|x|} \neq \lim_{x\to 0-} \frac{x^2+x}{|x|}.$$

Therefore $f(x)$ has no limit when x closes to 0.

Infinite Limits



We now consider the behaviour of function $f(x) = \dfrac{1}{x}$ for values of x near 0. In the graph, the x-values are taken closer and closer to 0 from the right side, the values of $f(x)$ are positive and increase larger and larger. Similarly the x-values are taken closer and closer to 0 from the left side, the values of $f(x)$ are negative and decrease smaller and smaller.

More precisely,

when $x = -1$,	$f(x) = -1$;	when $x = 1$,	$f(x) = 1$
when $x = -0.1$,	$f(x) = -10$;	when $x = 0.1$,	$f(x) = 10$
when $x = -0.01$,	$f(x) = -100$;	when $x = 0.01$,	$f(x) = 100$, etc.

Therefore $\displaystyle\lim_{x\to 0^-} \frac{1}{x} = -\infty.$ $\qquad \displaystyle\lim_{x\to 0^+} \frac{1}{x} = \infty.$

In writing these equations, we are not saying that the limits exist because there are no real number ∞ and $-\infty$. So we say that the function $f(x) = \dfrac{1}{x}$ has no limits as $x \to 0^-$ and as $x \to 0^+$.

Example 8.

Find the one-sided limits of (a) $\displaystyle\lim_{x\to 2^+} \frac{1}{4-2x}$ $\qquad$ (b) $\displaystyle\lim_{x\to 2^-} \frac{1}{4-2x}$

Solution

The domain of function $\dfrac{1}{4-2x}$ is $\{x \mid x \neq 2\}$.

(a) If $x > 2$ and x is getting close to 2, the value of $\frac{1}{4-2x}$ is negative and is getting smaller. That is

$$\lim_{x \to 2^+} \frac{1}{4-2x} = -\infty$$

(b) If $x < 2$ and x is getting close to 2, the value of $\frac{1}{4-2x}$ is positive and is getting larger. That is

$$\lim_{x \to 2^-} \frac{1}{4-2x} = \infty$$

Continuous Functions

The graph of a function can be described as a **continuous curve** if it has no breaks or holes. Therefore we now study what properties of a function can cause breaks or holes.

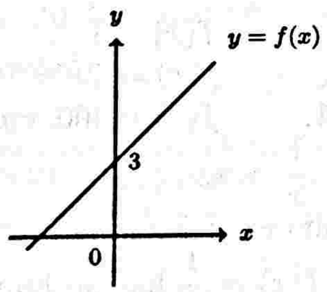
Continuity at a Point

Definition.

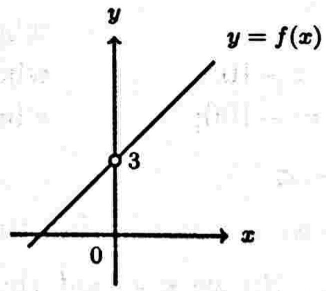
A function f is said to be **continuity at a point** $x = c$ if the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

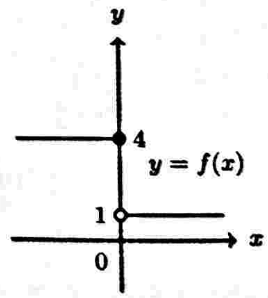
If one or more of the conditions of this definition fail to hold, then we will say that f is **discontinuous at** $x = c$ and c is **a point of discontinuous** of f.



Continuous
Fig. 11.1



Discontinuous at $x = 0$
Fig. 11.2

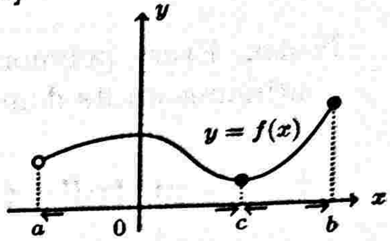


Jump at $x = 0$
Fig. 11.3

In figure 11.3, the function $f(x)$ has no limit as $x \to 0$ because its values **jump at** $x = 0$. For positive value of x arbitrarily close to zero, $f(x) = 1$. For the negative value of x arbitrarily close to zero, $f(x) = 4$. There is no single value approached by $f(x)$ as $x \to 0$.

Continuity on an Interval

A function f is continuous at each number in an open interval (a, b), then we say that f is **continuous on** (a, b). However when we consider the continuity of function on a closed interval $[a, b]$ or the end point of an interval of the form $[a, b)$ or $(a, b]$, we need to consider the one-sided limit at that endpoint. For example, the function graphed in the figure is continuous at the right endpoint of the interval $(a, b]$ because



$$\lim_{x \to b^-} f(x) = f(b)$$

but it is not continuous at the left endpoint because

$$\lim_{x \to a^+} f(x) \neq f(a).$$

In general, we say a function f is **continuous from the left** at c if

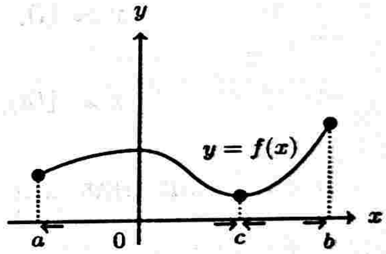
$$\lim_{x \to c^-} f(x) = f(c)$$

and is **continuous from the right** at c if

$$\lim_{x \to c^+} f(x) = f(c).$$

Definition.

A function f is said to be **continuous on a closed interval** $[a, b]$ if the following conditions are satisfied:



1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

A **continuous function** is one that is continuous at every point of its domain.

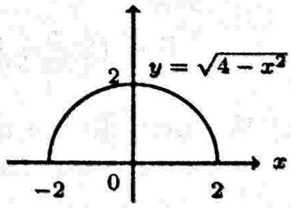
Example 9.

Show that a function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of it domain $[-2, 2]$.

Solution

If c is any point in the interval $(-2, 2)$, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{4 - x^2} = \sqrt{\lim_{x \to c}(4 - x^2)} = \sqrt{4 - c^2} = f(c).$$



Thus, f is continuous at each point in $(-2, 2)$.

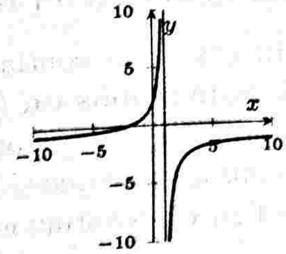
$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \sqrt{4 - x^2} = \sqrt{\lim_{x \to 2^-} (4 - x^2)} = 0 = f(2).$$

A function f is continuous from the left at 2.

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \sqrt{4 - x^2} = \sqrt{\lim_{x \to -2^+} (4 - x^2)} = 0 = f(-2).$$

A function f is continuous from the right at -2. Hence, f is continuous on the closed interval $[-2, 2]$.

Example 10.



The function $y = -\frac{x+2}{x-1}$ is continuous at every value of x except $x = 1$. It has a point of discontinuity at $x = 1$ because it is not defined there. Therefore the given rational function is continuous on its domain $(-\infty, 1)$ and $(1, \infty)$.

Note. Every polynomial function is continuous on $\mathbb{R}$ and every rational function is continuous on its domain.

Limits at Infinity

If the values of variable x increase arbitrarily large, then we write $x \to \infty$ and if the values of variable x decrease arbitrarily small, then we write $x \to -\infty$. Let us consider the limit of function $f(x) = \frac{1}{x}$ for both $x \to \infty$ and $x \to -\infty$.

That is

$$x = 1, \quad \frac{1}{x} = 1, \qquad x = -1, \quad \frac{1}{x} = -1$$

$$x = 10, \quad \frac{1}{x} = 0.1, \qquad x = -10, \quad \frac{1}{x} = -0.1$$

$$x = 100, \quad \frac{1}{x} = 0.01, \qquad x = -100, \quad \frac{1}{x} = -0.01, \text{ etc.}$$

It turns out that $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to-\infty} \frac{1}{x} = 0$. More generally, for $n = 1, 2, \ldots$

$$\lim_{x\to\infty} \frac{1}{x^n} = 0, \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

Example 11.

Find: (a) $\lim_{x\to\infty}(6x^3 - 3x^2 - 1)$ (b) $\lim_{x\to-\infty}(2x^5 + 3x^2 - x)$ (c) $\lim_{x\to\infty} \frac{x^2 - x + 1}{x^3 - 2}$

Solution

(a) $\lim_{x\to\infty}(6x^3 - 3x^2 - 1) = \lim_{x\to\infty}[x^3(6 - \frac{3}{x} - \frac{1}{x^3})] = \infty$

(b) $\lim_{x\to-\infty}(2x^5 + 3x^2 - x) = \lim_{x\to-\infty}[x^5(2 + \frac{3}{x^3} - \frac{1}{x^4})] = -\infty$

(c) We first divide the numerator and denominator by **the highest power of x in the denominator.**

$$\lim_{x\to\infty} \frac{x^2 - x + 1}{x^3 - 2} = \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}}{1 - \frac{2}{x^3}} = \frac{0 - 0 + 0}{1 - 0} = 0$$

Exercise 11.2

1. Find the one-sided limit:
   (a) $\lim_{x\to 1^+} \frac{1}{1-x}$ (b) $\lim_{x\to 1^-} \frac{1}{1-x}$ (c) $\lim_{x\to 2^+} \frac{1}{x-2}$ (d) $\lim_{x\to 2^-} \frac{1}{x-2}$
   (e) $\lim_{x\to 2^+} \frac{x+2}{x^2-4}$ (f) $\lim_{x\to 2^-} \frac{x+2}{x^2-4}$ (g) $\lim_{x\to 5^+} \frac{x}{x^2-5x}$ (h) $\lim_{x\to 5^-} \frac{x}{x^2-5x}$
2. Show that the limit do not exist for the following:
   (a) $\lim_{x\to 0} \frac{|x|}{x}$ (b) $\lim_{x\to 0} \frac{x^2-2x}{|x|}$ (c) $\lim_{x\to 1} \frac{x^2-1}{|x-1|}$
3. Find the limit of the following functions:
   (a) $\lim_{x\to\infty} \frac{3x-5}{12x+1}$ (b) $\lim_{x\to\infty} \frac{3x^2+7}{x^3-2}$ (c) $\lim_{x\to-\infty} (1-3x-4x^7)$
   (d) $\lim_{x\to\infty} \frac{x^2-x+10}{x-1}$ (e) $\lim_{x\to\infty} \frac{\sqrt{x^2-2}}{2x-1}$ (f) $\lim_{x\to-\infty} \left(1-\frac{2}{x}\right)$
4. Determine where f is continuous.
   (a) $f(x) = \frac{x-3}{2x+1}$ (b) $f(x) = \sqrt{x}$ (c) $f(x) = x^2+3x-1$
5. Find the value of x at which f is not continuous.
   (a) $f(x) = \frac{x^2+2x}{x+2}$ (b) $f(x) = \frac{1}{x} + \frac{x-1}{x^2-1}$ (c) $f(x) = \frac{x^2-4}{|x-2|}$

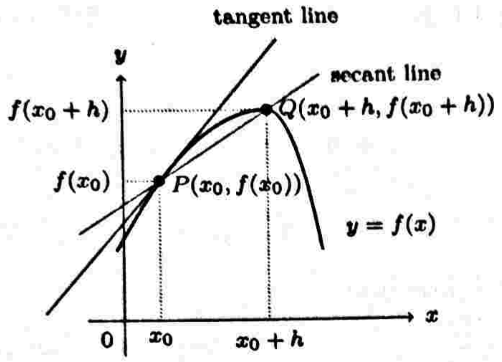
11.2 Derivatives

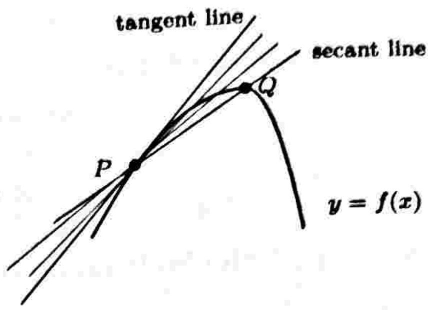
Let us consider the point $P(x_0, f(x_0))$ on the graph of the function $y = f(x)$ and $Q(x_0+h, f(x_0+h))$ is another point on the graph. The **slope of the curve at the point** P is

$$m = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

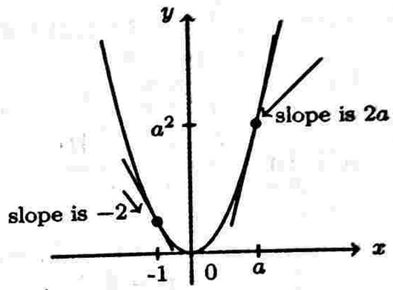
if the limit exists. The **tangent line** to the curve at P is the line through P with this slope.

Example 12.
Find the slope of the curve $y = x^2$ at any point $x = a$. What are the slope at the point $x = 2$ and $x = -1$? Where does the slope equal 1?





Solution
Here $f(x) = x^2$. The slope at (a, a^2) is



$$\begin{aligned}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{(a+h)^2-a^2}{h}\\ &= \lim_{h\to 0}\frac{a^2+2ah+h^2-a^2}{h}\\ &= \lim_{h\to 0}(2a+h) = 2a.\end{aligned}$$

Thus the slope at the point $x = a$ is $2a$. When $a = 2$, the slope is 4 and when $a = -1$, the slope is -2. If the slope is 1, then $2a = 1$. Thus $a = \frac{1}{2}$. Therefore the curve has slope 1 at the point $(\frac{1}{2}, \frac{1}{4})$.

Derivative at a Point

The expression $\dfrac{f(x_0+h)-f(x_0)}{h}$, $h \neq 0$ is called the **difference quotient of f at x_0 with an increment h.** If the difference quotient has a limit as h approaches zero, it is given a special name and notation.

Definition.
The derivative of a function f at a point x_0, denoted by $f'(x_0)$ is

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

if this limit exists.

Example 13.
Find the derivative of $f(x) = x^2 + 5$ at 3. That is, find $f'(3)$.

Solution
Since $f(3) = 3^2 + 5 = 14$, we have

$$\frac{f(3+h)-f(3)}{h} = \frac{(3+h)^2+5-14}{h} = \frac{h^2+6h}{h}$$

The derivative of f at 3 is

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{h(h+6)}{h} = \lim_{h\to 0} (h+6) = 6.$$

Example 14.

Find the derivative of $f(x) = x^3$ at c. That is, find $f'(c)$.

Solution

Since $f(c) = c^3$, we have

$$\frac{f(c+h) - f(c)}{h} = \frac{(c+h)^3 - c^3}{h} = \frac{3c^2h + 3ch^2 + h^3}{h}$$

The derivative of f at c is

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0} \frac{h(3c^2 + 3ch + h^2)}{h} = \lim_{h\to 0} (3c^2 + 3ch + h^2) = 3c^2.$$

The Derivative as a Function

We now discuss the derivative as a function derived from f by considering the limit at each point x in the domain of f.

Definition.

The function f' defined by the formula

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is called **the derivative of f with respect to x**. The domain of f' consists of all x in the domain of f for which the limit exists.

There are many ways to denote the derivative of a function $y = f(x)$. Some common notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x).$$

The process of calculating a derivative is called differentiation.

Example 15.

Differentiate $x^2 + 3x + 6$ with respective to x by using the definition.

Solution

Let $f(x) = x^2 + 3x + 6$.

$$f(x+h) = (x+h)^2 + 3(x+h) + 6 = x^2 + 2xh + h^2 + 3x + 3h + 6 = (x^2 + 3x + 6) + (2xh + h^2 + 3h)$$

Thus

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{2xh + h^2 + 3h}{h} = \lim_{h\to 0} (2x + h + 3) = 2x + 3.$$

Example 16.

Differentiate $f(x) = \dfrac{1}{x}$ with respective to x by using the definition.

Solution

Since $f(x) = \dfrac{1}{x}$, we have $f(x+h) = \dfrac{1}{x+h}$.

Then

$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{x - (x+h)}{(x+h)x} = \frac{-h}{(x+h)x}.$$

Therefore

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}.$$

Example 17.

Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$ by using the definition.

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}.$$

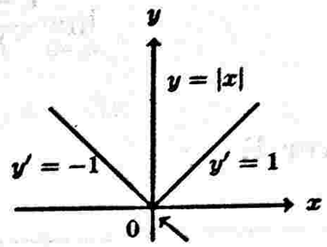
Note that some functions may fail to have the derivative. Thus we need to study the right-hand and left-hand limit of difference quotient of a function at any point of the function's domain.

Example 18.
Show that the function $y = |x|$ has no derivative at $x = 0$.

Solution
To the right of the origin, $\lim\limits_{x \to 0^+} \dfrac{f(0+h) - f(0)}{h} = \dfrac{|h|}{h} = \dfrac{h}{h} = 1.$

To the left, $\lim\limits_{x \to 0^-} \dfrac{f(0+h) - f(0)}{h} = \dfrac{|h|}{h} = \dfrac{-h}{h} = -1.$
We observe that

$$\lim_{x \to 0^+} \frac{f(0+h) - f(0)}{h} \neq \lim_{x \to 0^-} \frac{f(0+h) - f(0)}{h}.$$



y' not defined at x = 0

Hence there is no derivative at the origin.

Exercise 11.3

1. By using the definition, find the derivative of each function at the given number.
   (a) $f(x) = x^3 - x + 1$ at 3 (b) $f(x) = x^2 - 2x$ at 0
   (c) $f(x) = -5x + 1$ at -2 (d) $f(x) = \sqrt{x+2}$ at 4

2. Find the derivatives of each function with respective to x.
   (a) $f(x) = x^2 - 2x + 1$ (b) $f(x) = \dfrac{1}{x^2}$

3. Show that the function $y = |x - 5|$ has no derivative at $x = 5$.

11.3 Differentiation Rules

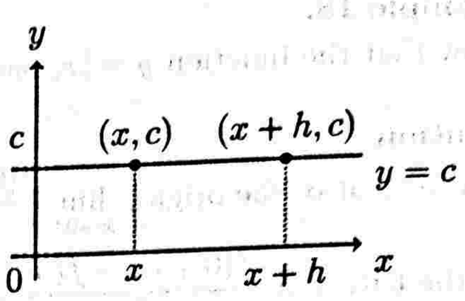
In this section, we introduce the differentiation formulas of constant functions and power functions. Then you will study the derivative of sum rule, product rule, quotient rule and chain rule.

Derivative of a Constant Function

If $f(x) = c$ is a constant function, then $\dfrac{df}{dx} = \dfrac{d}{dx}c = 0.$

Proof

At every value of x, we find that



$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

Power Rule

> If n is a positive integer, then $\frac{d}{dx}x^n = nx^{n-1}$.

Proof

Let $f(x) = x^n$ where n is a positive integer.

$$\begin{aligned} f(x+h) &= (x+h)^n \\ &= x^n + n\cdot x^{n-1}\cdot h + \frac{n(n-1)}{2}\cdot x^{n-2}\cdot h^2 + \ldots + h^n \\ &= x^n + h[n\cdot x^{n-1} + \frac{n(n-1)}{2}\cdot x^{n-2}\cdot h + \ldots] \end{aligned}$$

$$\therefore\ f(x+h) - f(x) = h[n\cdot x^{n-1} + \frac{n(n-1)}{2}\cdot x^{n-2}\cdot h + \ldots]$$

$$\begin{aligned} \frac{d}{dx}f(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{h[n\cdot x^{n-1} + \frac{n(n-1)}{2}\cdot x^{n-2}\cdot h + \ldots]}{h} \\ &= \lim_{h\to 0}[n\cdot x^{n-1} + \frac{n(n-1)}{2}\cdot x^{n-2}\cdot h + \ldots] = n\cdot x^{n-1}. \end{aligned}$$

Power Rule (General Version)

> If n is any rational number, then $\frac{d}{dx}x^n = nx^{n-1}$ for any x where the power x^n and x^{n-1} are defined.

Example 19.

Differentiate the following with respect to x.

(a) x^4 (b) $x^{\frac{2}{3}}$ (c) $x^{-\frac{4}{3}}$ (d) $\frac{1}{x^3}$

Solution

(a) $\frac{d}{dx}(x^4) = 4x^{4-1} = 4x^3$ (b) $\frac{d}{dx}(x^{\frac{2}{3}}) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}}$

(c) $\frac{d}{dx}(x^{-\frac{4}{3}}) = -\frac{4}{3}x^{-\frac{4}{3}-1} = -\frac{4}{3}x^{-\frac{7}{3}}$ (d) $\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}(x^{-3}) = -3x^{-3-1} = -3x^{-4}$

Derivative of Constant Multiple Rule

If f is a differentiable function of x and c is a constant, then $\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$.

Proof

$$\frac{d}{dx}(cf(x)) = \lim_{h\to 0}\frac{cf(x+h) - cf(x)}{h} = c\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = c\frac{d}{dx}f(x)$$

Example 20.

Find $\frac{dy}{dx}$ for (a) $y = 3x^2$ (b) $y = -x$ (c) $y = -2\sqrt{x}$.

Solution

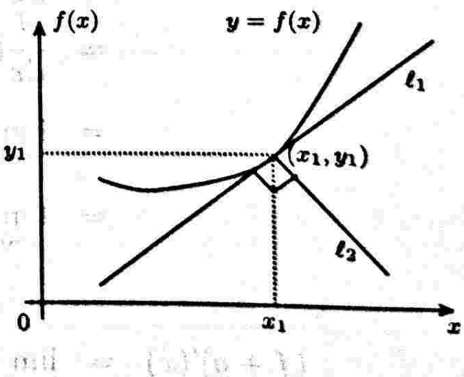
(a) $\frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2) = 3\cdot 2x = 6x$

(b) $\frac{d}{dx}(-x) = \frac{d}{dx}(-1\cdot x) = -1\cdot\frac{d}{dx}(x) = -1$

(c) $\frac{d}{dx}(-2\sqrt{x}) = -2\frac{d}{dx}(x^{\frac{1}{2}}) = -2\cdot\frac{1}{2}x^{(\frac{1}{2})-1} = -x^{-\frac{1}{2}}$

For the curve $y = f(x)$, the slope of the tangent ℓ_1 at the point (x_1, y_1) is the value of $\frac{dy}{dx}$ at $x = x_1$, hence the **equation of the tangent** at (x_1, y_1) is

$$y - y_1 = f'(x_1)(x - x_1).$$



The line ℓ_2 which is perpendicular to the tangent ℓ_1 at (x_1, y_1) is called the **normal** to the curve at (x_1, y_1). Hence its slope is the value of $-\frac{1}{f'(x_1)}$ where $f'(x_1) \neq 0$. The **equation of the normal** at (x_1, y_1) is

$$y - y_1 = -\frac{1}{f'(x_1)}(x - x_1).$$

Example 21.

Find equations of the tangent line and normal line to the curve $f(x) = x^2$ at the point $(1, 1)$.

Solution

$$f(x) = x^2, \text{then} \quad f'(x) = 2x.$$

At $(1, 1)$, $f'(1) = 2$. Therefore the equation of the tangent line at $(1, 1)$ is

$$\begin{aligned} y - 1 &= 2(x - 1) \\ y &= 2x - 1. \end{aligned}$$

The slope of normal line is $-\frac{1}{2}$. Thus the equation of the normal line at $(1, 1)$ is

$$\begin{aligned} y - 1 &= -\frac{1}{2}(x - 1) \\ y &= -\frac{1}{2}x + \frac{3}{2}. \end{aligned}$$

Sum Rule

If f and g are differentiable at x, then the sum $f + g$ is differentiable at x,

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

(or)

$$(f + g)'(x) = f'(x) + g'(x)$$

Proof

$$\begin{aligned} (f + g)'(x) &= \frac{d}{dx}(f + g)(x) \\ &= \frac{d}{dx}(f(x) + g(x)) \\ &= \lim_{h \to 0} \frac{[f(x + h) + g(x + h)] - [f(x) + g(x)]}{h} \\ &= \lim_{h \to 0} \left[\frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h}\right] \end{aligned}$$

$$\begin{aligned} (f + g)'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} \\ &= \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \\ &= f'(x) + g'(x). \end{aligned}$$

Combining the Sum Rule and the Constant Multiple Rule, we get

$$\begin{aligned}\frac{d}{dx}(f(x)-g(x)) &= \frac{d}{dx}f(x)+\frac{d}{dx}(-g(x))\\ &= \frac{d}{dx}f(x)+(-1)\frac{d}{dx}g(x)\\ &= \frac{d}{dx}f(x)-\frac{d}{dx}g(x)\end{aligned}$$

Example 22.

Find the derivative of $y = x^3 + 2x^2 - 3x - 6$.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^3+2x^2-3x-6)\\ &= \frac{dx^3}{dx}+2\frac{dx^2}{dx}-3\frac{dx}{dx}-\frac{d}{dx}6\\ &= 3x^2+2(2x)-3(1)-0 = 3x^2+4x-3\end{aligned}$$

Product Rule

If f and g are differentiable at x, then the product fg is differentiable at x,

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)$$

(or)

$$(fg)'(x) = f(x)g'(x)+g(x)f'(x).$$

Proof

$$\begin{aligned}\frac{d}{dx}(f(x)g(x)) &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{\{f(x+h)(g(x+h)-g(x))\}+g(x)(f(x+h)-f(x))}{h}\\ &= \lim_{h\to 0}\left\{f(x+h)\cdot\frac{g(x+h)-g(x)}{h}+g(x)\cdot\frac{f(x+h)-f(x)}{h}\right\}\\ &= \lim_{h\to 0}f(x+h)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)\end{aligned}$$

Since $\lim_{h\to 0} f(x+h) = f(x)$, we get

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x).$$

You may notice that while the derivative of the sum of two differentiable functions is the sum of their derivatives, the derivative of the product of two differentiable functions is not the product of their derivatives.

Example 23.

Differentiate $(x^3 - 2x)(3x^4 + 2)$ with respect to x.

Solution

$$\begin{aligned}\frac{d}{dx}[(x^3-2x)(3x^4+2)] &= (x^3-2x)\frac{d}{dx}(3x^4+2) + (3x^4+2)\frac{d}{dx}(x^3-2x)\\ &= (x^3-2x)(12x^3) + (3x^4+2)(3x^2-2)\\ &= 21x^6 - 30x^4 + 6x^2 - 4\end{aligned}$$

Quotient Rule

If f and g are differentiable at x and if $g(x) \neq 0$, then the quotient $\frac{f(x)}{g(x)}$ is differentiable at x,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}$$

(or)

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Proof

First we proof $\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{(g(x))^2}$.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \lim_{h\to 0}\frac{\left\{\frac{1}{g(x+h)} - \frac{1}{g(x)}\right\}}{h} &= \lim_{h\to 0}\frac{\frac{g(x)-g(x+h)}{g(x+h)g(x)}}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\cdot\frac{g(x)-g(x+h)}{g(x+h)g(x)}\\ &= \lim_{h\to 0}\left\{-\frac{g(x+h)-g(x)}{h}\cdot\frac{1}{g(x+h)g(x)}\right\}\\ &= -\frac{d}{dx}g(x)\lim_{h\to 0}\frac{1}{g(x+h)g(x)}.\end{aligned}$$

Since $\lim_{h\to 0} g(x+h) = g(x)$, we get

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{(g(x))^2}.$$

Next we use the product rule to obtain

$$\begin{aligned}\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\cdot\frac{1}{g(x)}\right) &= f'(x)\cdot\frac{1}{g(x)} + f(x)\left(\frac{1}{g(x)}\right)' \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{\{g(x)\}^2} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}\end{aligned}$$

Example 24.

Differentiate (a) $\frac{2x+1}{5x-2}$, (b) $\frac{1}{3x-4}$ and (c) $\frac{x-2}{x^2+3x-1}$ with respect to x.

Solution

(a) $$\frac{d}{dx}\frac{2x+1}{5x-2} = \frac{(5x-2)\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}(5x-2)}{(5x-2)^2} = \frac{(5x-2)2 - (2x+1)5}{(5x-2)^2} = \frac{-9}{(5x-2)^2}$$

(b) $$\frac{d}{dx}\frac{1}{3x-4} = \frac{(3x-4)\times 0 - \frac{d}{dx}(3x-4)}{(3x-4)^2} = \frac{-3}{(3x-4)^2}$$

(c) $$\begin{aligned}\frac{d}{dx}\frac{x-2}{x^2+3x-1} &= \frac{(x^2+3x-1)\frac{d}{dx}(x-2) - (x-2)\frac{d}{dx}(x^2+3x-1)}{(x^2+3x-1)^2} \\ &= \frac{(x^2+3x-1) - (x-2)(2x+3)}{(x^2+3x-1)^2} \\ &= \frac{(x^2+3x-1) - (2x^2-x-6)}{(x^2+3x-1)^2} \\ &= \frac{-x^2+4x+5}{(x^2+3x-1)^2}\end{aligned}$$

When we consider the differentiation of the function

$$F(x) = \sqrt{x^2 - x},$$

we cannot calculate $F'(x)$ directly.

Observe that F is a composite function, that is if we let $y = f(u) = \sqrt{u}$ and if $u = g(x) = x^2 - x$, we can write $y = F(x) = f(g(x)) = (f \circ g)(x)$. We have studied how to differentiate both f and g, so we can find the derivative of $F = f \circ g$ in terms of the derivatives of f and g. This is called **the Chain Rule.**

The Chain Rule
If $y = f(u)$ is differentiable at u and $u = g(x)$ is differentiable at x, then the function $y = f(g(x))$ is differentiable at x, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Since $y = f(g(x)) = (f \circ g)(x)$,

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)..$$

Example 25.

Find $y'(x)$ if $y(x) = \sqrt{x^2 - x}$.

Solution
Let $u = x^2 - x$. Then $y = \sqrt{u}$.

$$y'(x) = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x - 1) = \frac{1}{2\sqrt{x^2 - x}}(2x - 1) = \frac{x - \frac{1}{2}}{\sqrt{x^2 - x}}.$$

The Power Rule Combined with the Chain Rule
If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x).$$

Example 26.

Differentiate $y = (x^2 - 1)^5$ with respect to x.

Solution

$$y = (x^2 - 1)^5$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 - 1)^5 = 5(x^2 - 1)^4 \frac{d}{dx}(x^2 - 1) = 5(x^2 - 1)^4 \cdot 2x = 10x(x^2 - 1)^4.$$

Example 27.

Find $f'(x)$ if $f(x) = \left(\frac{x-1}{x+1}\right)^3$.

Solution

$$f(x) = \left(\frac{x-1}{x+1}\right)^3$$

$$\begin{aligned} f'(x) &= 3\left(\frac{x-1}{x+1}\right)^2 \frac{d}{dx}\left(\frac{x-1}{x+1}\right) \\ &= 3\left(\frac{x-1}{x+1}\right)^2 \frac{(x+1)\cdot 1-(x-1)\cdot 1}{(x+1)^2} = \frac{6(x-1)^2}{(x+1)^4} \end{aligned}$$

Example 28.

Find $f'(x)$ if $f(x) = (x^2-2x)^3(x^4-1)^2$.

Solution

$$f(x) = (x^2-2x)^3(x^4-1)^2$$

$$\begin{aligned} f'(x) &= [\frac{d}{dx}(x^2-2x)^3](x^4-1)^2 + (x^2-2x)^3[\frac{d}{dx}(x^4-1)^2] \\ &= [3(x^2-2x)^2\frac{d}{dx}(2x^2-2x)](x^4-1)^2 \\ &\quad +(x^2-2x)^3[2(x^4-1)\frac{d}{dx}(x^4-1)] \\ &= 3(x^2-2x)^2(4x-2)(x^4-1)^2 + 2(x^2-2x)^3(x^4-1)(4x^3) \end{aligned}$$

Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function of x. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. The function f'' is called the **second derivative** of f. It can be written as

$$f''(x) = \frac{df'}{dx} = \frac{d^2f}{dx^2} = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y''.$$

If $y = f(x)$, then successive derivatives can be denoted by

$$f', \quad f'', \quad f''', \quad f^{(4)}, \quad f^{(5)}, \ldots.$$

For example, if $f(x) = 3x^4$, then $f'(x) = 12x^3$ and we have

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}(12x^3) = 36x^2.$$

$$f'''(x) = \frac{d}{dx}f''(x) = \frac{d}{dx}(36x^2) = 72x.$$

Example 29.

Find y' and y'' for $y = (3x^2 - 2x + 1)^2$.

Solution

$$\begin{aligned} y &= (3x^2 - 2x + 1)^2 \\ y'(x) &= 2(3x^2 - 2x + 1)(6x - 2) \\ y''(x) &= 2(6x-2)\frac{d}{dx}(3x^2 - 2x + 1) + 2(3x^2 - 2x + 1)\frac{d}{dx}(6x - 2) \\ &= 2(6x-2)(6x-2) + 12(3x^2 - 2x + 1) = 4(27x^2 - 18x + 5) \end{aligned}$$

Exercise 11.4

1. Differentiate the following with respect to x.

   (a) $x^4 - 5x^3 + 2x - 1$ (b) $\dfrac{\sqrt{x+3}}{x+1}$ (c) $2x + \dfrac{1}{2x}$

   (d) $(2x^2 + 2)(x^2 - 3x)^2$ (e) $(3 + x^2)\sqrt{3 - x^2}$ (f) $\dfrac{1}{x+1}$

   (g) $\dfrac{3x - 5}{2x^2 + 7}$ (h) $\dfrac{2x - 7}{\sqrt{x+7}}$ (i) $\sqrt{\dfrac{x^2+1}{x^2-1}}$

2. Find the slope of the curve $y = x^3 - x$ at any point $x = a$. What is the slope at the point $x = 1$? Where does the slope equal 11?

3. Calculate the slope of the curve $y = \dfrac{3x^2 - 8}{5 - 2x}$ at the point $(2, 4)$.

4. Find equations of the tangent line and normal line to the curve $y = \dfrac{2x^2 + 1}{x}$ at the point $(1, 3)$.

5. Find y' and y'' for each of the following functions.

   (a) $y = \dfrac{x}{x-1}$ (b) $y = x\sqrt{x+2}$ (c) $y = \dfrac{x+1}{x^2}$ (d) $y = (3x+2)^{10}$

6. If $y = \dfrac{2x^2 + 3}{x}$, prove that $x^2y'' + xy' = y$.

7. If $y = \sqrt{2x - 1}$, show that $-y^3y'' + yy' = 2$.

11.4 Implicit Differentiation

We have been studied the differentiation of the equation of the form $y = f(x)$. However some functions like

$$x^2y + xy^2 = 7 \qquad y^2 = \frac{xy-1}{x+1} \qquad y^2 - 2xy = 3 - 2y$$

cannot be differentiated in a usual way. These equations define **an implicit** relation between the dependent and independent variables y and x. To calculate the derivative of implicitly defined functions, we proceed the following steps:

Suppose y as a differentiable function of x.

Step 1 Differentiate both sides of the equation with respect to x.

Step 2 Collect the terms with $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$.

Example 30.
If $x^2 - xy^2 - y^3 = 2$, find $\frac{dy}{dx}$.

Solution

$$x^2 - xy^2 - y^3 = 2$$

Differentiate with respect to x on both sides.

$$\begin{aligned}
2x - [x \cdot \frac{dy^2}{dx} + y^2 \cdot \frac{dx}{dx}] - 3y^2 \cdot \frac{dy}{dx} &= 0 \\
2x - x \cdot 2y\frac{dy}{dx} - y^2 - 3y^2 \cdot \frac{dy}{dx} &= 0 \\
(2xy + 3y^2)\frac{dy}{dx} &= 2x - y^2 \\
\frac{dy}{dx} &= \frac{2x - y^2}{2xy + 3y^2}
\end{aligned}$$

Example 31.
Find the equation of the tangent line to the curve $3x^2 + 2y^2 = 2xy + 23$ at the point $(3, 2)$.

Solution

Curve: $$3x^2 + 2y^2 = 2xy + 23$$

Differentiate with respect to x on both sides.

$$\begin{aligned}
6x + 4y\frac{dy}{dx} &= 2[x \cdot \frac{dy}{dx} + y] \\
(4y - 2x)\frac{dy}{dx} &= 2y - 6x \\
\frac{dy}{dx} &= \frac{y - 3x}{2y - x}
\end{aligned}$$

The gradient of tangent to the curve at $(3, 2)$ is

$$m = \frac{2 - 3(3)}{2(2) - 3} = -7.$$

Then equation of the tangent line is

$$\begin{aligned} y - 2 &= -7(x - 3) \\ 7x + y &= 23. \end{aligned}$$

Example 32.

If $x^2 - y^2 = 25$, find $\dfrac{d^2y}{dx^2}$.

Solution

$$x^2 - y^2 = 25$$

We differentiate both sides of the equation with respect to x, we get

$$\begin{aligned} 2x - 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{x}{y}. \end{aligned}$$

Next

$$\begin{aligned} \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{x}{y}\right) = \frac{y - x\frac{dy}{dx}}{y^2} &= \frac{1}{y} - \frac{x}{y^2} \cdot \frac{dy}{dx} \\ &= \frac{1}{y} - \frac{x}{y^2} \cdot \frac{x}{y} \\ &= \frac{1}{y} - \frac{x^2}{y^3} \end{aligned}$$

Exercise 11.5

1. Find $\dfrac{dy}{dx}$.
   (a) $xy = 5$ (b) $x(x + y) = y^2$ (c) $x^3 - 4xy + y^2 = 14$
   (d) $\dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{1}{4}$ (e) $x^3 + xy + y^2 = 2$ (f) $\sqrt{x} - \sqrt{y} = 1$

2. Find equations of the tangent line and normal line to the curve $y^2 = x^3$ at the point $(1, -1)$.

3. Find y' if $x^3 + y^3 = 4xy$. Find an equation of the tangent line to the curve $x^3 + y^3 = 4xy$ at the point $(2, 2)$.

4. Find y' if $x^2 - xy - y^2 = 1$. Find an equation of the tangent line to the curve $x^2 - xy - y^2 = 1$ at the point $(2, 1)$.

5. Find y'' if $x^4 + y^4 = 1$.

6. Show that the equation of the tangent to the curve $x^2 + xy + y = 0$ at the point (a, b) is $x(2a + b) + y(a + 1) + b = 0$.

7. Find the coordinates of the points on the curve $x^2 - y^2 = 3xy - 39$ at which the tangent is parallel to the line $x + y = 1$.