National Unity Government of Myanmar

Ministry of Education

Grade 12

Mathematics

Textbook

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Chapter 1
COMPLEX NUMBERS

In this chapter we introduce a new number system which is the extension of the real number system.

1.1 Pure Imaginary Unit i

Since we have known that $x^2 \geq 0$ for every real number x, the equation $x^2 + 4 = 0$ (or) $x^2 = -4$ has no real solution.

But if there is a pure imaginary unit i such that $i^2 = -1$ with acceptance of the usual operations on real numbers and i such as $(2i)^2 = 4i^2 = 4(-1) = -4$ and $(-2i)^2 = 4i^2 = 4(-1) = -4$ then $x^2 = -4$ has two solutions 2i and $-2i$.

If we try to solve $x^2 = -n$ for $n > 0$, like $x^2 = -4$, then $\sqrt{n}i$ and $-\sqrt{n}i$ are solutions because $(\pm\sqrt{n}i)^2 = ni^2 = n(-1) = -n$.

Example 1

Solve $x^2 + 2x + 5 = 0$.

Solution

Completing the square method

$$x^2 + 2x + 5 = 0$$

$$x^2 + 2x = -5$$

$$x^2 + 2x + 1^2 = -5 + 1^2$$

$$(x + 1)^2 = -4$$

$$(x + 1)^2 = 4i^2 \qquad (\because\ i^2 = -1)$$

$$x + 1 = \pm 2i$$

$$x = -1 \pm 2i$$

Using formula

$$x^2 + 2x + 5 = 0$$

$$ax^2 + bx + c = 0$$

$$a = 1, b = 2, c = 5$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm \sqrt{16i^2}}{2}$$

$$= \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$

Example 2

Solve $x^2 + 2x + 3 = 0$ and check your answer.

Solution

$x^2 + 2x + 3 = 0$

$$x^2 + 2x = -3$$

$$x^2 + 2x + 1 = -3 + 1$$

$$(x + 1)^2 = -2$$

$$(x + 1)^2 = 2i^2$$

$$x + 1 = \pm\sqrt{2}i$$

$$x = -1 \pm \sqrt{2}i.$$

Thus, $x = -1 + \sqrt{2}i$ (or) $x = -1 - \sqrt{2}i$.

For $x = -1 + \sqrt{2}i$,

$$x^2 + 2x + 3 = (-1 + \sqrt{2}i)^2 + 2(-1 + \sqrt{2}i) + 3$$

$$= 1 - 2\sqrt{2}i + 2i^2 - 2 + 2\sqrt{2}i + 3$$

$$= 1 - 2 - 2 + 3$$

$$= 0 .$$

For $\quad x = -1 - \sqrt{2}i,$

$$x^2 + 2x + 3 = (-1 - \sqrt{2}i)^2 + 2(-1 - \sqrt{2}i) + 3$$
$$= 1 + 2\sqrt{2}i + 2i^2 - 2 - 2\sqrt{2}i + 3$$
$$= 1 - 2 - 2 + 3$$
$$= 0 .$$

Exercises 1.1

1. Solve the following equations.

(a) $x^2 - 6x + 10 = 0$ (b) $-2x^2 + 4x - 3 = 0$

(c) $5x^2 - 2x + 1 = 0$ (d) $3x^2 + 7x + 5 = 0$

2. Solve the following equations and check your answers.

(a) $x^2 - 2x + 4 = 0$ (b) $x^2 - 4x + 5 = 0$

3. Find the value of i^n for every positive integer n, where $i^2 = -1$, $i^3 = i^2\, i$, $i^4 = i^2 i^2$, etc.

1.2 Complex Number (x, y)

As we have just seen in Section 1.1, there are numbers like $x_1 + y_1 i$ and $x_2 + y_2 i$

such that $(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i$ and

$$(x_1 + y_1 i)(x_2 + y_2 i) = x_1x_2 + (x_1y_2 + y_1x_2)i + y_1y_2\, i^2$$
$$= x_1x_2 + (x_1y_2 + y_1x_2)i + y_1y_2(-1)$$
$$= (x_1x_2 - y_1y_2) + (x_1y_2 + y_1x_2)i.$$

For real numbers x_1, x_2, y_1, y_2 , so we define the number $x_1 + y_1 i$ as follows.

A complex number is an ordered pair (x, y) of real numbers with equality and operations sum and product of two complex numbers $(x_1,\ y_1)$, $(x_2,\ y_2)$ are defined as follows:

Let $z_1 = x_1 + y_1 i = (x_1, y_1)$ and $z_2 = x_2 + y_2 i = (x_2, y_2)$.

Equality $z_1 = z_2$, if and only if $x_1 = x_2$ and $y_1 = y_2$.

Sum $z_1 + z_2 = (x_1 + y_1 i) + (x_2 + y_2 i)$

$$= (x_1 + x_2) + (y_1 + y_2)i$$

$$= (x_1 + x_2, y_1 + y_2).$$

Product $z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i)$

$$= x_1x_2 + x_1y_2 i + x_2y_1 i - y_1y_2$$

$$= (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i$$

$$= (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1).$$

Then we have, $(x, 0) + (y, 0) = (x + y, 0)$ and $(x, 0)(y, 0) = (xy, 0)$.

Real Numbers	Complex Numbers
x	$(x, 0)$
y	$(y, 0)$
$x + y$	$(x, 0) + (y, 0)$
xy	$(x, 0)(y, 0)$

Table 1.1

$i = (0,\ 1)$ and $i^2 = (0,\ 1)(0,\ 1)$

$$= (0 - 1,\ 0 + 0)$$

$$= (-1,\ 0) = -1.$$

We have

$$x + yi = (x,\ 0) + (y,\ 0)(0,\ 1)$$

$$= (x, 0)(0 - 0,\ y + 0)$$

$$= (x,\ 0) + (0,\ y)$$

$$x + yi = (x,\ y).$$

Note that sum and product of complex numbers satisfy commutative, associative and distributive properties.

Complex Number

A number that has both a real part and an imaginary part. The imaginary part is a multiple of the square root of minus one (i). Some algebraic equations cannot be solved with real numbers.

Example 3

Compute $(-2,\ 3)(1,\ -2) + (1, 1)(0,\ 1)$.

Solution

$$\begin{aligned}(-2,\ 3)(1,\ -2) + (1, 1)(0,\ 1) &= (-2 + 3\mathrm{i})(1 - 2\mathrm{i}) + (1 + \mathrm{i})\mathrm{i} \\ &= -2 + 4\mathrm{i} + 3\mathrm{i} - 6\mathrm{i}^2 + \mathrm{i} + \mathrm{i}^2 \\ &= -2 + 8\mathrm{i} + 6 - 1 \\ &= 3 + 8\mathrm{i} \\ &= (3,\ 8).\end{aligned}$$

Exercises 1.2

1. Compute:

   (a) $(2,\ 0)(3,\ 5) + (3,\ -2)(0,\ 1)$

   (b) $(2,\ -5)(-1,\ 0) + (1,\ 0)(5,\ 1)$

   (c) $(-3,\ -2)(-2,\ -3) + (-2,\ -3)(-3,\ -2)$

   (d) $(1,\ 0)(0,\ 1) + (0,\ 1)(1,\ 0)$.

2. Compute:

   (a) $(3 + 2\mathrm{i})(3 - 2\mathrm{i}) + (-5 + 7\mathrm{i})(-1 - \mathrm{i})$

   (b) $(-1 + \mathrm{i})(1 - \mathrm{i}) + (2 + 3\mathrm{i})$

   (c) $(1 + \mathrm{i})(1 - \mathrm{i}) + (-2 + \mathrm{i})(-2 + \mathrm{i})$

   (d) $(3 + 2\mathrm{i}) + (7 - \mathrm{i})(-3 + 3\mathrm{i})$.

1.3 Operations on Complex Numbers

Sum and product of complex numbers were defined in Section 1.2. Subtraction of complex numbers is defined as in real numbers as follows:

$$(x_1 + y_1 i) - (x_2 + y_2 i) = (x_1 + y_1 i) + (-x_2 - y_2 i)\ .$$

Therefore sum, product and subtraction of complex numbers can be performed as in the real numbers, except only that $i^2 = -1$. But the division of complex numbers is a little different from the division of real numbers. We need some notations to define the division.

Let us denote a complex number by z, so that

$$z = x + yi = (x, y).$$

The conjugate $\bar{z}$ of a complex number z is defined by $\bar{z} = x - yi = (x, -y)$.

Then we have

$$z\bar{z} = (x + yi)(x - yi) = x^2 - y^2 i^2 = x^2 + y^2.$$

Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$.

We will calculate $\frac{z_1}{z_2}$, $(z_2 \neq 0)$, as follows:

$$\frac{z_1}{z_2} = \frac{x_1 + y_1 i}{x_2 + y_2 i} \cdot \frac{x_2 - y_2 i}{x_2 - y_2 i}$$

$$= \frac{x_1 x_2 - x_1 y_2 i + x_2 y_1 i + y_1 y_2}{{x_2}^2 - {y_2}^2 i^2}$$

$$= \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{{x_2}^2 + {y_2}^2}$$

$$= \frac{x_1 x_2 + y_1 y_2}{{x_2}^2 + {y_2}^2} + i\ \frac{x_2 y_1 - x_1 y_2}{{x_2}^2 + {y_2}^2}.$$

Example 4

Calculate $\frac{2+3i}{3+i}$.

Solution

$$\frac{2+3i}{3+i} = \frac{2+3i}{3+i} \cdot \frac{3-i}{3-i}$$

$$= \frac{6-2i+9i-3i^2}{9-i^2}$$

$$= \frac{6+7i+3}{9+1}$$

$$= \frac{9+7i}{10}$$

$$= \frac{9}{10} + \frac{7}{10} i \, .$$

Exercises 1.3

1. Let $z_1 = -2+3i$, $z_2 = 5+2i$. Compute:

(a) $z_1 - 2z_2 + 1$ (b) $3{z_2}^2 + 2z_2 + 1$ (c) $z_1\overline{z_2} + z_2\overline{z_1}$

(d) $\frac{1}{z_1}$ (e) $\frac{1}{z_2}$ (f) $\frac{1}{z_1 z_2}$

(g) $\frac{z_1}{z_2}$ (h) $\frac{\overline{z_1}}{z_2}$ (i) $\frac{z_2}{z_1}$

(j) $\overline{\left(\frac{z_2}{z_1}\right)}$ (k) $\frac{\overline{z_1}\, z_2}{z_1 \overline{z_2}}$ (l) $\frac{z_2}{\overline{z_1}} + \frac{z_1}{\overline{z_2}}$.

2. Let $z_1 = 3-2i$, $z_2 = -1+4i$. Show that

(a) $\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}$ (b) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ (c) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.

1.4 Trigonometric Form

As a complex number $z = x + iy$ is an ordered pair (x, y) of real numbers, we can place z in xy −coordinate plane as usual. If the length of the line segment from O to z is r , we say that the **absolute value** of z is r and is denoted by $|z|$. Here angle θ, measured in radians, is an angle with positive x −axis and the line segment.

Let $z = x + iy = (x, y)$

Absolute value of $z = |z| = r = \sqrt{x^2 + y^2}$

$$x = r\cos\theta,\ y = r\sin\theta,\quad \tan\theta = \frac{y}{x}$$

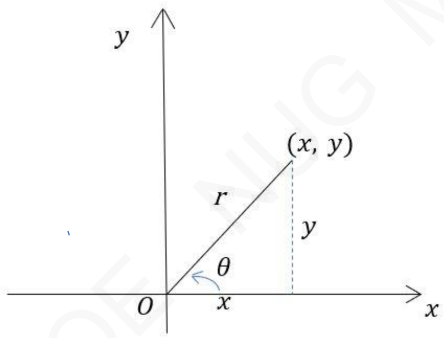


Figure 1.1

So, we have $\quad z = x + iy$

$= r\cos\theta + i\, r\sin\theta$

$= r(\cos\theta + i\sin\theta)$ **(Trigonometric Form)**

$= (r, \theta)$. **(Polar Form)**

Here angle θ, measured in radians, is an angle with positive x-axis and the line segment.

Argument of a complex number (θ)

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}(\frac{y}{x})$

Argument of $\theta = \tan^{-1}\left(\frac{y}{x}\right), \quad (-\pi < \theta \le \pi)$

The angle θ is called the **argument of z, arg z**.

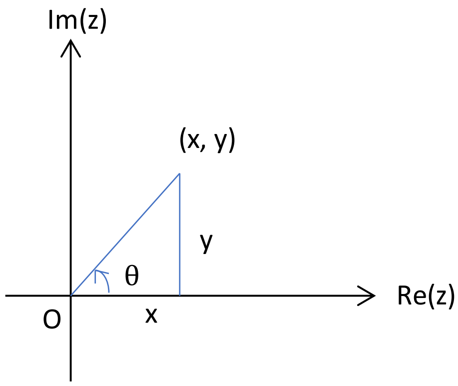


Figure 1.2

Principle argument of z

The value of θ such that $-\pi < \theta \leq \pi$ is called the **principal argument of z**.

θ is the 1st quadrant

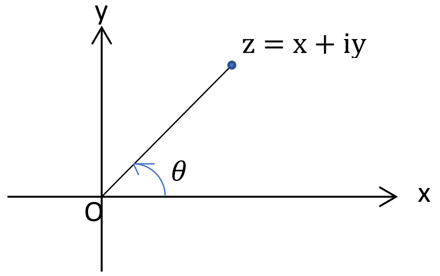


Figure 1.3

θ is the 2nd quadrant

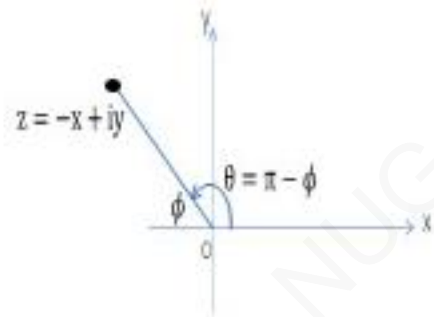


Figure 1.4

θ is the 3rd quadrant

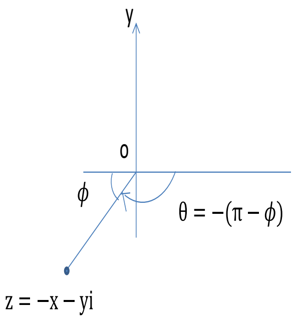


Figure 1.5

θ is the 4th quadrant

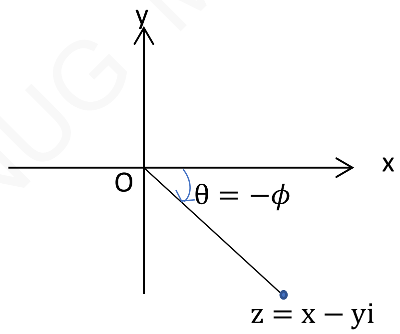


Figure 1.6

Example 5

Find the trigonometric form with $-\pi < \theta \leq \pi$.

(a) $z = 1 + \sqrt{3}i$ (b) $z = -1 + i$ (c) $z = -\sqrt{3} - i$ (d) $z = -1$

Solution

(a) $z = 1 + \sqrt{3}i = (1, \sqrt{3}) = (x, y)$

$x = 1,\ y = \sqrt{3}$

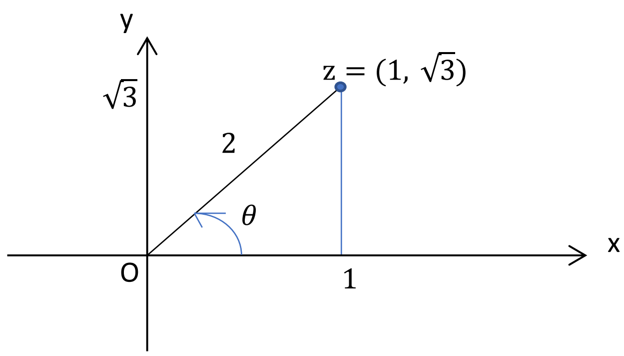
$r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2$

$\cos\theta = \frac{x}{r} = \frac{1}{2},\ \sin\theta = \frac{\sqrt{3}}{2}$

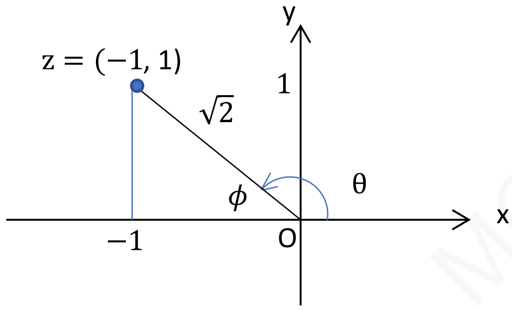
$\theta = \frac{\pi}{3},\ -\pi < \theta \leq \pi.$

$z = r(\cos\theta + i\sin\theta)$

$= 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}).$



(b) $z = -1 + i = (-1,\ 1) = (x,\ y)$



$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$

$\cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{2}}, \sin\theta = \frac{1}{\sqrt{2}}$

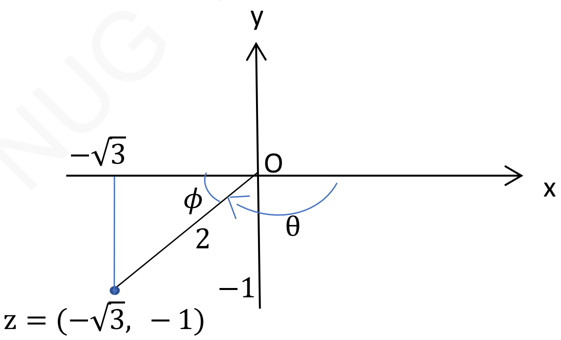
Basic acute angle $= \phi = \frac{\pi}{4}$

$\theta = \pi - \phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$

$z = r(\cos\theta + i\sin\theta)$

$= \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}).$

(c) $z = -\sqrt{3} - i = \left(-\sqrt{3},\ -1\right) = (x,\ y)$



$r = \sqrt{3+1} = \sqrt{4} = 2$

$\cos\theta = \frac{-\sqrt{3}}{2}, \sin\theta = -\frac{1}{2}$

Basic acute angle $= \phi = \frac{\pi}{6}$

$\theta = -(\pi - \phi) = -\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}.$

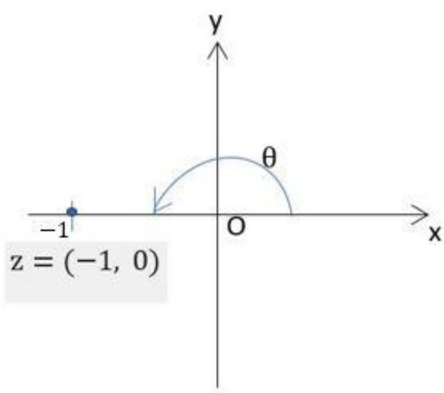
$z = r(\cos\theta + i\sin\theta)$

$= 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right).$

(d) $z = -1 = -1 + 0i = (-1,\ 0) = (x,\ y)$

$r = \sqrt{1+0} = \sqrt{1} = 1$

$\cos\theta = \frac{-1}{1} = -1, \sin\theta = \frac{0}{1} = 0$



$\theta = \pi$

$z = 1(\cos\pi + i\sin\pi)$

$= \cos\pi + i\sin\pi.$

Product in Trigonometric Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1).\ r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2(\cos\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$

$= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$

So $\quad z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$.

Example 6

Given that $z_1 = 1 + \sqrt{3}i$, $z_2 = -1 + i$ find z_1z_2 by using trigonometric forms. Check your answer by direct multiplication.

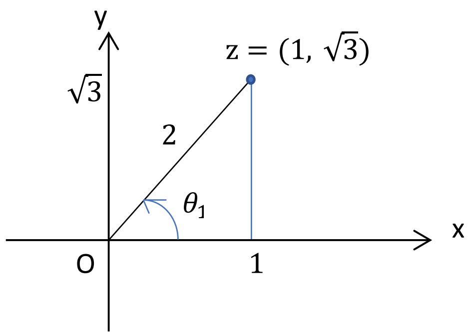
Solution

$z_1 = 1 + \sqrt{3}i = (1, \sqrt{3}) = (x_1, y_1)$

$r_1 = \sqrt{1+3} = \sqrt{4} = 2$

$\sin\theta_1 = \frac{\sqrt{3}}{2}, \cos\theta_1 = \frac{1}{2}$

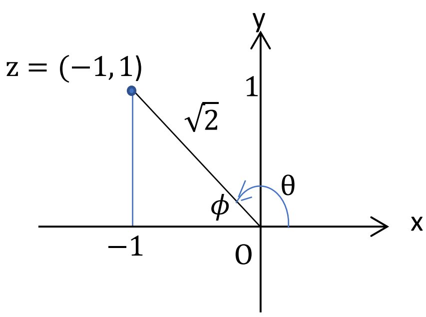
$\theta_1 = \frac{\pi}{3}$



$z_1 = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

$z_2 = -1 + i = (-1, 1) = (x_2, y_2)$

$r_2 = \sqrt{1+1} = \sqrt{2},$



$\sin\theta_2 = \frac{1}{\sqrt{2}}, \cos\theta_2 = \frac{-1}{\sqrt{2}}$

$\phi = \frac{\pi}{4}$

$\theta_2 = \pi - \phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$z_2 = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$

$$z_1z_2 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= 2\sqrt{2}(\cos\left(\frac{\pi}{3} + \frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{3} + \frac{3\pi}{4}\right))$$

$$= 2\sqrt{2}\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)$$

$$= 2\sqrt{2}(\frac{-\sqrt{6}-\sqrt{2}}{4} + \frac{-\sqrt{6}+\sqrt{2}}{4}i) \quad \text{(use calculator)}$$

$$= \left(-1-\sqrt{3}\right) + \left(1-\sqrt{3}\right)i \quad \text{(use calculator)}$$

$$z_1z_2 = \left(1+\sqrt{3}i\right)(-1+i)$$

$$= -1 + i - \sqrt{3}i + \sqrt{3}i^2$$

$$= -1 + i - \sqrt{3}i - \sqrt{3}$$

$$= \left(-\sqrt{3}-1\right) + \left(-\sqrt{3}+1\right)i\ .$$

Multiplicative Inverse in Trigonometric Form

Since $z^{-1} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$ for $z = x + iy$,

$$z^{-1} = \frac{1}{x^2+y^2}(x - iy)$$

$$= \frac{1}{r^2}r\,(\cos\theta - i\sin\theta\,)$$

$$= \frac{1}{r}\,(\cos\theta - i\sin\theta\,)$$

$$z^{-1} = \frac{1}{r}\,(\cos(-\theta) + i\sin(-\theta\,)).$$

Division in Trigonometric Form

Let $z_1 = r_1(\cos\ \theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\ \theta_2 + i\sin\theta_2)$.

Then $\frac{z_1}{z_2} = z_1{z_2}^{-1}$

$$= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)).$$

Example 7

Given that $z = -\sqrt{3} - i$, using trigonometric form of z, find z^{-1}. Check your answer by showing that $zz^{-1} = 1$.

Solution

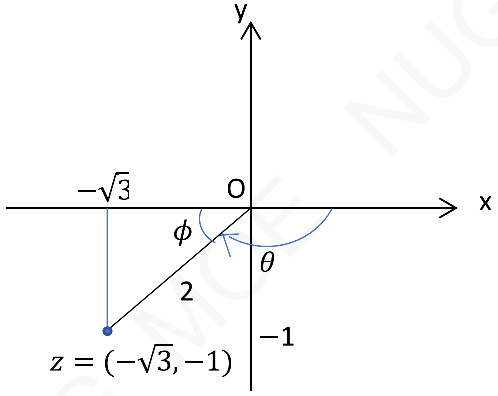
$z = -\sqrt{3} - i = (-\sqrt{3},\ -1) = (x, y)$

$r = \sqrt{3+1} = \sqrt{4} = 2$

$\sin\theta = \frac{-1}{2}$, $\cos\theta = \frac{-\sqrt{3}}{2}$

$\phi = \frac{\pi}{6}$

$\theta = -\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}$



$z = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$

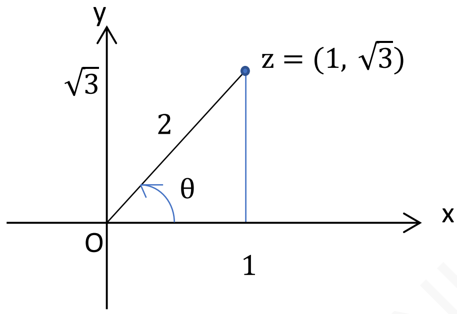
$$z^{-1} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$$
$$= \frac{1}{2}\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$
$$= \frac{1}{2}\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$
$$= \frac{-\sqrt{3}}{4} + \frac{1}{4}i\,.$$

$$zz^{-1} = (-\sqrt{3} - i)\left(-\frac{\sqrt{3}}{4} + \frac{1}{4}i\right)$$
$$= \frac{3}{4} - \frac{\sqrt{3}}{4}i + \frac{\sqrt{3}}{4}i - \frac{1}{4}i^2$$
$$= \frac{3}{4} + \frac{1}{4} = 1.$$

Example 8

Given that $z_1 = 1 + \sqrt{3}i$, $z_2 = -1 + i$, find $\frac{z_1}{z_2}$ by using trigonometric forms.

Check your answer by direct calculation.



Solution

$$z_1 = 1 + \sqrt{3}i = (1, \sqrt{3}) = (x_1, y_1)$$

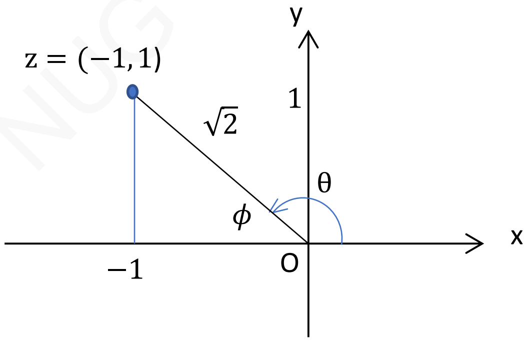
$$r_1 = \sqrt{1+3} = \sqrt{4} = 2$$

$$\sin\theta_1 = \frac{\sqrt{3}}{2},\ \cos\theta_1 = \frac{1}{2}$$

$$\theta_1 = \frac{\pi}{3}$$

$$z_1 = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$

$$z_2 = -1 + i = (-1, 1) = (x_2, y_2)$$



$$r_2 = \sqrt{1+1} = \sqrt{2}$$

$$\sin\theta_2 = \frac{1}{\sqrt{2}},\ \cos\theta_2 = \frac{-1}{\sqrt{2}}$$

$$\phi = \frac{\pi}{4}$$

$$\theta = (\pi - \phi) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$z_2 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$= \frac{2}{\sqrt{2}}\left[\cos\left(\frac{\pi}{3} - \frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{3} - \frac{3\pi}{4}\right)\right]$$

$$= \frac{2}{\sqrt{2}}\left[\cos\left(\frac{\pi}{3} - \frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{3} - \frac{3\pi}{4}\right)\right]$$

$$= \frac{2}{\sqrt{2}}\left[\cos\left(-\frac{5\pi}{12}\right) + i\sin\left(-\frac{5\pi}{12}\right)\right]$$

$$= \frac{-1+\sqrt{3}}{2} - \frac{1+\sqrt{3}}{2}i \quad . \qquad \text{(use calculator)}$$

$$\frac{z_1}{z_2} = \frac{1+\sqrt{3}i}{-1+i} \times \frac{-1-i}{-1-i}$$

$$= \frac{-1-i-\sqrt{3}i-\sqrt{3}i^2}{1-i^2}$$

$$= \frac{-1-i-\sqrt{3}i+\sqrt{3}}{1+1}$$

$$= \frac{-1+\sqrt{3}}{2} - \frac{1+\sqrt{3}}{2}i.$$

Powers of Complex Numbers

The power of a complex number $z = r(\cos\theta + i\sin\theta)$ is given by

$z^n = r^n(\cos n\,\theta + i\sin n\,\theta)$, n is an integer.

Example 9

Given that $z = 1+\sqrt{3}i$, find (a) z^{10} (b) z^{-10}.

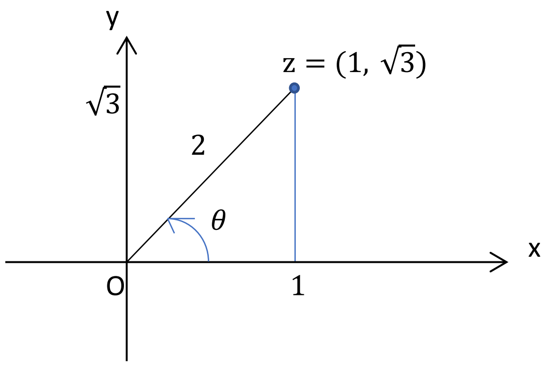
Solution

$z = 1+\sqrt{3}i = (1, \sqrt{3}) = (x, y)$

$r = \sqrt{1+3} = \sqrt{4} = 2$, $\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3}$

$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.



(a) $z^{10} = 2^{10}\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right)$

$= 2^{10}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$, (use calculator)

$= 1024\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -512 - 512\sqrt{3}i.$

(b) $z^{-10} = 2^{-10}\left(\cos\frac{-10\pi}{3} + i\sin\frac{-10\pi}{3}\right)$

$= 2^{-10}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$, (Use Calculator)

$= \frac{1}{1024}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= -\frac{1}{2048} + \frac{\sqrt{3}}{2048}i$.

Exercises 1.4

1. Find the trigonometric form with $-\pi < \theta \leq \pi$.

(a) $z = 1 - \sqrt{3}i$ (b) $z = -\sqrt{2} + \sqrt{2}i$ (c) $z = -2 - 2i$

(d) $z = \sqrt{3} - 1$ (e) $z = i$ (f) $z = -3i$

2. Given that $z_1 = 2 - 2\sqrt{3}i$, $z_2 = -1 - i$, find the following complex numbers by using trigonometric forms. Check your answer by direct calculation.

(a) $z_1 z_2$ (b) ${z_1}^{-1}$ (c) ${z_2}^{-1}$ (d) $\frac{z_1}{z_2}$ (e) $\frac{z_2}{z_1}$

3. Given that $z = -2\sqrt{3} - 2i$, find (a) z^5 (b) z^{-5}.

1.5 Roots of Complex Numbers

An n^{th} root of a complex number z is a complex number w that satisfies the equation

$w^n = z$, since $z = r(\cos\theta + i\sin\theta)$ and $w_k = r^{\frac{1}{n}}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$

where $k = 0, 1, 2, \ldots, n-1$.

Let $w^n = z \Leftrightarrow z^{\frac{1}{n}} = w$

$w_k = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$, $k = 0, 1, 2, \ldots, n-1$.

When $k = 0$, $w_0 = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$

When $k = n$, $w_n = \sqrt[n]{r}\left(\cos\frac{\theta+2n\pi}{n} + i\sin\frac{\theta+2n\pi}{n}\right) = \sqrt[n]{r}\ \left(\cos\frac{\theta}{n} + 2\pi\right) + i\sin\frac{\theta}{n} + 2\pi)$

So, the root for $k = n$ is the same as the root for $k = 0$. The same is also true for $k > n$ as the root for $k = n+1$ is the same as the root for $k = 1$, and so on.

Therefore, the roots of z, denoted by w_k of z are

$$w_k = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right),\ k = 0,\ 1, \ldots, n-1.$$

Example 10

Find the cube roots of $z = -2 - 2i$. (OR) Solve $z^3 = -2 - 2i$.

Solution

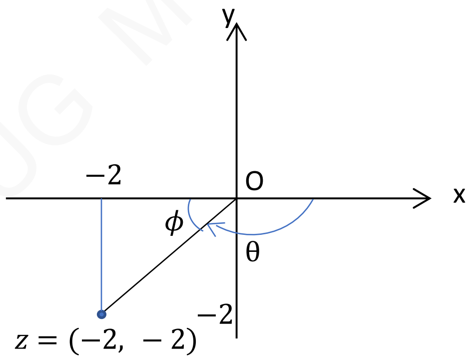
$z = -2 - 2i = (-2,\ -2) = (x,\ y)$

$r = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\sin\theta = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}},\ \cos\theta = \frac{-2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}$

$\phi = \frac{\pi}{4}$

$\theta = -\left(\pi - \frac{\pi}{4}\right) = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$



$z = 2\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$

$w_k = \sqrt[3]{2\sqrt{2}}\ [\cos\frac{-\frac{3\pi}{4}+2k\pi}{3} + i\sin\frac{-\frac{3\pi}{4}+2k\pi}{3}]$, $k = 0, 1, 2$.

Therefore, the cube roots are

$w_0 = \sqrt{2}\left(\cos\left(\frac{-3\pi/4}{3}\right) + i\sin\left(\frac{-3\pi/4}{3}\right)\right)$

$= \sqrt{2}(\cos\left(\frac{-3\pi}{12}\right) + i\sin\left(\frac{-3\pi}{12}\right))$

$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\ \right)$

$= \sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = 1 - i.$

$$w_1 = \sqrt{2}\ \left[\cos\frac{-\frac{3\pi}{4}+2\pi}{3} + i\sin\frac{-\frac{3\pi}{4}+2\pi}{3}\right]$$

$$= \sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right)$$

$$= \sqrt{2}\left(\frac{\sqrt{6}-\sqrt{2}}{4} + \frac{\sqrt{6}+\sqrt{2}}{4}i\right)$$

$$= \frac{-1+\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}i\ .$$

$$w_2 = \sqrt{2}\ \left[\cos\frac{-\frac{3\pi}{4}+4\pi}{3} + i\sin\frac{-\frac{3\pi}{4}+4\pi}{3}\right]$$

$$= \sqrt{2}\left(\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right)$$

$$= \sqrt{2}\left(\frac{-\sqrt{6}-\sqrt{2}}{4} + \frac{-\sqrt{6}+\sqrt{2}}{4}i\right)$$

$$= \frac{-\sqrt{3}-1}{2} + \frac{-\sqrt{3}+1}{2}i\ .$$

Example 11

Solve $z^6 = 1$. (OR) Find the sixth roots of unity.

Solution

$1 = (1,\ 0)$, $r = \sqrt{1+0} = 1$.

$\cos\theta = 1$, $\sin\theta = 0 \Longrightarrow \therefore\ \theta = 0$.

$$1\ = 1(\cos 0 + i\sin 0)$$

$$= 1\big(\cos(0+2k\pi) + i\sin(0+2k\pi)\big)$$

$$= 1(\cos 2k\pi + i\sin 2k\pi).$$

$$z^6 = 1 = 1(\cos 2k\pi + i\sin 2k\pi)$$

$$z = 1^{1/6}\left[\cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}\right],\ k = 0,\ 1,\ 2,\ 3,\ 4,\ 5.$$

$$z = \cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}\ .$$

When $k = 0$, $z = \cos 0 + i \sin 0 = 1$.

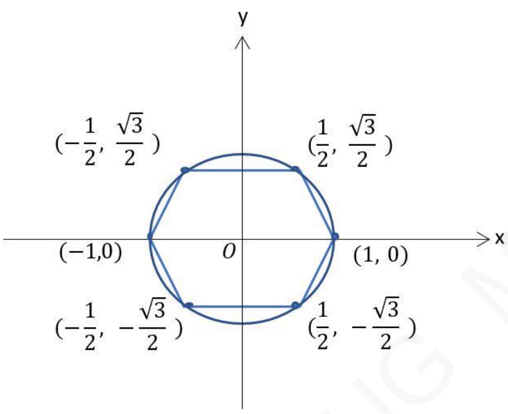
When $k = 1$, $z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

When $k = 2$, $z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

When $k = 3$, $z = \cos\pi + i\sin\pi = -1$.

When $k = 4$, $z = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

When $k = 5$, $z = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$.



Exercise 1.5

1. Find the square roots of the following complex numbers.

(a) $1 + \sqrt{3}i$ (b) i (c) $-\sqrt{3} + i$

(d) $-1 - \sqrt{3}i$ (e) $-i$ (f) $\sqrt{3} - i$

2. Find the cube roots of the following complex numbers.

(a) $1 + i$ (b) i (c) $-1 + i$

(d) $-1 - i$ (e) $-i$ (f) $1 - i$

3. Solve the following equations.

(a) $z^4 = -i$ (b) $z^4 = -1$ (c) $z^4 = -8 - 8\sqrt{3}i$ (d) $z^6 = -1$

Chapter 2
MATHEMATICAL INDUCTION

In this chapter, we will study a technique of proving a statement, theorem or formula which is thought to be true, for each and every natural number n. By generalizing this in form of a principle which we would use to prove any mathematical statement is "Principle of Mathematical Induction".

- **What is the statement in Mathematics?**

  In mathematics, a statement is a sentence that is either true or false but not both.

- **Which of the following sentences are statements?**

  (1) $3 + 4 = 8$.

  (2) $2x + 5 = 10$.

  (3) 12 is a multiple of 4.

  (4) $x^2 - y^2 = (x + y)(x - y)$ for all real numbers x and y.

  (5) The sum of a and b is greater than 0.

  In the above (1), (3) and (4) are statements but others are not.

2.1 Introduction

One of the techniques to prove mathematical statements discussed in this chapter is the Principle of Mathematical Induction. This method is a powerful and elegant technique for proving certain type of mathematical statements which assert that something is true for all natural numbers or for all natural numbers from some point on. The actual term mathematical induction was first used by De Morgan, even though the method was used by Fermat, Pascal and others before him. The method is used in many branches of Higher Mathematics.

Before we state the principle of mathematical induction, let us consider an example. Consider the sum of the first n odd positive integers. That is,

if	$n = 1$,	1			$=$	1^2,
if	$n = 2$,	$1 + 3$	$=$	4	$=$	2^2,
if	$n = 3$,	$1 + 3 + 5$	$=$	9	$=$	3^2,
if	$n = 4$,	$1 + 3 + 5 + 7$	$=$	16	$=$	4^2,
if	$n = 5$,	$1 + 3 + 5 + 7 + 9$	$=$	25	$=$	5^2,
if	$n = 6$,	$1 + 3 + 5 + 7 + 9 + 11$	$=$	36	$=$	6^2,

From the result above, it looks as if the sum of the first n odd natural numbers is always given by n^2.

To express this statement symbolically, first observe that the n^{th} odd number is $2n - 1$. Then the statement can be expressed as:

$$1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) = n^2 \qquad \ldots.. (1)$$

Although from this pattern we might conjecture that the statement (1) is true for any choice of n, can we really be sure that it does not fail for some choice of n? We have seen that the statement is true for n = 1, 2, 3, 4, 5, 6 by direct calculation. Is the statement true for any natural number n? Actually, no matter how many cases we check, we can never prove that the statement is always true because there are infinitely many cases and direct calculation cannot check them all. The method of proof by mathematical induction will, in fact, prove that any mathematical statement like (1) is true for all natural numbers n.

2.2 Principle of Mathematical Induction

For each natural numbers n, let P(n) be a statement depending on n. Suppose that the following two conditions are satisfied.

1. The statement is true for n = 1.
2. For any natural number k, if the statement is true for
   n = k, then the statement is true for n = k + 1.

Then the statement P(n) is true for all natural numbers n.

Motivation : To understand the principle of mathematical induction, suppose a set of thin rectangular dominos are placed on one end as shown in Figure (2.1).

Figure 2.1

When the first domino is pushed in the indicated direction, all the dominos will fall. To be absolutely sure that all the dominos will fall, it is sufficient to know that

(a) The first domino falls, and

(b) In the event that any domino falls its successor necessarily falls.

This phenomenon is called the Domino Effect or chain reaction.
This is the underlying principle of mathematical induction.

To apply the principle of mathematical induction, there are two steps:

The initial step : Prove that the statement is true for $n = 1$.

The inductive step: Assume that the statement is true for $n = k$, and use this assumption to prove that the statement is true for $n = k + 1$.

Conclusion: The statement has been proved for $n = 1$. By the inductive step, since the statement is true for $n = 1$, it is also true for $n = 2$. Again, by the inductive step, since the statement is true for $n = 2$, it is also true for $n = 3$. And since the statement is true for $n = 3$, it is also true for $n = 4$, and so on. Finally, we conclude that the statement is true for all natural numbers n.

Remark: In the inductive step we do not prove that the statement is true for $n = k$.

We only show that if the statement is true for $n = k$, then the statement is also true for $n = k + 1$. The assumption that the statement is true for $n = k$, is called the inductive hypothesis.

To see the important of these two steps, we will study the following two examples:

1st example :

Consider the following table.

n	1	2	3	4	5	6	7
$n^2 + n + 11$	13	17	23	31	41	53	67

Can we conclude that $n^2 + n + 11$ is prime for all natural numbers.

Let P(n) be the statement $n^2 + n + 11$ is prime, for all natural numbers n. It can be easily seen that P(1) is true. But we must prove that if P(k) is true, then P(k + 1) is true for any $k \geq 1$. This inductive step breaks down when k = 9. For,

if $k = 9$, $9^2 + 9 + 11 = 101$ is prime and

if $k = 10$, $10^2 + 10 + 11 = 121$ is not prime.

So, we cannot conclude that the statement P(n) is true for all natural numbers n.

2nd example :

Consider the statement 3n + 2 is a multiple of 3 for all natural numbers n.
Let P(n) be the statement 3n + 2 is a multiple of 3 for all natural numbers n.
We will only prove the inductive step.

Assume that P (k) is true.

i.e, 3k + 2 is a multiple of 3.

This can be written as the equation

$3k + 2 = 3m$ for some natural number m.

We want to show that P (k + 1) is true.

For $n = k + 1$,

$$\begin{aligned} 3(k+1) + 2 &= 3k + 3 + 2 \\ &= 3k + 2 + 3 \\ &= 3m + 3 \\ &= 3(m+1). \end{aligned}$$

Since m + 1 is a natural number,

3(k + 1) + 2 is a multiple of 3.

∴ P (k + 1) is true.

The above proof shows that the inductive step is true. Does it follow that P(n) is true for all natural numbers n ? No, the statement P(n) is false because 3n + 2 cannot be written as a multiple of 3.

Although the inductive step is true, we cannot conclude that the statement is true. Because we have not established the basis step, the induction fails. In fact, P(1) is false. Since the first domino does not fall, we cannot even start the chain reaction.

The above two examples show that both steps in mathematical induction are equally important.

Example 1

Use the mathematical induction principle to prove that

$1 + 3 + 5 + \cdots + (2n - 1) = n^2$, for all natural numbers n.

Solution

Let P(n) denote the statement $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all natural numbers n.

For n = 1, L.H.S = 1,

R.H.S = $1^2 = 1$.

∴ L.H.S = R.H.S

∴ P (1) is true.

Assume that P(k) is true.

i.e., $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. . . . . . (1)

We will show that P (k + 1) is true.

For n = k + 1,

$$\begin{aligned} \text{L.H.S} &= 1 + 3 + 5 + \ldots + (2(k + 1) - 1) \\ &= 1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) \\ &= k^2 + 2k + 1 \quad \text{(From (1))} \\ &= (k + 1)^2 \\ \text{R.H.S} &= (k + 1)^2 \end{aligned}$$

∴ L.H. S = R.H.S

∴ P (k + 1) is true.

Hence, by the principle of mathematical induction,

the statement P(n) is true for all natural numbers n.

Example 2

Use the mathematical induction principle to prove that $1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}$, for all natural numbers n.

Solution

Let P(n) denote the statement $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ for all natural numbers n.

For n = 1, L.H.S = 1,

$$\text{R.H.S} = \frac{1(1+1)}{2} = 1.$$

$$\therefore \text{L.H.S} = \text{R.H.S}$$

$\therefore$ P(1) is true.

Assume P(k) is true.

i.e., $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$ ...... (1)

We want to show that P(k + 1) is true.

For n = k + 1,

$$\begin{aligned}
\text{L.H.S} &= 1 + 2 + 3 + \ldots + (k + 1) \\
&= 1 + 2 + 3 + \ldots + k + (k + 1) \\
&= \frac{k(k+1)}{2} + (k + 1) \text{ (From (1))} \\
&= (k + 1)(\frac{k}{2} + 1) \\
&= \frac{(k+1)(k+2)}{2} \\
\text{R.H.S} &= \frac{(k+1)(k+1+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

$$\therefore \text{L.H.S} = \text{R.H.S}$$

$\therefore$ P (k +1) is true

Hence, by mathematical induction, the statement P(n) is true for all natural numbers n.

Example 3

Use the mathematical induction principle to prove that $(ab)^n = a^n b^n$ for every natural numbers n.

Solution

Let P(n) denote the statement $(ab)^n = a^n b^n$.

For n = 1, $\text{L.H.S} = (ab)^1 = ab$

$\text{R.H.S} = a^1 b^1 = ab.$

$$\therefore \text{L.H.S} = \text{R.H.S}$$

$\therefore$ P (1) is true.

Assume that P(k) is true.

i.e. $(ab)^k = a^k b^k$ . . . . . (1)

We want to show that P(k + 1) is true.

For n = k + 1, L.H.S $= (ab)^{k+1}$

$= (ab)^k (ab)$

$= (a^k b^k)(ab)$ (From (1))

$= (a^k a)(b^k b)$

$= a^{k+1} b^{k+1}$

R.H.S $= a^{k+1} b^{k+1}$

∴L.H.S = R.H.S

∴ P(k + 1) is true.

Hence, by mathematical induction, the statement P(n) is true for every natural numbers n.

Example 4

Prove that $1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + n \cdot 3^n = \frac{(2n-1)3^{n+1}+3}{4}$

for all natural numbers n, by the use of the mathematical induction principle.

Solution

Let P(n) denote the statement $1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + n \cdot 3^n = \frac{(2n-1)3^{n+1}+3}{4}$

for all natural numbers n.

For n = 1, L.H.S $= 1 \cdot 3 = 3$,

R.H.S $= \frac{(2-1)3^2+3}{4} = \frac{12}{4} = 3$.

∴P(1) is true.

Assume that P(k) is true.

i.e., $1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + k \cdot 3^k = \frac{(2k-1)3^{k+1}+3}{4}$ . . . . . (1)

We want to show that P(k + 1) is true.

For n = k + 1, L.H.S $= 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + (k+1).\, 3^{k+1}$

$= 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + k.3^k + (k+1)3^{k+1}$

$= \frac{(2k-1)3^{k+1}+3}{4} + (k+1)3^{k+1}$ (From (1))

$= \frac{(2k-1)3^{k+1}+3+(4k+4)3^{k+1}}{4}$

$= \frac{(2k-1+4k+4)3^{k+1}+3}{4}$

$= \frac{(6k+3)3^{k+1}+3}{4}$

L.H.S $= \frac{(2k+1).3.3^{k+1}+3}{4}$

$= \frac{(2k+1).3^{k+2}+3}{4}$.

$$\text{R.H.S} = \frac{[2(k+1)-1]3^{k+1+1}+3}{4}$$

$$= \frac{(2k+1)\cdot 3^{k+2}+3}{4}.$$

$\therefore$L.H.S = R.H.S

$\therefore$ P(k + 1) is true.

Hence, by the principle of mathematical induction,

the statement P(n) is true for all natural numbers n.

In some cases, a statement involving a variable n holds when the natural numbers $n \geq m$, $m \in N$ and the statement does not hold when $n < m$. In this case, we will prove that the statement is true for $n = m$ in the initial step.

Example 5

Use the mathematical induction principle to prove that $4n < 2^n$ for all natural numbers $n \geq 5$.

Solution

Let P(n) denote the statement $4n < 2^n$, for all natural numbers $n \geq 5$

For $n = 5$, L.H.S $= 4(5) = 20$,

R.H.S $= 2^5 = 32$.

$\therefore$ L.H.S < R.H.S

$\therefore$ P(5) is true.

Assume that P(k) is true.

i.e., $4k < 2^k$. .....(1)

We want to show that P(k + 1) is true.

For $n = k + 1$, L.H.S $= 4(k+1)$

$= 4k + 4$

$< 2^k + 4$ (by(1))

$< 2^k + 4k$ (since $4 < 4k$)

$< 2^k + 2^k$ (by (1))

L.H.S $< 2 . 2^k$

$= 2^{k+1}$.

R.H.S $= 2^{k+1}$

$\therefore$L.H.S < R.H.S

$\therefore$ P(k + 1) is true.

Hence, by the principle of mathematical induction, the statement P(n) is true for all natural numbers $n \geq 5$.

Example 6

Use the mathematical induction principle to prove that $3^n - 1$ is a multiple of 2, for all natural numbers n.

Solution

Let P(n) denote the statement $3^n - 1$ is a multiple of 2 for all natural numbers n.

For n = 1, $3^1 - 1 = 2$ which is a multiple of 2.

∴P(1) is true.

Assume that P(k) is true.

i.e., $3^k - 1$ is a multiple of 2.

For this we have $3^k - 1 = 2m$ for some natural number m. ------- (1)

We will show that P(k + 1) is true.

For n = k + 1,

$$\begin{aligned} 3^{k+1} - 1 &= 3 \cdot 3^k - 1 \\ &= 3 \cdot 3^k - 3^k + 3^k - 1 \\ &= (3-1) \cdot 3^k + 3^k - 1 \\ &= 2 \cdot 3^k + 3^k - 1 \\ &= 2 \cdot 3^k + 2m \quad (\text{by}(1)) \\ &= 2(3^k + m) \end{aligned}$$

Since 3^k and m are natural numbers, $3^k + m$ is also a natural number.

∴ $2(3^k + m)$ is a multiple of 2.

∴ $3^{k+1} - 1$ is a multiple of 2.

∴ P(k + 1) is true.

Hence, by the principle of mathematical induction, the statement P(n) is true for all natural numbers n.

Example 7

Use the mathematical induction principle to prove that $a - b$ is a factor of $a^n - b^n$, for all natural numbers n.

Solution

Let P(n) denote the statement $a - b$ is a factor of $a^n - b^n$ for all natural numbers n.

For n = 1, $a^1 - b^1 = a - b = (a - b) \times 1$

$\therefore$ a – b is a factor of $a^1 - b^1$.

$\therefore$ P(1) is true.

Assume the P(k) is true.

i.e., a – b is a factor of $a^k - b^k$ . . . . . . (1)

We will show that P(k + 1) is true.

For n = k + 1, $a^{k+1} - b^{k+1} = a^{k+1} - a^k b + a^k b - b^{k+1}$

$= a^k . a - a^k b + a^k b - b^k . b$

$= a^k (a - b) + b(a^k - b^k)$.

a – b is a factor of $a^k(a - b)$ and from (1), a – b is also a factor of $b(a^k - b^k)$.

So, (a – b) is a fator of $a^k (a - b) + b(a^k - b^k)$.

$\therefore$ a – b is a factor of $a^{k+1} - b^{k+1}$.

$\therefore$ P (k + 1) is true.

Hence, by the principle of mathematical induction, the statement P(n) is true for all natural numbers n.

Exercises 2.1

1.(a) Prove that $1 + 3 + 3^2 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}$ by using the principle of mathematical induction for all natural numbers n.

(b) Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ by using the principle of mathematical induction for all natural numbers n.

(c) Prove that $2^3 + 4^3 + 6^3 + \cdot \cdot \cdot + (2n)^3 = 2n^2 (n + 1)^2$ by using the principle of mathematical induction for all natural numbers n.

(d) Prove that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$ by using the principle of mathematical induction for all natural numbers n.

(e) Prove that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$ by using the principle of mathematical induction for all natural numbers n.

(f) Prove that $1 + 2 + 2^2 + \cdot \cdot \cdot + 2^{n-1} = 2^n - 1$ by using the principle of mathematical induction for all natural numbers n.

(g) Prove that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ by using the principle of mathematical induction for all natural numbers n.

2. Prove that $(n + 1)^2 < 2n^2$ for all natural numbers $n \geq 3$ by using the mathematical induction.

3. Prove that $(2n + 7) < (n + 3)^2$ for all natural numbers n by using the mathematical induction.

4. Prove that 3 is a factor of $4^n - 1$ for all natural numbers n by using the mathematical induction.

5. Prove that $3^{2n} - 1$ is divisible by 8 for all natural number n by using the mathematical induction.

6. Prove that $n^3 - n + 3$ is divisible by 3 for all natural numbers n by using the mathematical induction.

7. Prove that $x^{2n} - y^{2n}$ is divisible by $x + y$ for all natural numbers n by using the mathematical induction.

Chapter 3

ANALYTICAL SOLID GEOMETRY

We have learned about points and lines in two-dimensional rectangular coordinate system. In this chapter, we will extend the system to three dimensions.

3.1 Coordinates of a Point in Space

In the plane, each point is associated with an ordered pair of real numbers.
In space, each point is associated with an ordered triple of real numbers. Through a fixed point, called the **origin O**, draw three mutually perpendicular lines: the **x-axis**, the **y-axis** and the **z-axis**.

A point P in space is determined by an ordered triple **(x, y, z)** of real numbers as shown in the diagram. The numbers **x**, **y**, **z** are called the **coordinates of P**.

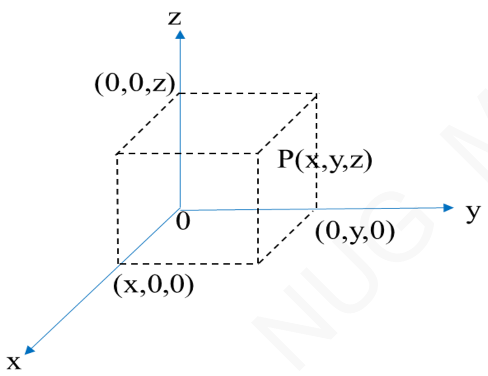


Figure 3.1

Plane	Equation	Coordinates
xy-plane	z = 0	(x, y, 0)
yz-plane	x = 0	(0, y, z)
xz-plane	y = 0	(x, 0, z)

Table 3.1

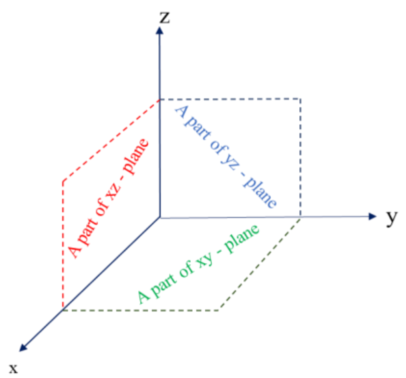


Figure 3.2

Plane	Equation	Coordinates	Example	
			Equation	Coordinates
Parallel to xy-plane	z = c	(x, y, c)	z = 1	(x, y, 1)
Parallel to yz-plane	x = a	(a, y, z)	x = 5	(5, y, z)
Parallel to zx-plane	y = b	(x, b, z)	y = 2	(x, 2, z)

Table 3.2

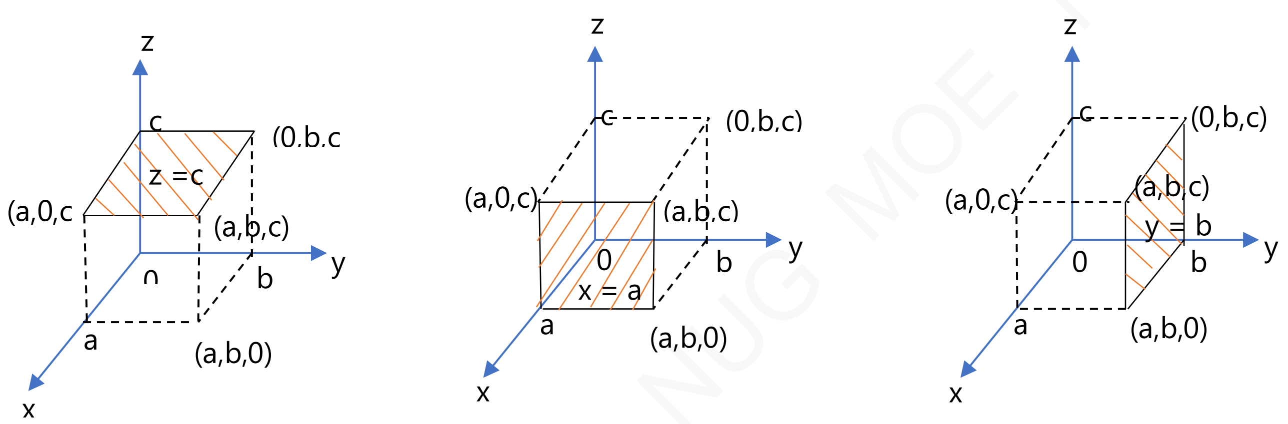


Figure 3.3

Line perpendicular to the xy-plane

Equation of any line **perpendicular** to the **xy-plane** and passing through the point (**a**, **b**, c) is **x = a**, **y = b** and the coordinates of the points on that line are of the form (**a**, **b**, z).

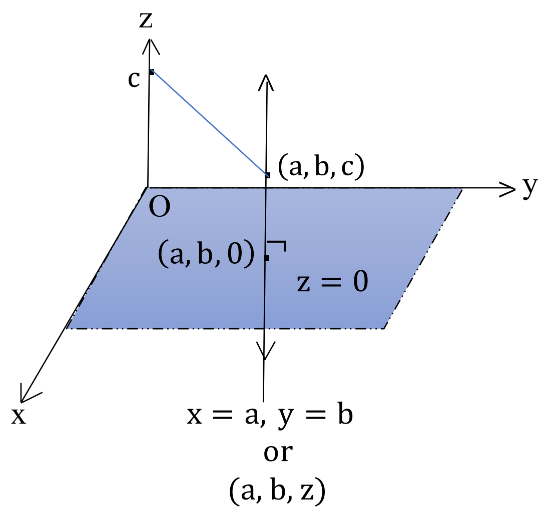


Figure 3.4

Line perpendicular to the yz-plane

Equation of any line **perpendicular** to the **yz-plane** and passing through the point (a, **b**, **c**) is **y = b**, **z = c** and the coordinates of the points on the line are of the form (x, **b**, **c**).

Line perpendicular to the zx-plane

Equation of any line **perpendicular** to the **zx-plane** and passing through the point (**a**, b, **c**) is **x = a**, **z = c** and the coordinates of the points on the line are of the form (**a**, y, **c**).

Example 1

Find the equation of the line through the point $(-3, 5, 7)$ and perpendicular to

(a) xy-plane (b) yz-plane (c) zx-plane.

Find the point of intersection of the line and plane.

Solution

(a) The equation of the line through the point $(-3, 5, 7)$ and perpendicular to xy-plane is $x = -3, y = 5$ or $(-3, 5, z)$.

The point of intersection of the line and xy-plane is $(-3, 5, 0)$.

(b) The equation of the line through the point $(-3, 5, 7)$ and perpendicular to yz-plane is $y = 5, z = 7$ or $(x, 5, 7)$.

The point of intersection of the line and yz-plane is $(0, 5, 7)$.

(c) The equation of the line through the point $(-3, 5, 7)$ and perpendicular to zx-plane is $z = 7, x = -3$ or $(-3, y, 7)$.

The point of intersection of the line and zx-plane is $(-3, 0, 7)$.

Note that

- **Equation of x-axis: $\mathbf{y = 0, z = 0}$.**
- **Equation of y-axis: $\mathbf{x = 0, z = 0}$.**
- **Equation of z-axis: $\mathbf{x = 0, y = 0}$.**

Distance between two points

The **distance** between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\,.$$

Exercise 3.1

1. Find the equation of the plane containing the point $(1, -2, 3)$ and parallel to the

   (a) xy-plane (b) yz-plane (c) zx-plane.

2. Find the equation of the line through the point $(2, 3, -4)$ and perpendicular to

   (a) xy-plane (b) yz-plane (c) zx-plane.

   Find the point of intersection of the line and plane.

3. Find the distance between the points $(2, -3, 5)$ and $(7, 5, -2)$.
4. Show that the points $(-1, 2, 5)$, $(1, 1, 6)$ and $(0, 5, 6)$ form a right triangle.
5. Show that the points $(1, -1, 2)$, $(3, -2, 3)$ and $(5, -3, 4)$ are collinear.

3.2 Lines

Directed value of a line segment

For a line segment PQ, **directed values** $\langle l, m, n\rangle$ of PQ where P is (x_1, y_1, z_1) and Q is (x_2, y_2, z_2) is defined by

$$\langle PQ\rangle = \langle l, m, n\rangle = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$$

The **length of the segment PQ** is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{l^2 + m^2 + n^2}.$$

Real numbers and points on the line

The coordinates of point R on the line PQ with respect to the point P and a real number k are $(x, y, z) = (x_1 + kl,\ y_1 + km,\ z_1 + kn)$.

The equation is called **coordinate form** of the equation of line PQ and k is called a **parameter**.

In general, if a segment of a line through (x_1, y_1, z_1) has directed values $\langle l, m, n\rangle$ which are not equal to zero, then the equation of the line can be written as

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}.$$

Since $\langle PR\rangle = \langle x_1 + kl - x_1, y_1 + km - y_1,\ z_1 + kn - z_1\rangle = \langle kl,\ km,\ kn\rangle$ and R is any point on the line PQ with $\langle PQ\rangle = \langle l, m, n\rangle$, we will defined directed values of a line PQ as $\langle kl,\ km,\ kn\rangle$ where k is a real number.

Example 2

Given $P(1,\ 2,\ 3)$ and $Q(3,\ 6,\ 5)$, find the coordinates of point $R(x, y, z)$ on the line PQ with respect to the point P and the following parameters.

(a) $k = \frac{1}{2}$ (b) $k = 2$ (c) $k = -2$.

Solution

By given $P(1,\ 2,\ 3)$ and $Q(3,\ 6,\ 5)$,

$\langle PQ\rangle = \langle l, m, n\rangle = \langle 3-1,\ 6-2,\ 5-3\rangle = \langle 2,\ 4,\ 2\rangle$.

(a) For $k = \frac{1}{2}$, $(x, y, z) = \left(1 + \frac{1}{2}(2), 2 + \frac{1}{2}(4), 3 + \frac{1}{2}(2)\right) = (2,\ 4,\ 4)$.

(b) For $k = 2$, $(x, y, z) = (1 + 2(2), 2 + 2(4), 3 + 2(2)) = (5,\ 10,\ 7)$.

(c) For $k = -2$, $(x, y, z) = (1 + (-2)(2), 2 + (-2)(4), 3 + (-2)(2))$

$= (-3, -6, -1)$.

Example 3

Given $P(-1,\ 2,\ 3)$ and $Q(\ 3,\ 5, -2)$ determine whether or not the following points are on the line PQ. If the point is on the line PQ, find the corresponding parameter with respect to the point P.

(a) $\left(1, \frac{7}{2}, \frac{1}{2}\right)$ (b) $(\ 7,\ 8, -7)$ (c) $(-5, -1,\ 8)$ (d) $(\ 7,\ 8, -2)$

Solution

By given $P(-1,\ 2,\ 3)$ and $Q(\ 3,\ 5, -2)$,

$\langle PQ\rangle = \langle l, m, n\rangle = \langle 3+1,\ 5-2, -2-3\rangle = \langle 4,\ 3, -5\rangle$.

Thus, the equation of the line PQ is $\frac{x+1}{4} = \frac{y-2}{3} = \frac{z-3}{-5}$.

(a) If $(x, y, z) = \left(1, \frac{7}{2}, \frac{1}{2}\right)$, then $\frac{x+1}{4} = \frac{1+1}{4} = \frac{1}{2}$

$$\frac{y-2}{3} = \frac{\frac{7}{2}-2}{3} = \frac{1}{2}$$

$$\frac{z-3}{-5} = \frac{\frac{1}{2}-3}{-5} = \frac{1}{2}.$$

So $\frac{x+1}{4} = \frac{y-2}{3} = \frac{z-3}{-5}$ for $\left(1, \frac{7}{2}, \frac{1}{2}\right)$.

Hence the point $\left(1, \frac{7}{2}, \frac{1}{2}\right)$ is on the line PQ with corresponding parameter is $\frac{1}{2}$.

(b) If $(x, y, z) = (7, 8, -7)$, then $\frac{x+1}{4} = \frac{7+1}{4} = 2$

$$\frac{y-2}{3} = \frac{8-2}{3} = 2$$

$$\frac{z-3}{-5} = \frac{-7-3}{-5} = 2.$$

So $\frac{x+1}{4} = \frac{y-2}{3} = \frac{z-3}{-5}$ for $(7, 8, -7)$.

Hence the point $(7, 8, -7)$ is on the line PQ with corresponding parameter is 2.

(c) If $(x, y, z) = (-5, -1, 8)$, then $\frac{x+1}{4} = \frac{-5+1}{4} = -1$

$$\frac{y-2}{3} = \frac{-1-2}{3} = -1$$

$$\frac{z-3}{-5} = \frac{8-3}{-5} = -1.$$

So $\frac{x+1}{4} = \frac{y-2}{3} = \frac{z-3}{-5}$ for $(-5, -1, 8)$.

Hence the point $(-5, -1, 8)$ is on the line PQ with corresponding parameter is -1.

(d) If $(x, y, z) = (7, 8, -2)$, then $\frac{x+1}{4} = \frac{7+1}{4} = 2$

$$\frac{y-2}{3} = \frac{8-2}{3} = 2$$

$$\frac{z-3}{-5} = \frac{-2-3}{-5} = 1.$$

So $\frac{x+1}{4} = \frac{y-2}{3} \neq \frac{z-3}{-5}$ for $(7, 8, -2)$.

Hence the point $(7, 8, -2)$ is not on the line PQ.

Example 4

Given $P(2, 1, 3)$ and $Q(6, -5, 3)$, determine whether or not the following points are on the line PQ, if the point is on the line PQ, find the corresponding parameter with respect to the point P.

(a) $(4, -2, 3)$ (b) $(-2, 7, 3)$ (c) $(10, -11, 3)$ (d) $(1, 1, 3)$

Solution

By given $P(2, 1, 3)$ and $Q(6, -5, 3)$,

$$\langle PQ \rangle = \langle l, m, n \rangle = \langle 6-2, -5-1, 3-3 \rangle = \langle 4, -6, 0 \rangle.$$

Then the coordinates of the point (x, y, z) on the line PQ are

$$(x, y, z) = (2+4k, 1-6k, 3).$$

This means that the line PQ is on the plane $z = 3$.

(a) If $(x, y, z) = (4, -2, 3)$, then $(4, -2, 3) = (2+4k, 1-6k, 3)$

$\therefore 2 + 4k = 4, \quad 1 - 6k = -2$

$\therefore k = \frac{1}{2}.$

Thus the point $(4, -2, 3)$ is on the line PQ with corresponding parameter $\frac{1}{2}$.

(b) If $(x, y, z) = (-2, 7, 3)$, then $(-2, 7, 3) = (2+4k, 1-6k, 3)$

$\therefore 2 + 4k = -2, \quad 1 - 6k = 7$

$\therefore k = -1.$

Thus the point $(-2, 7, 3)$ is on the line PQ with corresponding parameter -1.

(c) If $(x, y, z) = (10, -11, 3)$, then $(10, -11, 3) = (2+4k, 1-6k, 3)$

$\therefore 2 + 4k = 10, \quad 1 - 6k = -11$

$\therefore k = 2.$

Thus, the point $(10, -11, 3)$ is on the line PQ with corresponding parameter 2.

(d) If $(x, y, z) = (1, 1, 3)$, then $(1, 1, 3) = (2+4k, 1-6k, 3)$

$\therefore 2 + 4k = 1, \quad 1 - 6k = 1$

$\therefore k = -\frac{1}{4}, \quad k = 0.$

There is no value k that satisfies this condition. Therefore, the point $(1, 1, 3)$ is not the line PQ.

Exercise 3.2

1. Given $P(3,\ 1,\ 5)$ and $Q(-3,7,-2)$, find the coordinates of the point $R(x,\ y,\ z)$ on the line PQ with respect to the point P and the following parameters.

(a) $k=\frac{1}{2}$ (b) $k=3$ (c) $k=-2$

2. Given $P(-2,\ 1,\ 3)$ and $Q(4,4,-3)$, determine whether or not the following points are on the line PQ. If the point is on the line PQ, find the corresponding parameter with respect to the point P.

(a) $(6,3,-6)$ (b) $(6,5,\ -5)$ (c) $(-4,\ 0,\ 5)$ (d) $(7,\ 8,-2)$

3. Given $P(-2,\ 1,\ 3)$ and $Q(4,4,-3)$, determine whether or not the following points are on the line PQ. If the point is on the line PQ, find the corresponding real number with respect to point P.

(a) $(5,\ 2,\ 11)$ (b) $(2,\ 2,-7)$ (c) $\left(\frac{7}{2},\ 2,\ 2\right)$ (d) $(6,\ 2,\ 10)$

4. Find the points of intersection of the line joining the two points $(2,\ 4,\ 5)$ and $(3,\ 5,-4)$ with the following planes.

(a) xy-plane (b) yz-plane (c) zx-plane

3.3 Parallel, Skew and Perpendicular Lines

Parallel Lines

Two lines are **parallel** if and only if their directed values are **multiples of each other** by some real number.

Skew Lines

In space, there are pairs of lines that are neither **parallel** nor **intersect**. These pairs of lines are called **skew lines**.

The Measure of $\angle$PAQ

Consider $P(x_1, y_1, z_1)$, $A(a, b, c)$ and $Q(x_2, y_2, z_2)$.

$$\langle AP \rangle = \langle l_1, m_1, n_1 \rangle = \langle x_1 - a,\ y_1 - b,\ z_1 - c \rangle$$

$$\langle AQ \rangle = \langle l_2, m_2, n_2 \rangle = \langle x_2 - a,\ y_2 - b,\ z_2 - c \rangle$$

$$\langle PQ \rangle = \langle l_3, m_3, n_3 \rangle = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$$

$$= \langle l_2 - l_1,\ m_2 - m_1,\ z_2 - z_1 \rangle$$

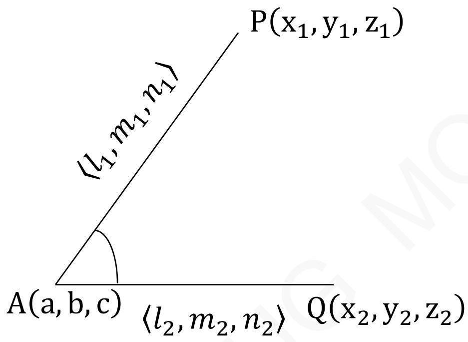


Figure 3.5

$$AP^2 + AQ^2 - PQ^2 = l_1^2 + m_1^2 + n_1^2 + l_2^2 + m_2^2 + n_2^2 - (l_2 - l_1)^2 + (m_2 - m_1)^2 + (n_2 - n_1)^2$$

$$= 2(l_1 l_2 + m_1 m_2 + n_1 n_2).$$

By the law of cosines, $\cos \angle PAQ = \frac{AP^2 + AQ^2 - PQ^2}{2AP.\ AQ}$

$$= \frac{2(l_1 l_2 + m_1 m_2 + n_1 n_2)}{2AP\,.\,AQ}$$

$$= \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}$$

If $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$, then $\cos \angle PAQ = 0$ and hence $\angle PAQ = 90^{\circ}$.

Perpendicular Lines

Two lines are **perpendicular** if and only if they are intersect and $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ for any directed values $\langle l_1, m_1, n_1 \rangle$ and $\langle l_2, m_2, n_2 \rangle$ of the lines.

Example 5

Given $P(2,\ 1,\ 3)$, $Q(6, -5,\ 4)$, $R(2,\ 3,\ 4)$ and $S(-1,\ 5,\ 1)$, determine whether the lines PQ and RS are parallel or skew or intersect.

Solution

For $P(2,\ 1,\ 3)$ and $Q(6, -5,\ 4)$, $\langle PQ\rangle = \langle 4, -6,\ 1\rangle$.

For $R(2,\ 3,\ 4)$ and $S(-1,\ 5,\ 1)$, $\langle RS\rangle = \langle -3,\ 2, -3\rangle$.

Since $\frac{4}{-3} \neq \frac{-6}{2} \neq \frac{1}{-3}$, directed values of $\langle PQ\rangle$ are not multiple of $\langle RS\rangle$.

So, two lines are not parallel.

If a point (x, y, z) is on the lines PQ and RS, then

$$x = 2 + 4s \qquad x = 2 - 3t$$

$$y = 1 - 6s \qquad y = 3 + 2t$$

$$z = 3 + s \qquad z = 4 - 3t$$, for real numbers s and t.

Thus, $2 + 4s = 2 - 3t$

$$1 - 6s = 3 + 2t$$

$$3 + s = 4 - 3t$$

Solving first two of these equations, we have

$$s = -\frac{3}{5} \text{ and } t = \frac{4}{5}.$$

But $3 + \left(-\frac{3}{5}\right) \neq 4 - 3\left(\frac{4}{5}\right)$ these values of *s* and *t* do not satisfy the last equation. So, the system of equations has no solution and hence the given lines do not intersect.

Therefore, the given lines are skew.

Example 6

Given $P(0,\ 0,\ 1)$, $Q(3,\ 6,\ 4)$, $R(0,\ 3,\ 1)$ and $S(3,\ 0,\ 4)$, show that the lines PQ and RS are perpendicular.

Solution

$P(0,\ 0,\ 1)$, $Q(3,\ 6,\ 4)$, $R(0,\ 3,\ 1)$, $S(3,\ 0,\ 4)$

$\langle PQ\rangle = \langle l_1, m_1, n_1\rangle = \langle 3,\ 6,\ 3\rangle$ and $\langle RS\rangle = \langle l_2, m_2, n_2\rangle = \langle 3, -3,\ 3\rangle$.

If a point (x, y, z) is on the lines PQ and RS, then

$$x = 0 + 3s \qquad x = 0 + 3t$$

$$y = 0 + 6s \qquad y = 3 - 3t$$

$$z = 1 + 3s \qquad z = 1 + 3t,$$ for real numbers s and t.

$$3s = 3t \Rightarrow s = t$$

$$6s = 3 - 3t \Rightarrow 6s = 3 - 3s \Rightarrow s = \frac{1}{3}$$

$$1 + 3s = 1 + 3t$$

$$\therefore\ t = \frac{1}{3}$$

$$\therefore\ x = 3\left(\frac{1}{3}\right) = 1, \qquad y = 6\left(\frac{1}{3}\right) = 2, \qquad z = 1 + 3\left(\frac{1}{3}\right) = 2.$$

The point of intersection is $(1, 2, 2)$ and two lines intersect.

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = (3)(3) + (6)(-3) + (3)(3) = 0$$

Hence PQ and RS are perpendicular.

Example 7

Find the equation of the line passing through the point $(-4, 7, -3)$ and perpendicular to the line $(x, y, z) = (3 + 2k, -1 + 3k, 1 - k)$. Find also the point of intersection of two lines.

Solution

Directed values of the given line are $\langle 2, 3, -1 \rangle$.

Directed values of the require line are

$$\langle -4 - (3 + 2k),\ 7 - (-1 + 3k),\ -3 - (1 - k) \rangle$$

$$= \langle -7 - 2k,\ 8 - 3k,\ -4 + k \rangle,$$ for some real number k.

If two lines are perpendicular, then

$$2(-7-2k)+3(8-3k)+(-1)(-4+k)=0$$

$$-14-4k+24-9k+5-k=0$$

$$14k=14$$

$$k=1.$$

So, directed values of required line are $\langle -7-2(1),\ 8-3(1),\ -4+1\rangle = \langle -9,\ 5, -3\rangle$

and the equation of the line is $(x, y, z) = (-4-9t,\ 7+5t,\ -3-3t)$.

The point of intersection is $(x, y, z) = (5,\ 2,\ 0)$.

Exercise 3.3

1. Find $\cos\angle PAQ$ for the followings.

(a) $P(1,\ 2, -1)$, $A(-2,\ 1,\ 5)$, $Q(2, -1,\ 0)$

(b) $P(0,\ 2, -3)$, $A(2, -1,\ 5)$, $Q(-2,\ 3, -1)$

2. Determine whether the lines PQ and RS are parallel or skew or intersect. If PQ and RS intersect, are they perpendicular?

(a) $P\ (1,\ 2,\ 3), Q(4,\ 5,\ 6), R(-2,\ 3,\ 5), S(4,\ 9,\ 11)$

(b) $P\ (3, -1, -3), Q(2, -3,\ 1), R(3, -2,\ 5), S(-1, -2,\ 1)$

(c) $P\ (4, -2,\ 5), Q(-2,\ 6,\ 1), R(-1,\ 1,\ 4), S(3,\ 3,\ 2)$

(d) $P\ (-3, -1, 6)$, $Q\ (-1, 3, 0)$, $R\ (0, 6, 7)$, $S\ (-4, -4, -1)$

3. Find the equation of the line passing through the point $(8, -1, -10)$ and perpendicular to the line $(x, y, z) = (1+2k, 2-k, 3-7k)$. Find also the point of intersection of two lines.

3.4 Planes

A **plane** is determined by three points which are not on the same line.

Let P (x, y, z) be any point on the plane through A, B, C. Since A, B, C are not on the same line, line segment joining any two points will intersect each other. Let AB intersect AC at A. Draw a line through P parallel to AC.

This line will meet AB at R $(x_1 + sl_1, y_1 + sm_1, z_1 + sn_1)$ for some parameter s. As shown in the given figure, coordinates of any point $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ on the plane are

$$x = x_1 + sl_1 + tl_2$$
$$y = y_1 + sm_1 + tm_2$$
$$z = z_1 + sn_1 + tn_2,$$

for some parameter t.

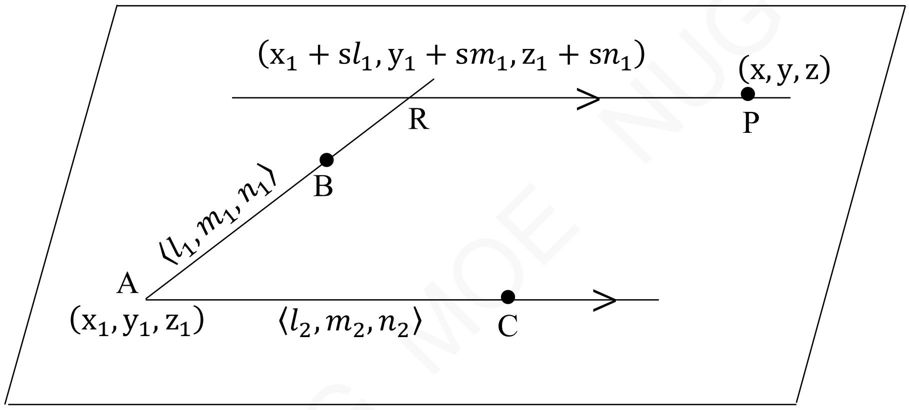
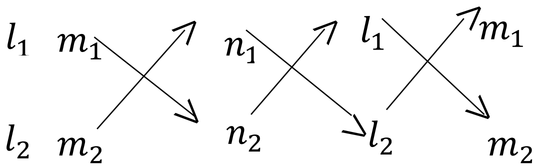


Figure 3.6

Let $a = m_1n_2 - m_2n_1$

$b = n_1l_2 - n_2l_1$

$c = l_1m_2 - l_2m_1.$



Then $al_1 + bm_1 + cn_1 = 0$ and $al_2 + bm_2 + cn_2 = 0$.

Thus, the plane equation is

$$ax + by + cz = ax_1 + by_1 + cz_1.$$

Cartesian form: $ax + by + cz = d$ where $d = ax_1 + by_1 + cz_1$.

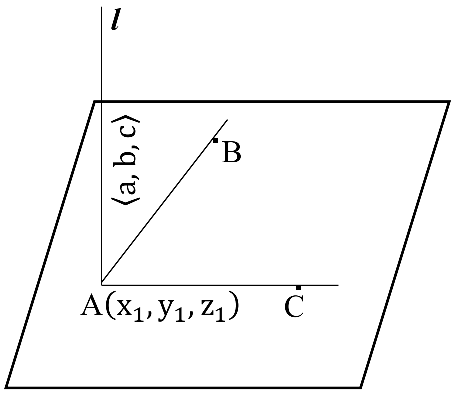


Figure 3.7

The line ***l*** with equation $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

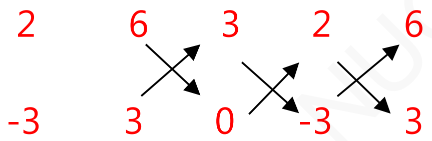
is perpendicular to both of the lines AB and AC, so the line ***l*** is perpendicular to the plane ABC. Hence any line with directed values $\langle ka, kb,\ kc\rangle$, for some parameter k, is perpendicular to the plane ABC.

Example 8

Find the equation of the plane containing $A(1,\ 0,\ 1)$, $B(3,\ 6,\ 4)$ and $C(-2,\ 3,\ 1)$.

Solution

$A(1,\ 0,\ 1),\ B(3,\ 6,\ 4), C(-2,\ 3,\ 1)$



$\langle AB\rangle = \langle 2,\ 6,\ 3\rangle$

$\langle AC\rangle = \langle -3,\ 3,\ 0\rangle$

$\therefore\ 2a + 6b + 3c = 0$

$-3a + 3b + 0c = 0$

$a = 6(0) - 3(3) = -9$

$b = 3(-3) - 0(2) = -9$

$c\ = 2(3) - (-3)6 = 24$

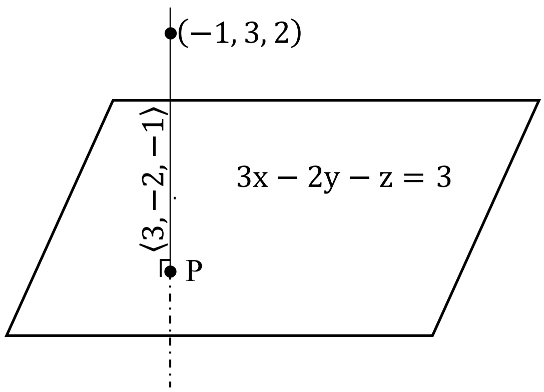
$d = ax_1 + by_1 + cz_1 = -9(1) + (-9)(0) + 24(1) = 15$

$\therefore$ The equation of the plane is $-9x - 9y + 24z = 15$ or $3x + 3y - 8z = -5$.

Example 9

Find the equation of the line that passes through the point $(-1,\ 3,\ 2)$ and perpendicular to the plane $3x - 2y - z = 3$. Find the point of intersection of the line and the given plane.

Solution



$\langle a, b, c\rangle = \langle 3,\ -2,\ -1\rangle$.

The equation of the line is

$$\frac{x-(-1)}{3} = \frac{y-3}{-2} = \frac{z-2}{-1}$$

$$\frac{x+1}{3} = \frac{y-3}{-2} = \frac{z-2}{-1} = k \quad \text{(say)}$$

$\therefore\ x = 3k - 1, \quad y = -2k + 3, \quad z = -k + 2$

$3x - 2y - z = 3$

$3(3k - 1) - 2(-2k + 3) - (-k + 2) = 3$

$$9k - 3 + 4k - 6 + k - 2 = 3$$

$$14k = 14$$

$$k = 1.$$

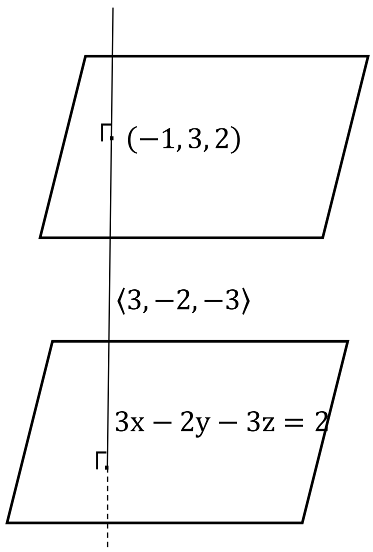
$\therefore \quad x = 3 - 1 = 2, \quad y = -2 + 3 = 1, \quad z = -1 + 2 = 1.$

Hence the point of intersection is $(2,\ 1,\ 1)$.

Example 10

Find the equation of the plane containing the point $(-1,\ 3,\ 2)$ and parallel to the plane $3x - 2y - 3z = 2$.

Solution



$\langle a, b, c\rangle = \langle 3,\ -2,\ -3\rangle$.

Thus the equation of the required plane is

$$3x - 2y - 3z = d.$$

The point $(-1,\ 3,\ 2)$ lies on the required plane.

$\therefore \quad 3(-1) - 2(3) - 3(2) = d$

$$\therefore \quad d = -15.$$

The equation of the required plane is

$$3x - 2y - 3z = -15.$$

Exercise 3.4

1. Find the equation of the plane containing

   (a) $A(2, -5, 4)$, $B(-5, 2, 4)$ and $C(-2, 3, -1)$

   (b) $A(4, 2, -3)$, $B(1, -2, 4)$ and $C(-1, 0, 3)$.

2. Find the equation of the line passing through the point $(3, -2, -2)$ and perpendicular to the plane $-2x + 3y - z = 4$. Find the point of intersection of the line and the plane.

3. Find the equation of the plane containing the point $(2, 3, -1)$ and parallel to the plane $-2x + y + 3z = 6$.

3.5 Sphere

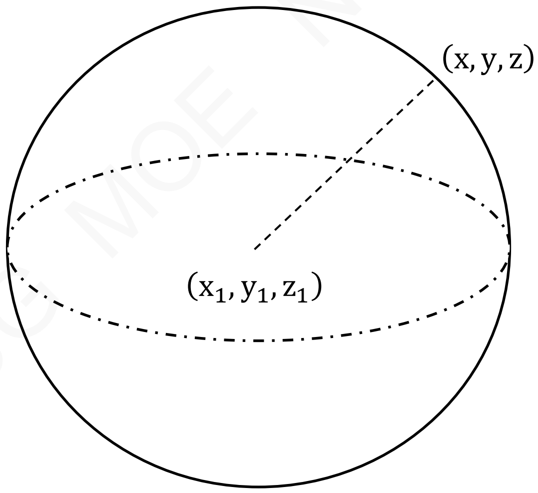


Figure 3.8

The distance between center (x_1, y_1, z_1) and any point (x, y, z) of a **sphere** is **radius r**.

$$r = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}$$

The equation of the sphere with center (x_1, y_1, z_1) and radius r is

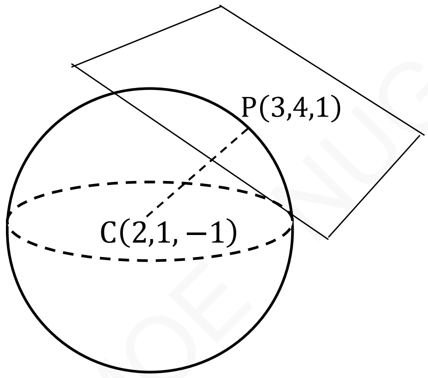
$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$$

Example 11

Find the equation of the plane tangent to the sphere

$(x-2)^2+(y-1)^2+(z+1)^2=14$ at the point $(3,\ 4,\ 1)$.

Solution



$\langle CP\rangle = \langle 1,\ 3,\ 2\rangle$

The equation of the plane is

$$x+3y+2z=d.$$

Since $P(3,\ 4,\ 1)$ is on the plane, so we get

$$3+3(4)+2(1)=d$$

$$d=17.$$

The equation of the plane is $x+3y+2z=17$.

Example 12

Find the equation of the sphere with center $(0,1,0)$ and touching the plane

$x-2y+2z+5=0$.

Solution

The equation of the line that passes through the center $C(0,\ 1,\ 0)$ and perpendicular to the plane $x-2y+2z+5=0$ is

$$\frac{x-0}{1}=\frac{y-1}{-2}=\frac{z-0}{2}$$

$$\frac{x}{1}=\frac{y-1}{-2}=\frac{z}{2}=s \quad \text{(say)}$$

$\therefore \quad x=s, \qquad y=-2s+1, \qquad z=2s.$

If one of these points P is and the plane, then

$$s-2(1-2s)+2(2s)+5=0$$

$$s-2+4s+4s+5=0$$

$$9s = -3$$

$$s = -\frac{1}{3}$$

$\therefore \quad x = -\frac{1}{3}, \qquad y = -2\left(-\frac{1}{3}\right) + 1 = \frac{5}{3}, \qquad z = 2\left(-\frac{1}{3}\right) = -\frac{2}{3}$.

The point of intersection $= \left(-\frac{1}{3}, \frac{5}{3}, -\frac{2}{3}\right)$.

$$\text{Radius} = CP = \sqrt{\left(-\frac{1}{3} - 0\right)^2 + \left(\frac{5}{3} - 1\right)^2 + \left(-\frac{2}{3} - 0\right)^2}$$

$$= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}}$$

$$= 1.$$

The equation of the sphere is $(x - 0)^2 + (y - 1)^2 + (z - 0)^2 = 1$

$$x^2 + (y - 1)^2 + z^2 = 1.$$

Example 13

Find the equation of a sphere that passes through the points $(9, 0, 0)$, $(3, 13, 5)$ and $(11, 0, 10)$, given that its center lies on the yz-plane.

Solution

The equation of the sphere with center (x_1, y_1, z_1) and radius r is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2.$$

$(x - 0)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$, since its center lies on the yz-plane.

At $(9, 0, 0)$, $\quad (9 - 0)^2 + (0 - y_1)^2 + (0 - z_1)^2 = r^2$

$81 + y_1^2 + z_1^2 = r^2$ -------- (1)

At $(3, 13, 5)$, $\quad (3 - 0)^2 + (13 - y_1)^2 + (5 - z_1)^2 = r^2$

$9 + (13 - y_1)^2 + (5 - z_1)^2 = r^2$ ------------ (2)

At $(11, 0, 10)$, $\quad (11 - 0)^2 + (0 - y_1)^2 + (10 - z_1)^2 = r^2$

$121 + y_1^2 + (10 - z_1)^2 = r^2$ ----------- (3)

From (1) and (3),

$$81 + y_1^2 + z_1^2 = 121 + y_1^2 + (10 - z_1)^2$$

$$81 + z_1^2 = 121 + 100 - 20z_1 + z_1^2$$

$$20z_1 = 140$$

$$z_1 = 7.$$

From (1) and (2),

$$81 + y_1^2 + z_1^2 = 9 + (13 - y_1)^2 + (5 - z_1)^2$$

$$81 + y_1^2 + 7^2 = 9 + 169 - 26y_1 + y_1^2 + (5 - 7)^2$$

$$81 + 49 = 178 - 26y_1 + 4$$

$$130 = 182 - 26y_1$$

$$26y_1 = 52$$

$$y_1 = 2.$$

$\therefore$ Center $= (0, 2, 7)$

Substitute $x_1 = 0$, $y_1 = 2$ and $z_1 = 7$ in equation (1),

$$81 + (2)^2 + 7^2 = r^2$$

$$81 + 4 + 49 = r^2$$

$$r^2 = 134.$$

Hence the equation of the sphere is

$$x^2 + (y - 2)^2 + (z - 7)^2 = 134.$$

Exercise 3.5

1. Find the equation of the sphere with center C and radius r.

   (a) $C(1, -2, 4)$, $r = 3$ (b) $C(2, 6, -3)$, $r = 2$ (c) $C(2, 3, 5)$, $r = 5$

2. Check whether the given point P lies inside, outside or on a sphere.

   (a) Center $C(0, 0, 0)$, radius $r = 3$ and point $P(1, 1, 1)$.

   (b) Center $C(0, 0, 0)$, radius $r = 3$ and point $P(2, 1, 2)$.

   (c) Center $C(0, 0, 0)$, radius $r = 3$ and point $P(10, 10, 10)$.

3. Find the equation of the sphere on the join of $(1, -1, 1)$ and $(-3, 4, 5)$ as diameter.

4. Find the equation of the plane tangent to the sphere

   $(x + 2)^2 + (y - 1)^2 + (z + 3)^2 = 27$ at the point $(3, 2, -2)$.

5. Find the equation of the sphere with center $(6, -7, -3)$ and touching the plane

   $4x - 2y - z = 17$.

6. What is the equation of the sphere which passes through the points

   $(3, 0, 2)$, $(-1, 1, 1)$ and $(2, -5, 4)$ and whose center lies on the plane

   $2x + 3y + 4z = 6$?

Chapter 4

VECTOR ALGEBRA

We have learned vectors in two-dimensional rectangular coordinate system. In this chapter, we will extend the system to three dimensions and we will use vectors to calculate angles, areas of triangle and parallelogram. We will also applied vectors to geometrical problems of lines and planes.

4.1 Vectors in Three Dimensions

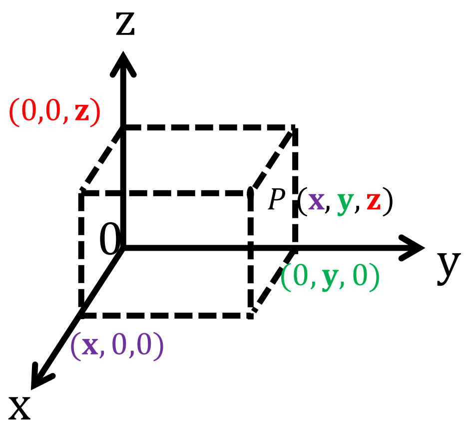


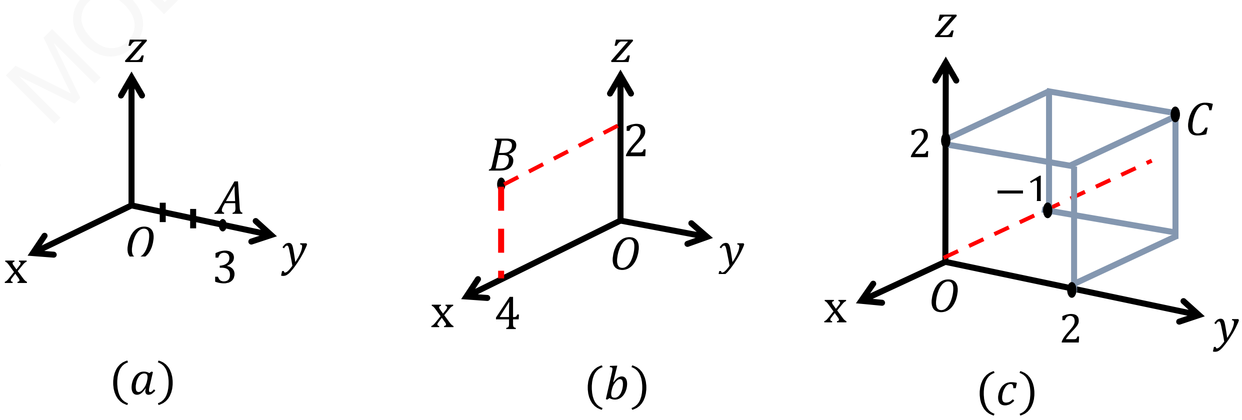
Figure 4.1

In space, each point is associated with an ordered triple of real numbers. Through a fixed point (**origin** O), draw three mutually perpendicular lines the x-axis, the y-axis and the z-axis. A point P in space is determined by $P(x, y, z)$ as shown in the diagram. Those numbers x, y, z are called the **coordinates** of P.

Example 1

Illustrate the points (a) $A(0, 3, 0)$ (b) $B(4, 0, 2)$ (c) $C(-1, 2, 2)$.

Solution



Standard Unit Vectors in Three Dimensions

$$\hat{\imath} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \hat{\jmath} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \hat{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

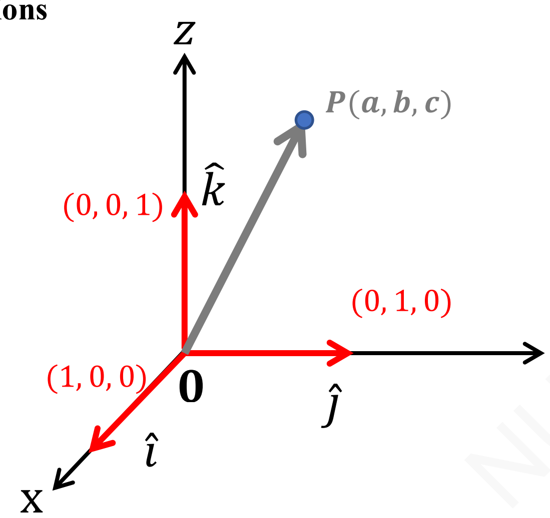


Figure 4.2

Position Vectors in Three Dimensions

A vector whose initial point at the origin is called a position vector.

$$\overrightarrow{OP} = \vec{p} = \begin{pmatrix}a\\b\\c\end{pmatrix} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$$

Magnitude of a vector $\overrightarrow{\mathbf{OP}} = |\overrightarrow{\mathbf{OP}}| = |\vec{\mathbf{p}}| = \sqrt{\mathbf{a^2 + b^2 + c^2}}$.

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then

$$\overrightarrow{\mathbf{AB}} = \overrightarrow{\mathbf{OB}} - \overrightarrow{\mathbf{OA}} = \begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix} - \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} = \begin{pmatrix}\mathbf{x_2 - x_1}\\\mathbf{y_2 - y_1}\\\mathbf{z_2 - z_1}\end{pmatrix}$$

and $\overrightarrow{AB}$ is called the position vector of B relative to A.

Example 2

If P is $(-3,1,2)$ and Q is $(1,-1,3)$, find:

(a) $\overrightarrow{OP}$ (b) $\overrightarrow{PQ}$ (c) $|\overrightarrow{PQ}|$ (d) $\overrightarrow{QP}$ (e) $|\overrightarrow{QP}|$.

Solution

(a) $\overrightarrow{OP} = \begin{pmatrix}-3\\1\\2\end{pmatrix}$.

(b) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}1\\-1\\3\end{pmatrix} - \begin{pmatrix}-3\\1\\2\end{pmatrix} = \begin{pmatrix}1+3\\-1-1\\3-2\end{pmatrix} = \begin{pmatrix}4\\-2\\1\end{pmatrix}$.

(c) $|\overrightarrow{PQ}| = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$.

(d) $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3-1 \\ 1+1 \\ 2-3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ -1 \end{pmatrix}$.

(e) $|\overrightarrow{QP}| = \sqrt{(-4)^2 + 2^2 + (-1)^2} = \sqrt{21}$.

Negative Vector ; The vector $-\vec{a}$ has the same length as the vector $\vec{a}$ but the opposite direction.

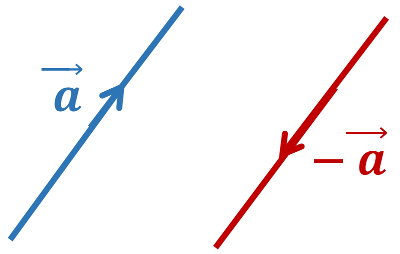


Figure 4.3

for example, if $\vec{a} = \begin{pmatrix} 11 \\ -6 \\ 5 \end{pmatrix}$, then $-\vec{a} = -\begin{pmatrix} 11 \\ -6 \\ 5 \end{pmatrix} = \begin{pmatrix} -11 \\ 6 \\ -5 \end{pmatrix}$.

Zero Vector ; The zero vector is denoted by $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

The Rules for Algebra with Vectors

If $\vec{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, then

1. $\vec{a} + \vec{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$.

2. $\vec{a} - \vec{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$.

3. $k\,\vec{a} = k\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} k\,x_1 \\ k\,y_1 \\ k\,z_1 \end{pmatrix}$ for any scalar k.

Example 3

If $\overrightarrow{p}=\begin{pmatrix}1\\-1\\4\end{pmatrix}$ and $\overrightarrow{q}=\begin{pmatrix}-2\\0\\2\end{pmatrix}$, find:

(a) $\overrightarrow{p}+\overrightarrow{q}$ (b) $\overrightarrow{p}-\frac{1}{2}\overrightarrow{q}$ (c) $\frac{3}{2}\overrightarrow{q}-\overrightarrow{p}$.

Solution

(a) $\overrightarrow{p}+\overrightarrow{q}=\begin{pmatrix}1\\-1\\4\end{pmatrix}+\begin{pmatrix}-2\\0\\2\end{pmatrix}=\begin{pmatrix}1-2\\-1+0\\4+2\end{pmatrix}=\begin{pmatrix}-1\\-1\\6\end{pmatrix}$.

(b) $\overrightarrow{p}-\frac{1}{2}\overrightarrow{q}=\begin{pmatrix}1\\-1\\4\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-2\\0\\2\end{pmatrix}=\begin{pmatrix}1+1\\-1-0\\4-1\end{pmatrix}=\begin{pmatrix}2\\-1\\3\end{pmatrix}$.

(c) $\frac{3}{2}\overrightarrow{q}-\overrightarrow{p}=\frac{3}{2}\begin{pmatrix}-2\\0\\2\end{pmatrix}-\begin{pmatrix}1\\-1\\4\end{pmatrix}=\begin{pmatrix}-3-1\\0+1\\3-4\end{pmatrix}=\begin{pmatrix}-4\\1\\-1\end{pmatrix}$.

Equal Vectors

Two vectors are equal if they have the same magnitude and direction.

If $\overrightarrow{a}=\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$ and $\overrightarrow{b}=\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}$, then

$$\overrightarrow{a}=\overrightarrow{b} \quad \Leftrightarrow \quad x_1=x_2,\ y_1=y_2,\ z_1=z_2.$$

Parallel Vectors

Two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel $\Leftrightarrow$ $\overrightarrow{a}=k\,\overrightarrow{b}$ for some scalar k.

Note that equal vectors are parallel and equal in length.

Example 4

Find u and v given that $\overrightarrow{a}=\begin{pmatrix}-1\\-1\\u\end{pmatrix}$ is parallel to $\overrightarrow{b}=\begin{pmatrix}v\\2\\-2\end{pmatrix}$.

Solution

Since $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel, we have $\overrightarrow{a}=k\,\overrightarrow{b}$ for some scalar k.

$$\begin{pmatrix}-1\\-1\\u\end{pmatrix}=k\begin{pmatrix}v\\2\\-2\end{pmatrix}$$

$-1=kv$, $\quad -1=2k$ $\quad$ and $\quad u=-2k$.

We have $k=-\frac{1}{2}$. Thus $u=1$ and $v=2$.

Example 5

ABCD is a parallelogram. A is $(-1, 1, 1)$, B is $(2, 0, -2)$ and D is $(3, 1, 4)$. Find the coordinates of C.

Solution

Let C be (x, y, z).

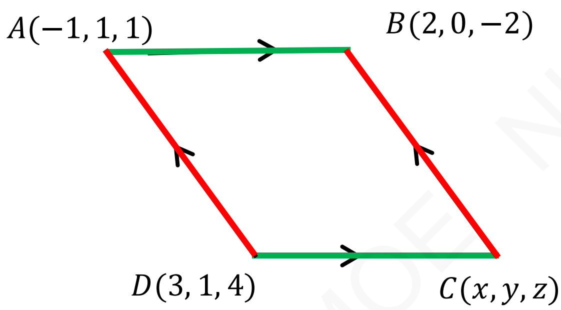
Since ABCD is a parallelogram, AB // DC, and they have the same length, so

$$\overrightarrow{DC} = \overrightarrow{AB}$$

$$\overrightarrow{OC} - \overrightarrow{OD} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x-3 \\ y-1 \\ z-4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix}$$



$$x - 3 = 3, \quad y - 1 = -1, \quad z - 4 = -3.$$

$$x = 6, \quad y = 0, \quad z = 1.$$

Therefore C is $(6, 0, 1)$.

Unit Vector ; A vector magnitude 1 is called a unit vector. For a nonzero vector $\vec{a}$, the unit vector in the direction of $\vec{a}$, denoted by $\hat{a}$ is given by

$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|}.$$

Example 6

(a) Find the unit vector in the same direction as $\vec{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

(b) Find a vector of magnitude 5 that is parallel to $\vec{a}$.

Solution

(a) $|\vec{a}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3.$

The unit vector of $\vec{a}$ is $\quad \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$

(b) Let $\vec{b}$ be parallel to $\vec{a}$ and $|\vec{b}| = 5$.

Then $\vec{b} = 5\,\hat{a} = \begin{pmatrix} \frac{10}{3} \\ -\frac{10}{3} \\ \frac{5}{3} \end{pmatrix}$ or $\vec{b} = -5\,\hat{a} = \begin{pmatrix} -\frac{10}{3} \\ \frac{10}{3} \\ -\frac{5}{3} \end{pmatrix}$.

Collinear Points

Three or more points are said to be collinear if they lie on the same straight line. A, B and C are collinear if $\overrightarrow{\mathbf{AB}} = \mathbf{k}\,\overrightarrow{\mathbf{BC}}$ for some scalar k.

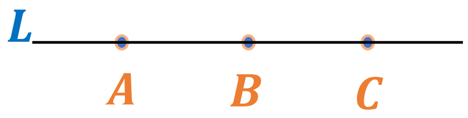


Figure 4.4

Example 7

Prove that A(8, 2, 2), C(20, 5, 5) and B(12, 3, 3) are collinear.

Solution

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 12 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 20 \\ 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 12 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = 2\,\overrightarrow{AB}.$$

Therefore BC is parallel to AB.

Since B is common to both, it follows that A, B and C are collinear.

Exercise 4.1

1. Let $\vec{a} = \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 0 \\ 7 \\ -1 \end{pmatrix}$. Find the following vectors.

(a) $3\,\vec{a}$ (b) $4\,\vec{b}$ (c) $\vec{a} - \vec{b}$ (d) $\vec{b} + \vec{c}$

(e) $2\,\vec{b} + \vec{c}$ (f) $\vec{a} - 2\,\vec{b}$ (g) $\vec{a} + \vec{b} - 2\,\vec{c}$ (h) $3\,\vec{a} - \vec{b} + \vec{c}$

2. Given vectors $\vec{a} = \begin{pmatrix}1\\2\\7\end{pmatrix}$, $\vec{b} = \begin{pmatrix}-3\\4\\2\end{pmatrix}$ and $\vec{c} = \begin{pmatrix}-2\\p\\q\end{pmatrix}$.

(a) Find the values of p and q such that $\vec{c}$ is parallel to $\vec{a}$.

(b) Find the value of scalar k such that $\vec{a} + k\vec{b}$ is parallel to vector $\begin{pmatrix}0\\10\\23\end{pmatrix}$.

3. Points A, B, C and D have position vectors $\vec{a} = \begin{pmatrix}3\\-1\\1\end{pmatrix}$, $\vec{b} = \begin{pmatrix}5\\0\\3\end{pmatrix}$, $\vec{c} = \begin{pmatrix}7\\8\\-3\end{pmatrix}$ and

$\vec{d} = \begin{pmatrix}4\\3\\-2\end{pmatrix}$ respectively. Point E is the midpoint of BC.

(a) Find the position vector of E.

(b) Show that ABED is a parallelogram.

4. Points A, B and C have position vectors $\vec{a} = \begin{pmatrix}2\\-1\\4\end{pmatrix}$, $\vec{b} = \begin{pmatrix}5\\1\\2\end{pmatrix}$ and $\vec{c} = \begin{pmatrix}3\\1\\4\end{pmatrix}$ respectively.

Find the position vector of point D such that ABCD is a parallelogram.

5. $K(1, -1, 0)$, $L(4, -3, 7)$ and $M(a, 2, b)$ are collinear. Find a and b.

4.2 Angle between Two Vectors and Scalar Product

If $\vec{a} = \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$ and $\vec{b} = \begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}$,

then scalar product (or) dot product of vector $\vec{a}$ and $\vec{b}$ is

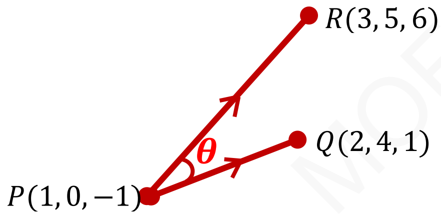
$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \mathbf{x_1 x_2 + y_1 y_2 + z_1 z_2}.$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}.$$

Example 9

Given points $P(1,0,-1)$, $Q(2,4,1)$ and $R(3,5,6)$, find $\angle QPR$.



Solution

The angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is given by θ in $\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}|\,|\overrightarrow{PR}|}$.

Since $\overrightarrow{PQ} = \begin{pmatrix}2\\4\\1\end{pmatrix} - \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\4\\2\end{pmatrix}$, we get $|\overrightarrow{PQ}| = \sqrt{1^2+4^2+2^2} = \sqrt{21}$.

Again $\overrightarrow{PR} = \begin{pmatrix}3\\5\\6\end{pmatrix} - \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}2\\5\\7\end{pmatrix}$, we get $|\overrightarrow{PR}| = \sqrt{2^2+5^2+7^2} = \sqrt{78}$.

$$\overrightarrow{PQ}\cdot\overrightarrow{PR} = \begin{pmatrix}1\\4\\2\end{pmatrix}\cdot\begin{pmatrix}2\\5\\7\end{pmatrix} = (1)(2)+(4)(5)+(2)(7) = 2+20+14 = 36.$$

$$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}|\,|\overrightarrow{PR}|} = \frac{36}{\sqrt{21}\sqrt{78}}$$

$$\theta = \cos^{-1}\frac{36}{\sqrt{21}\sqrt{78}}$$

$$\theta = 27.2^\circ.$$

Algebraic Properties of the Scalar Product

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors in space and k is a scalar, then

1. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.
2. $(-\vec{a})\cdot\vec{b} = \vec{a}\cdot(-\vec{b}) = -(\vec{a}\cdot\vec{b})$.
3. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$.
4. $(k\vec{a})\cdot\vec{b} = k(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(k\vec{b})$.
5. $\vec{0}\cdot\vec{a} = 0$.

Geometric Properties of the Scalar Product

Let $\vec{a}$ and $\vec{b}$ be nonzero vectors.

1. If $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a} \cdot \vec{b} = 0$.

2. If $\vec{a}$ and $\vec{b}$ are parallel in same direction, then

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|, \text{ in particular } \vec{a} \cdot \vec{a} = |\vec{a}|^2.$$

3. If $\vec{a}$ and $\vec{b}$ are parallel in opposite direction, then

$$\vec{a} \cdot \vec{b} = -|\vec{a}|\,|\vec{b}|.$$

Example 10

Given that $\vec{a}$ and $\vec{b}$ are perpendicular vectors such that $|\vec{a}| = 3$ and $|\vec{b}| = 1$, evaluate $(\vec{a} - \vec{b}) \cdot (\vec{a} + 5\,\vec{b})$.

Solution

Since $\vec{a}$ and $\vec{b}$ are perpendicular, so $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = 0$.

$$\begin{aligned}(\vec{a} - \vec{b}) \cdot (\vec{a} + 5\,\vec{b}) &= \vec{a} \cdot \vec{a} + 5(\vec{a} \cdot \vec{b}) - \vec{b} \cdot \vec{a} - 5(\vec{b} \cdot \vec{b}) \\ &= \vec{a} \cdot \vec{a} - 5(\vec{b} \cdot \vec{b}) \\ &= |\vec{a}|^2 - 5\,|\vec{b}|^2 \\ &= 3^2 - 5 \times 1^2 \\ &= 4.\end{aligned}$$

Example 11

Points A, B and C have position vectors $\vec{a} = k\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$.

(a) Find $\overrightarrow{BC}$.

(b) Find $\overrightarrow{AB}$ in terms of k.

(c) Find the value of k for which $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$.

Solution

(a) $\overrightarrow{BC} = \vec{c} - \vec{b} = \begin{pmatrix}1\\1\\4\end{pmatrix} - \begin{pmatrix}3\\2\\-2\end{pmatrix} = \begin{pmatrix}-2\\-1\\6\end{pmatrix}.$

(b) $\overrightarrow{AB} = \vec{b} - \vec{a} = \begin{pmatrix}3\\2\\-2\end{pmatrix} - k\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}3-2k\\2+k\\-2-k\end{pmatrix}.$

(c) Since $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$, it follows that

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$$

$$\begin{pmatrix}3-2k\\2+k\\-2-k\end{pmatrix} \cdot \begin{pmatrix}-2\\-1\\6\end{pmatrix} = 0$$

$$-6 + 4k - 2 - k - 12 - 6k = 0$$

$$-3k = 20$$

$$k = -\frac{20}{3}.$$

Exercise 4.2

1. For $\vec{a} = \begin{pmatrix}2\\1\\3\end{pmatrix}$, $\vec{b} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$ and $\vec{c} = \begin{pmatrix}0\\-1\\1\end{pmatrix}$. Find

(a) $\vec{a} \cdot \vec{b}$ (b) $\vec{b} \cdot \vec{a}$ (c) $|\vec{a}|^2$

(d) $\vec{a} \cdot \vec{a}$ (e) $\vec{a} \cdot (\vec{b} + \vec{c})$ (f) $\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.

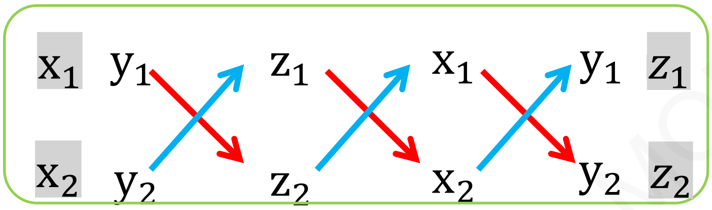
2. Find the angle between $\vec{m}$ and $\vec{n}$ if

(a) $\vec{m} = \begin{pmatrix}2\\-1\\-1\end{pmatrix}$ and $\vec{n} = \begin{pmatrix}-1\\3\\2\end{pmatrix}$ (b) $\vec{m} = 2\hat{j} - \hat{k}$ and $\vec{n} = \hat{i} + 2\hat{k}$.

4.3 Area of a Parallelogram and Vector Product

Vector Product (or) Cross Product

Let $\vec{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$.



$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{pmatrix} \mathbf{x_1} \\ \mathbf{y_1} \\ \mathbf{z_1} \end{pmatrix} \times \begin{pmatrix} \mathbf{x_2} \\ \mathbf{y_2} \\ \mathbf{z_2} \end{pmatrix} = \begin{pmatrix} \mathbf{y_1z_2 - y_2z_1} \\ \mathbf{z_1x_2 - z_2x_1} \\ \mathbf{x_1y_2 - x_2y_1} \end{pmatrix}$$

$\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \begin{pmatrix} y_1z_2 - y_2z_1 \\ z_1x_2 - z_2x_1 \\ x_1y_2 - x_2y_1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$= y_1z_2x_1 - y_2z_1x_1 + z_1x_2y_1 - z_2x_1y_1 + x_1y_2z_1 - x_2y_1z_1$$

$$= 0.$$

$$\therefore (\vec{a} \times \vec{b}) \perp \vec{a}.$$

Similarly, $(\vec{a} \times \vec{b}) \cdot \vec{b} = \begin{pmatrix} y_1z_2 - y_2z_1 \\ z_1x_2 - z_2x_1 \\ x_1y_2 - x_2y_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \cdots = 0.$

$$\therefore (\vec{a} \times \vec{b}) \perp \vec{b}.$$

Area of a parallelogram

$\boldsymbol{\alpha}(\mathbf{OACB}) = |\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$

and $\boldsymbol{\alpha}(\triangle\,\mathbf{OAB}) = \frac{1}{2}\,|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|.$

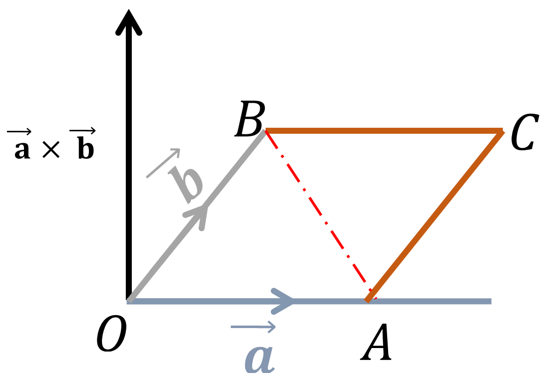


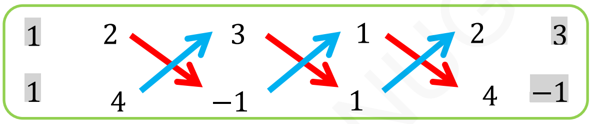
Figure 4.5

Example 12

Find the area of the parallelogram determined by the vectors

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}.$$

Solution



We can find that $\vec{a} \times \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} 2 \times (-1) - 3 \times 4 \\ 3 \times 1 - 1 \times (-1) \\ 1 \times 4 - 2 \times 1 \end{pmatrix} = \begin{pmatrix} -14 \\ 4 \\ 2 \end{pmatrix}.$$

$|\vec{a} \times \vec{b}| = \sqrt{(-14)^2 + 4^2 + 2^2} = \sqrt{196 + 16 + 4} = \sqrt{216}$.

Therefore, the area of the parallelogram is $\sqrt{216}$ unit^2.

Example 13

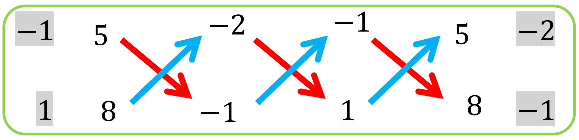
Find the area of the triangle ABC with vertices $A(1, -1, 3)$, $B(0, 4, 1)$ and $C(2, 7, 2)$.

Solution

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix}$ and

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ 7 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ -1 \end{pmatrix}.$

Then, we find their vector product



$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 8 \\ -1 \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \\ -13 \end{pmatrix}.$$

Next $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{11^2 + (-3)^2 + (-13)^2} = \sqrt{299}$.

Therefore, we have

Area of triangle ABC $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{299}$ unit^2.

Algebraic Properties of Vector Product

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors in space and k is a scalar, then

(1) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.

(2) $(k\,\vec{a}) \times \vec{b} = k\,(\vec{a} \times \vec{b})$.

(3) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$.

(4) $\vec{a} \times \vec{0} = \vec{0}$.

Geometric Properties of the Vector Product

1. If two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel, then $\vec{a} \times \vec{b} = \vec{0}$.

   In particular, $\vec{a} \times \vec{a} = \vec{0}$.

2. If two nonzero vectors $\vec{a}$ and $\vec{b}$ are perpendicular, then

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|.$$

Example 14

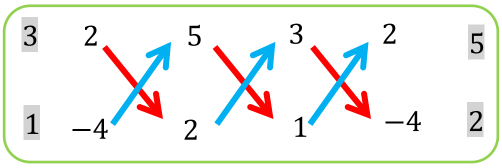
(a) Calculate $\vec{a} \times \vec{b}$ when $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 5\hat{k}$ and $\vec{b} = \hat{\imath} - 4\hat{\jmath} + 2\hat{k}$.

(b) Find a unit vector $\hat{n}$ that is perpendicular to both $\vec{a}$ and $\vec{b}$.

Solution

(a) $$\vec{a} \times \vec{b} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times 2 - 5 \times (-4) \\ 5 \times 1 - 3 \times 2 \\ 3 \times (-4) - 2 \times 1 \end{pmatrix} = \begin{pmatrix} 24 \\ -1 \\ -14 \end{pmatrix}.$$



(b) $\vec{a} \times \vec{b} = \begin{pmatrix} 24 \\ -1 \\ -14 \end{pmatrix}$. This vector is perpendicular to $\vec{a}$ and $\vec{b}$.

$$|\vec{a} \times \vec{b}| = \sqrt{24^2 + (-1)^2 + (-14)^2} = \sqrt{773}\,.$$

So, a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\hat{n} = \frac{1}{\sqrt{773}} \begin{pmatrix} 24 \\ -1 \\ -14 \end{pmatrix}$.

Example 15

Given that $|\vec{a}| = 4$, $|\vec{b}| = 5$ and that $\vec{a}$ and $\vec{b}$ are perpendicular,

evaluate $|(2\vec{a} - \vec{b}) \times (\vec{a} + 3\vec{b})|$.

Solution

$$\begin{aligned}(2\vec{a} - \vec{b}) \times (\vec{a} + 3\vec{b}) &= 2\vec{a} \times \vec{a} + 2\vec{a} \times 3\vec{b} - \vec{b} \times \vec{a} - \vec{b} \times 3\vec{b} \\ &= 2(\vec{a} \times \vec{a}) + 6(\vec{a} \times \vec{b}) - (\vec{b} \times \vec{a}) - 3(\vec{b} \times \vec{b}) \\ &= 6(\vec{a} \times \vec{b}) - (\vec{b} \times \vec{a}) \\ &= 6(\vec{a} \times \vec{b}) + (\vec{a} \times \vec{b}) \\ &= 7(\vec{a} \times \vec{b}).\end{aligned}$$

$$\begin{aligned}|(2\vec{a} - \vec{b}) \times (\vec{a} + 3\vec{b})| &= 7\,|\vec{a} \times \vec{b}| \\ &= 7\,|\vec{a}|\,|\vec{b}| \\ &= 7 \times 4 \times 5 \\ &= 140\,.\end{aligned}$$

Exercise 4.3

1. Find a vector perpendicular to the following pair of vectors:

   (a) $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (b) $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$.

2. Consider $\vec{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

   (a) Find $\vec{a} \times \vec{b}$.

   (b) Find $\sin\theta$ using $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$.

3. The points $A(3, 1, 2)$, $B(-1, 1, 5)$ and $C(7, 2, 3)$ are vertices of a parallelogram ABCD.
   (a) Find the coordinates of D.
   (b) Calculate the area of the parallelogram.

4.4 Lines and Planes in Space

Lines in Three-Dimensional Space

A **direction vector** of a straight line is a vector parallel to the line.

$R(x, y, z)$ is any point on the line and $A(a_1, a_2, a_3)$ is the **fixed point** on the line. $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is the direction vector of the line, that is $\overrightarrow{AR} = t\,\vec{b}$ for some $t \in R$.
So, $\vec{r} = \vec{a} + t\vec{b}$, $t \in R$ is the **vector equation of the line**.

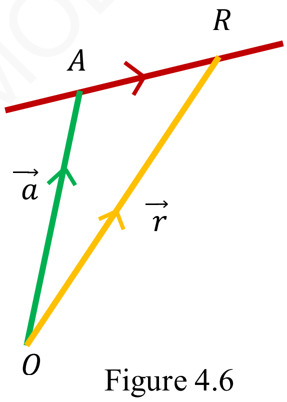


Figure 4.6

i.e., Vector Equation of the Line is

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathbf{a_1} \\ \mathbf{a_2} \\ \mathbf{a_3} \end{pmatrix} + \mathbf{t}\begin{pmatrix} \mathbf{b_1} \\ \mathbf{b_2} \\ \mathbf{b_3} \end{pmatrix}.$$

Parametric Equation

$x = a_1 + tb_1$, $y = a_2 + tb_2$, $z = a_3 + tb_3$, where $t \in R$ is the parameter.

Note : Each point on the line corresponds to exactly one value of t.

Cartesian Equation of line is

$$\frac{\mathbf{x - a_1}}{\mathbf{b_1}} = \frac{\mathbf{y - a_2}}{\mathbf{b_2}} = \frac{\mathbf{z - a_3}}{\mathbf{b_3}}.$$

Example 16

Find the Cartesian equation of the line with vector equation

$$\vec{r} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}.$$

Solution

The vector equation is $\vec{r} = \vec{a} + t\,\vec{b}$, therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}, \ t \in \mathbb{R}.$$

The parametric equations are:

$x = 1 + 3t, \quad t = \frac{x-1}{3},$

$y = 4 + 2t, \quad t = \frac{y-4}{2},$

$z = -1 + 5t, \quad t = \frac{z+1}{5}.$

The Cartesian equation of the line is $\frac{x-1}{3} = \frac{y-4}{2} = \frac{z+1}{5}$.

Example 17

Does the point $A(3, -2, 2)$ lie on the line with equation $\frac{x+1}{2} = \frac{4-y}{3} = \frac{2z}{3}$?

Solution

Consider the given Cartesian equation passing through the point $A(3, -2, 2)$.

$$\frac{x+1}{2} = \frac{3+1}{2} = 2$$

$$\frac{4-y}{3} = \frac{4+2}{3} = 2$$

$$\frac{2z}{3} = \frac{2 \times 2}{3} = \frac{4}{3}$$

and so $2 = 2 \neq \frac{4}{3}$.

The coordinates do not satisfy the Cartesian equation. Therefore, the point does not lie on the line.

Planes in Three Dimensions

Consider A, B and C be three non-collinear points on the plane with $\overrightarrow{OA} = \vec{a}$. Let $\vec{d_1}$ and $\vec{d_2}$ are two nonparallel vectors on the plane, then the position vector $\vec{r}$ of any point R on the plane is given by following equation.

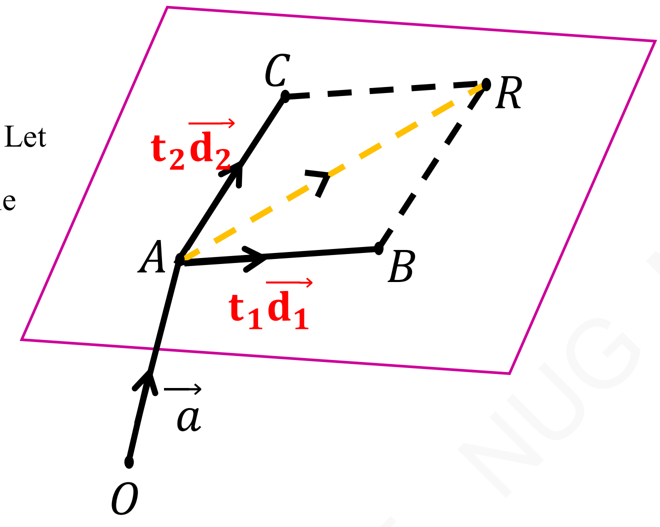


Figure 4.7

Vector Equation of the Plane is $\vec{\mathbf{r}} = \vec{\mathbf{a}} + \mathbf{t_1}\vec{\mathbf{d_1}} + \mathbf{t_2}\vec{\mathbf{d_2}}$ where $\mathbf{t_1}, \mathbf{t_2} \in$ R.

$$\overrightarrow{AR} = \mathbf{t_1}\vec{\mathbf{d_1}} + \mathbf{t_2}\vec{\mathbf{d_2}}.$$

Normal Vector

Any vector that is perpendicular to a plane is called a normal vector (normal to the plane).

$$\overrightarrow{AR} \cdot \vec{n} = 0$$

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = d$$

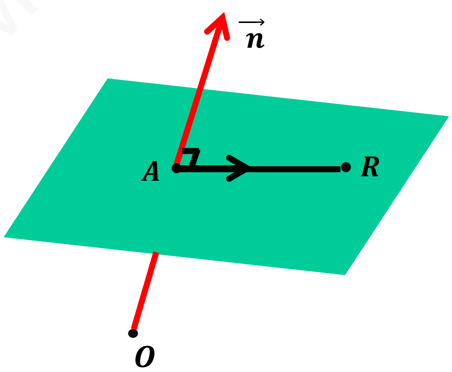


Figure 4.8

where $d = \vec{a} \cdot \vec{n}$, d is constant.

Cartesian equation of a plane : $ax + by + cz = d$ where $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is normal vector of the plane.

Example 18

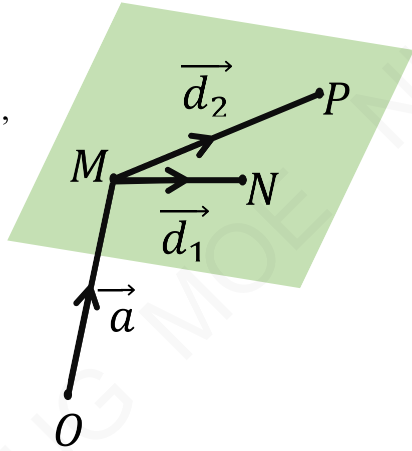
Find the vector equation of the plane containing points $M(2,2,-2)$, $N(1,-1,3)$ and $P(4,0,2)$.

Solution

Let $\vec{a} = \overrightarrow{OM} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$. We can find that

$$\vec{d_1} = \overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix},$$

$$\vec{d_2} = \overrightarrow{MP} = \overrightarrow{OP} - \overrightarrow{OM} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}.$$



Therefore, the vector equation of the plane is

$$\vec{r} = \vec{a} + t_1\vec{d_1} + t_2\vec{d_2}$$

$$= \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} + t_1\begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix} + t_2\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}.$$

Example 19

Find the Cartesian equation of the plane containing points $M(2,2,-2)$, $N(1,-1,3)$ and $P(4,0,2)$.

Solution

The equation of the plane containing M, N and P is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} + t_1\begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix} + t_2\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}.$$ [See **Example 18**]

$x = 2 - t_1 + 2t_2$ ---------- (1)

$y = 2 - 3t_1 - 2t_2$ ---------- (2)

$z = -2 + 5t_1 + 4t_2$ ---------- (3)

Add equations (1) and (2),

$$x + y = 4 - 4t_1 \qquad \text{---------- (4)}$$

Multiply equation (2) by 2 and then add (3),

$$2y + z = 2 - t_1 \qquad \text{---------- (5)}$$

From equations (4) and (5), we obtain

$$\frac{x + y - 4}{-4} = \frac{2y + z - 2}{-1}$$

$$-x - y + 4 = -8y - 4z + 8$$

$$x - 7y - 4z = -4.$$

Example 20

Determine whether points $A(3, -1, 4)$, $B(2, 1, 1)$, $C(4, 3, 1)$ and $D(-3, 1, 4)$ lie in the same plane.

Solution

The equation of the plane containing A, B and C is

$$\vec{r} = \overrightarrow{OA} + t_1\overrightarrow{AB} + t_2\overrightarrow{AC} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + t_1\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + t_2\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 - t_1 + t_2 \\ -1 + 2t_1 + 4t_2 \\ 4 - 3t_1 - 3t_2 \end{pmatrix}$$

If D is on the plane, then $\vec{r} = \overrightarrow{OD}$,

$$3 - t_1 + t_2 = -3 \qquad \text{----------- (1)}$$

$$-1 + 2t_1 + 4t_2 = 1 \qquad \text{----------- (2)}$$

$$4 - 3t_1 - 3t_2 = 4 \qquad \text{----------- (3)}$$

Multiply equation (1) by 2, we get

$$-2t_1 + 2t_2 = -12 \qquad \text{-----------} \ (4)$$

From (2) and (4), we obtain

$$t_1 = \frac{13}{3}, \ t_2 = -\frac{5}{3}.$$

Substitute the values of t_1 and t_2 in left hand side of equation (3), we get

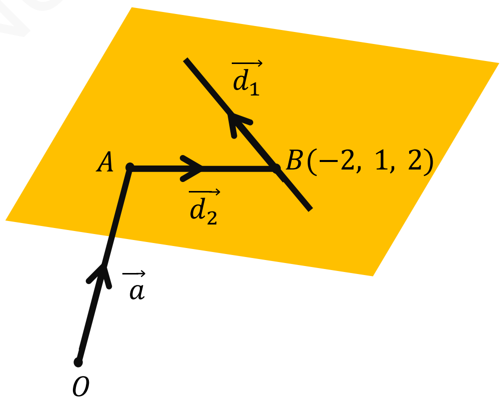
$$4 - 3\left(\frac{13}{3}\right) - 3\left(-\frac{5}{3}\right) = -4 \neq 4.$$

So, D does not lie on the same plane as A, B and C.

Example 21

Find a vector equation of the plane containing the line $\vec{r} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and point $A(3, -1, 2)$.

Solution



Let $\overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

The vector $\vec{d_1} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is on the plane.

$B(-2, 1, 2)$ is on the plane and so

$$\vec{d_2} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix}.$$

We see that $\vec{d_2}$ is also on the plane. Therefore, the vector equation of the plane is

$$\vec{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + t_1\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + t_2\begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix}.$$

Example 22

Vector $\vec{n} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$ is perpendicular to the plane which contains point $A(1, -5, 2)$.

(a) Write an equation of the plane in the form $\vec{r} \cdot \vec{n} = d$.

(b) Find the Cartesian equation of the plane.

Solution

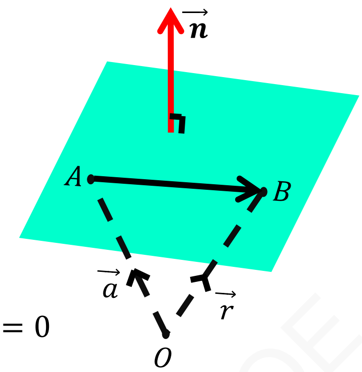
(a) The vector equation of the plane is $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$.

$$\vec{r} \cdot \vec{n} = \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = 2 - 20 - 4 = -22$$

(b) $\vec{r} \cdot \vec{n} = 22$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = -22$$

$$2x + 4y - 2z = -22$$



$\overrightarrow{AB} \cdot \vec{n} = 0$

Exercise 4.4

1. Find the vector equation of the line:

   (a) parallel to $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and through the point $(1, 3, -7)$.

   (b) through $(0, 1, 2)$ and with direction vector $\hat{\imath} + \hat{\jmath} - 2\hat{k}$.

   (c) parallel to the x-axis and through the point $(-2, 2, 2)$.

2. (a) Find the Cartesian equation of the line with parametric equation $x = 3t + 1$, $y = 4 - 2t$, $z = 3t - 1$.

   (b) Find the unit vector in the direction of the line.

3. Find the equation of the plane through $A(-1, 2, 1)$, $B(4, 1, 1)$ and $C(2, 0, 3)$:

   (a) in vector form

   (b) in Cartesian form.

4. Find the Cartesian equation of the plane with vector equation

$$\vec{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}.$$

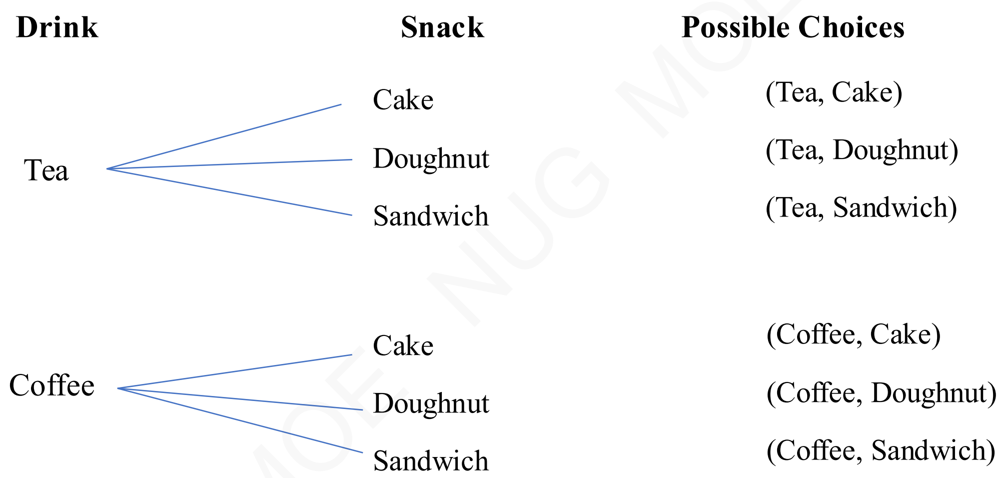
Chapter 5

PERMUTATIONS AND COMBINATIONS

In this chapter, the concepts of "permutation" and "combination" are described in accordance with some counting principles. Counting techniques are indispensable to apply probability when the sample space is finite and outcomes are equally likely. They are also beneficial and applicable to many areas of Mathematics, Statistics, Science and Engineering.

5. 1 Counting Principles

Suppose a coffee shop offers 2 choices of drinks (Tea and Coffee) and 3 choices of snacks (Cake, Doughnut and Sandwich). How many different choices are possible to have a drink and a snack? We can see the complete list of all possible choices in the tree diagram shown below.



In this problem, there are 2 ways to choose the type of drink. For each such choice, there are 3 ways to choose the type of snack. The number of ways to have a drink and a snack is 6, which can be computed as $2 \times 3 = 6$.

Now, let us have a look at another example. Suppose a list of 5 topics is given in an essay contest. A student must select one of. the topics on which to write a short essay, and then select a different topic from the list for a long essay. How many ways can the topics be chosen for the two essays? In this example, there are 5 ways to choose a topic for a short essay. After choosing one for the short essay, a long essay can be written on one of the remaining 4 topics. So the number of ways to choose the topics is $5 \times 4 = 20$.

The two examples illustrate the following multiplication principle.

5.1.1 The Multiplication Principle

Suppose a task can be performed in m ways, and no matter how the task has been performed another task can then be performed in n ways. Then the number of ways to perform these two tasks in succession is mn.The principle can be extended to any finite number of such successive tasks.

Example 1

Suppose that there are 6 roads between town A and town B, and that 4 roads between town B and town C. Find the number of ways a person can drive from A to C by passing through B ?

Solution

There are 6 ways to drive from A to B, and for each such way there are 4 ways to drive from B to C.

So, the number of ways to drive from A to C through B is $6 \times 4 = 24$.

Example 2

There are four blood types, namely A, B, AB and O. Blood can also be $RH(+\text{ ve })$ or $RH(-\text{ve})$. A blood donor can be classified as either male or female. How many different possible ways can a donor have his or her blood labeled?

Solution

There are 4 possibilities for the blood type, 2 possibilities for the RH factor and 2 possibilities for the gender of the donor.

So, the number of ways to label is $4 \times 2 \times 2 = 16$.

Example 3

There are 3 picture nails on a wall. If there are 5 different pictures and each nail can hold only one picture, in how many different ways can the pictures be hung on all the nails?

Solution

Any one of the 5 pictures can be chosen for the first nail, then any one from the remaining four for the second, and finally any one from the remaining three for the third.

The number of ways to choose the pictures for individual nails are 5,4 and 3 respectively.

So, the number of ways to hang the pictures is $5 \times 4 \times 3 = 60$.

Multiplication principle has been applied in solving counting problems in above examples. To introduce another counting principle, consider the following example.

Suppose a manufacturer makes shirts in 5 different sizes $\mathrm{S}, \mathrm{M}, \mathrm{L}, \mathrm{XL}$ and XXL. Shirts having sizes S, M and L are made in 4 different designs, while those having sizes XL and XXL are in 3 different designs. If each design has 5 different color schemes, how many different types of shirts are possible to make?

We need to consider the two cases as follows:

(i) Shirts have sizes S, M or L.

(ii) Shirts have sizes XL or XXL.

For the first case, the number of types is $3 \times 4 \times 5 = 60$.

For the second case, the number of types is $2 \times 3 \times 5 = 30$.

So, the total number of types must be $60 + 30 = 90$.

Notice that the two cases we mentioned above have no arrangement in common and so these are said to be disjoint. Moreover, every possible arrangement falls into some of these cases. The principle we applied above can generally be stated as follows:

5.1.2 The Addition Principle

Suppose there are k pairwise disjoint cases, into which the elements in a counting problem fall. If there are n_1 elements in the first case, n_2 elements in the second case, n_3 elements in the third case, ..., and n_k elements in the k^{th} case. then the number of elements in the problem is

$$\mathrm{n}_1 + \mathrm{n}_2 + \mathrm{n}_3 + \cdots + \mathrm{n}_\mathrm{k}.$$

Example 4

How many different numbers can be formed using the digits $3, 5, 6, 8$ and 9 in such a way that the numbers contain two or three digits without any repetition?

Solution

The two-digit numbers can be formed in $5 \times 4 = 20$ ways.

The three-digit numbers can be formed in $5 \times 4 \times 3 = 60$ ways.

So, there are $20 + 60 = 80$ different numbers.

Example 5

How many integers, having digit 5 only once, are there between 0 and 100?

Solution

The integers, that we want to form, must be one of the followings:

(i) One-digit integer, namely 5,

(ii) Two-digit integers having 5 as tens' digit (ones' digit may be anyone except 5),

(iii) Two-digit integers having 5 as ones' digit (tens' digit may be anyone except 0 and 5).

For the first case, the number of integers $= 1$.

For the second case, the number of integers $= 1 \times 9 = 9$.

For the third case, the number of integers $= 8 \times 1 = 8$.

The required number of integers $= 1 + 9 + 8 = 18$.

5.1.3 Factorial Notation

The problem in Example 3 can be modified as follows, to introduce a notation. There are 7 picture nails on a wall. If there are 7 different pictures and each nail can hold only one picture, how many different ways can the pictures be hung on all the nails?
By the multiplication principle, the pictures can be hung in $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ different ways. The number is the product of the first seven positive integers and can be expressed as 7!. In general, ! (factorial or factorial of n)stands for the product of the first n positive integers n, $(n-1), (n-2), \ldots, 1$ if n is a positive integer. We define 0 ! to be 1 .

If n is a positive integer

$$n! = n(n-1)(n-2)\cdots 1$$
$$0! = 1.$$

Notice that

$$n! = n \cdot (n-1)!$$
$$n! = n \cdot (n-1) \cdot (n-2)!$$

and so on.

For examples, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = \underbrace{4 \cdot (3 \cdot 2 \cdot 1)}_{4 \cdot 3!} = \underbrace{4 \cdot 3 \cdot (2 \cdot 1)}_{4 \cdot 3 \cdot 2!}$.

Example 6

Evaluate: (a) $\frac{7!}{4!\cdot 3!}$ (b) $\frac{6!+5!-4!}{4!}$.

Solution

(a) $\frac{7!}{4!\cdot 3!} = \frac{7\cdot 6\cdot 5\cdot 4!}{4!\cdot 3\cdot 2\cdot 1} = 35.$

(b) $\frac{6!+5!-4!}{4!} = \frac{6\cdot 5\cdot 4!+5\cdot 4!-4!}{4!} = \frac{(30+5-1)\cdot 4!}{4!} = 34.$

Example 7

Express $\frac{13\cdot 12\cdot 11}{3\cdot 2}$ in factorial form.

Solution

$$\frac{13\cdot 12\cdot 11}{3\cdot 2} = \frac{13\cdot 12\cdot 11\cdot 10!}{(3\cdot 2\cdot 1)\cdot 10!} = \frac{13!}{3!\cdot 10!}.$$

Exercise 5.1

1. A store sells men's wear. It has 6 kinds of shirts, 4 kinds of pants and 3 kinds of coats. If a man wants to buy a shirt, a pant and a coat, in how many ways can this be done? (Assume that any choice meets his requirement.)

2. A television news director wishes to use three of the 7 news stories on an evening show. How many possible ways can the program be set up, if the three stories are to be classified as the lead story, the second story and the closing story?

3. If a garage door opener has a 10-digit keypad, containing $0, 1, 2, \ldots, 9$, and the code to open the door must be a 4-digit code, how many codes are possible to create?

4. There are 3 restaurants X, Y and Z, we can go for lunch, in a certain area. Suppose that no two restaurants have the same menu, and the numbers of choices for appetizer, main dish and dessert available are shown in the table. Find the number of ways to have a lunch, consisting of an appetizer, a main dish and a dessert.

Restaurants	No. of Appetizers	No. of Main dishes	No. of Desserts
X	4	10	2
Y	3	8	5
Z	5	12	3

5. A registration code consists of two of the 12 different capital letters $A, B, C, \dots, L$, followed by one of the ten digits $0, 1, 2, \dots, 9$, for example ID5. How many codes are possible to generate:

(a) if the repetition of the letters is allowed?

(b) if the repetition of the letters is not allowed?

(c) if two letters in the codes must be the same?

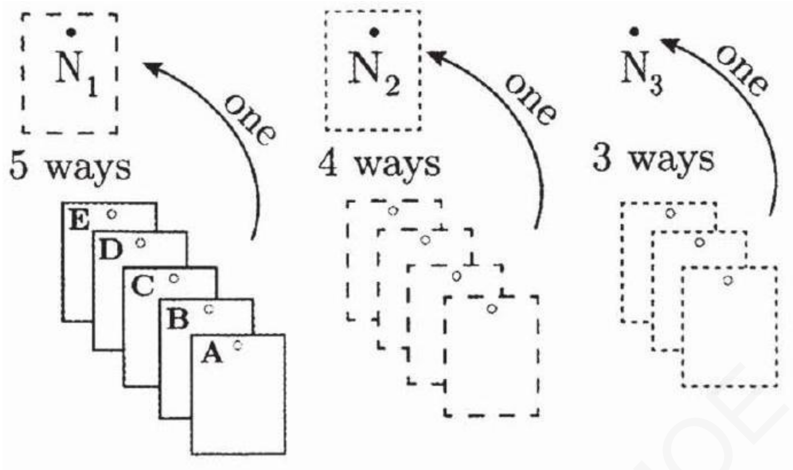
(d) if two letters are different, but must be both vowels or both consonants?

6. (a) Express $\frac{1}{5\cdot 4\cdot 3}$ in factorial form. (b) Evaluate $2\cdot 9! + 82\cdot 8!$.

7. Prove that $\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}=\frac{50}{7!}$.

5.2 Permutations

In Example 3 in Section 5.1, we consider the arrangements of some pictures on a wall. In that problem, there are 5 different pictures (say A, B, C, D and E) and 3 nails (say N_1, N_2 and N_3)



We have known that there are $5 \cdot 4 \cdot 3$ ways to hang the pictures. If the pictures C, E and A are hung on the nails in that order, we can express it as CEA. We can notice that AEC and ACE are in different orders.

Definition: A permutation is an arrangement of the objects in a specific order. If there are n objects, then a permutation of any such r objects, with $r \leq n$, is said to be a permutation of n objects taken r at a time. The number of permutations of n distinct objects taken r at a time is usually denoted by ${}^{n}P_{r}$.

In the above example, the number of ways to hang the pictures is the number of permutations of 5 pictures taken 3 at a time, and so we write

$${}^{5}P_{3} = \underbrace{5 \cdot 4 \cdot 3}_{3\text{ factors}}.$$

- If n and r are positive integers with $r \leq n$,
$${}^{n}P_{r} = \underbrace{n(n-1)(n-2)\cdots(n-(r-1))}_{r\text{ factors}}.$$
- We define ${}^{n}P_{0}$ to be 1, for a nonnegative integer n.

If n and r are positive integers with $r \leq n$, we have

$$\begin{aligned} {}^{n}P_{r} &= n(n-1)(n-2)\cdots(n-(r-1)) \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)\cdot(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!} \end{aligned}$$

Generally, for integers n and r with $0 \leq r \leq n$,

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Notice that ${}^{n}P_{n} = n!$.

Example 8

Evaluate $^{10}P_5 + {}^{10}P_0$.

Solution

$$^{10}P_5 + {}^{10}P_0 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 + 1 = 30240 + 1 = 30241.$$

Example 9

Solve the equations for n. (a) $^{n}P_2 = 9n$ (b) $^{n}P_3 = 12 \cdot {}^{n}P_2$.

Solution

$$\begin{aligned} &\text{(a)} \quad n \geq 2 \text{ and } n(n-1) &&= 9n \\ &n-1 &&= 9 \\ &n &&= 10. \\ &\text{(b)} n \geq 3 \text{ and } n(n-1)(n-2) &&= 12n(n-1) \\ &n-2 &&= 12 \\ &n &&= 14. \end{aligned}$$

Example 10

In how many ways can a president, a treasurer and a secretary for a committee be selected from a group of 15 people?

Solution

The number of ways to select people for three different positions (ranks) is the number of permutations of 15 people taken 3 at a time, and hence it is

$$^{15}P_3 = 15 \cdot 14 \cdot 13 = 2730.$$

Example 11

In how many ways can all the letters of the word PENCIL be arranged, without repeating any letters?

Solution

There are 6 distinct letters. So, the number of ways to arrange the letters is

$$^{6}P_6 = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

Example 12

There are 2 buses, which have 5 and 4 vacant seats respectively, and 4 people at a bus stop. In how many ways can all these people be seated on either of the buses, but not both?

Solution

The first bus has 5 vacant seats, so the number of ways to be seated there is

$$^{5}P_{4} = 5 \cdot 4 \cdot 3 \cdot 2 = 120.$$

The second bus has 4 vacant seats, so the number of ways to be seated there is

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$$

Therefore, the required number of ways is $120 + 24 = 144$.

Example 13

In how many ways can 6 different books be arranged along a line on a shelf if one of the books is a dictionary and it must be at one end?

Solution

The dictionary must be fixed at the 1^{st} or the 6^{th} position in the arrangements.

If the dictionary is in the 1^{st} position, the number of ways to arrange all the other books in remaining positions is

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

If the dictionary is in the 6^{th} position, the number of ways to arrange all the other books in remaining positions is

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

So, the required number of ways $= (1 \times 120) + (120 \times 1) = 240$.

Exercise 5.2

1. Solve the equations for n. (a) ${}^{n}P_2 = 42$ (b) ${}^{n}P_3 = 9 \cdot {}^{n}P_2$.

2. A newspaper has 14 reporters available to cover 3 different stories. In how many ways can the reporters be assigned to cover the stories, if no reporter can be assigned to cover more than one story?

3. Suppose we have to make a signal by choosing 4 different flags out of 9 different coloured flags and arranging them in a row. How many different signals can we do?

4. The manager of 4 movie theaters is deciding which of 12 available movies to show. The theaters have different seating capacities. How many ways can he show 4 different movies in the theaters at the same time?

5. A classroom has two rows of eight seats each. There are 10 students, 5 of whom want to sit in the front row, 4 want to sit in the back row and the remaining student can sit in any seat. In how many ways can the students be seated?

5.3 Combinations

Definition: A combination is a selection, in which the order does not matter, of objects from a collection. The number of combinations of n distinct objects taken r at a time is denoted by ${}^{n}C_r$.

The number of permutations of objects is related to the number of corresponding combinations. For example, the combinations of 4 objects A, B, C and D taken 3 at a time and the corresponding permutations are listed in the following table.

Combinations	Permutations					
$\{A, B, C\}$	ABC	ACB	BAC	BCA	CAB	CBA
$\{A, B, D\}$	ABD	ADB	BAD	BDA	DAB	DBA
$\{A, C, D\}$	ACD	ADC	CAD	CDA	DAC	DCA
$\{B, C, D\}$	BCD	BDC	CBD	CDB	DBC	DCB

Table 5.1

We can see from the table that each combination gives rise to $3!\,(= 6)$ permutations. Since there are ${}^{4}C_3$ combinations, the number of permutations must be ${}^{4}C_3 \cdot 3\,!$. Thus, we have

$${}^{4}C_3 \times 3! = {}^{4}P_3.$$

Generally, for integers n and r with $0 \leq r \leq n$

$$ {}^{n}C_{r} \times r! = {}^{n}P_{r}. $$

If n and r are integers with $0 \leq r \leq n$, then

$$ {}^{n}C_{r} = \frac{{}^{n}P_{r}}{r!} = \frac{n!}{r!\,(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 1}. $$

Example 14

In how many ways can a committee of 4 people be selected from a group of 10 people?

Solution

The number of ways is ${}^{10}C_{4} = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = 210$.

Example 15

Evaluate ${}^{21}C_{1}$, ${}^{21}C_{21}$, ${}^{21}C_{19}$ and ${}^{21}C_{2}$.

Solution:

$$ {}^{21}C_{1} = 21. $$

$$ {}^{21}C_{21} = \frac{21!}{21!\,(21-21)!} = \frac{21!}{21!\,0!} = \frac{21!}{21!\cdot 1} = 1. $$

$$ {}^{21}C_{19} = \frac{21!}{19!\,(21-19)!} = \frac{21!}{19!\,2!} = \frac{21\cdot 20\cdot 19!}{19!\cdot 2\cdot 1} = 210. $$

$$ {}^{21}C_{2} = \frac{21!}{2!\,(21+2)!} = \frac{21!}{2!\,19!} = \frac{21\cdot 20\cdot 19!}{2\cdot 1\cdot 19!} = 210. $$

We notice that ${}^{21}C_{1} = 21$, ${}^{21}C_{21} = 1$ and ${}^{21}C_{19} = {}^{21}C_{2} = {}^{21}C_{21-19}$.

Generally, for integers n and r with $0 \leq r \leq n$

$$ {}^{n}C_{1} = n,\ {}^{n}C_{n} = 1 \text{ and } {}^{n}C_{r} = {}^{n}C_{n-r}. $$

Example 16

A music class consists of 5 piano players, 7 guitarists and 4 violinists. A band of 1 piano player, 3 guitarists and 2 violinists must be chosen to play at a school concert. In how many ways can the band be chosen?

Solution

From the music class,

1 piano player can be chosen in ${}^{5}C_{1}$ ways,

3 guitarists can be chosen in ${}^{7}C_{3}$ ways, and

2 violinists can be chosen in ${}^{4}C_{2}$ ways.

So, the number of ways to choose a band is

$$^{5}C_1 \cdot {}^{7}C_3 \cdot {}^{4}C_2 = 5 \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} \cdot \frac{4 \cdot 3}{2 \cdot 1} = 1050.$$

Example 17

Suppose there are 4 black cars and 7 white cars. If all the cars are distinguishable, in how many ways can 3 cars of the same color be chosen?

Solution

To get the same color, the 3 cars must be all black or all white.

Out of 4 black cars, three can be chosen in $^{4}C_3$ ways.

Out of 7 white cars, three can be chosen in $^{7}C_3$ ways.

So, the required number of ways is

$$^{4}C_3 + {}^{7}C_3 = \frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1} + \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 4 + 35 = 39.$$

Example 18

There are 6 different books. In how many ways can the books be given to 3 children, if the youngest wants to receive 3 books, the elder 1 and the eldest 2 respectively?

Solution

For the youngest child, the books can be chosen in $^{6}C_3 = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$ ways.

For the elder child, the books can be chosen in $^{3}C_1 = 3$ ways.

For the eldest child, the books can be chosen in $^{2}C_2 = 1$ ways.

So, the required number of ways is $20 \cdot 3 \cdot 1 = 60$.

Example 19

In how many ways can 4 fruits be selected out of 9 fruits, so as always to:

(a) include the largest fruit? (Assume that such a largest fruit exists.)

(b) exclude the smallest fruit? (Assume that such a smallest fruit exists.)

Solution

(a) The largest one is included in every selection. The number of ways only depends on the choice of the other three from the remaining 8 fruits, and hence it is

$$^{8}C_3 = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56.$$

(b) Since the smallest one is excluded in every selection, we have to select 4 fruits from the remaining 8 fruits. So the required number ways is

$$^{8}C_{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70.$$

Exercise 5.3

1. Show that $^{12}C_5 \cdot {}^{7}C_4 = {}^{12}C_4 \cdot {}^{8}C_5$.

2. There are 10 candle holders, which are fixed in different locations along a line and each can hold only one candle. In how many ways can 7 identical candles be put in these holders?

3. There are 3 parts in a test. Each of the first two parts contains 5 questions, but the last part only 4. If a student must answer all from the first part, 4 and 3 questions from the second and the last parts respectively, in how many ways can this be done?

4. How many games can be played in a 9-team sport league if each team plays all other teams once?

5. How many lines are determined by 8 points, if no 3 such points are collinear? How many triangles are determined by these points?

6. In how many ways can 4 fruits be selected out of 9 fruits, having different sizes, so as always to include the largest fruit and exclude the smallest fruit?

5.4 Techniques for Some Counting Problems

5.4.1 Permutations with Repetitions

We had solved Example 11 about permutations of letters of the word PENCIL containing no repeated letters. Now consider the word KEENNESS. It has 8 letters consisting of one K, three E's, two N's and two S's. We want to determine the number of permutations of these 8 letters. It can be regarded as determining the number of ways to place these letters in 8 different places such that each letter occupies exactly one place.

Firstly, we need to decide which place is to be occupied by the letter K, and this can be done in $^{8}C_1$ ways. Then there are 7 places left, among them we need to decide which places are to be occupied by 3E's and this can be done in $^{7}C_3$ ways. After that, the places for 2N's can be chosen in ${}^{4}C_{2}$ ways. Finally, there are ${}^{2}C_{2}$ ways to place 2 S's. By the multiplication principle. the number of permutations of all the letters is

$$ {}^{8}C_{1} \cdot {}^{7}C_{3} \cdot {}^{4}C_{2} \cdot {}^{2}C_{2} = \frac{8!}{1! \cdot 7!} \cdot \frac{7!}{3! \cdot 4!} \cdot \frac{4!}{2! \cdot 2!} \cdot 1 = \frac{8!}{1!\,3!\,2!\,2!} \cdot $$

We can generalize the above result as follows.

Suppose we are given n objects, in which n_1 objects are of the first kind, n_2 objects are of the second kind, n_3 objects are of the third kind, ..., and n_r objects are of the r^{th} kind, such that $n_1 + n_2 + \cdots + n_r = n$. Then the number of permutations of all these n objects is $\frac{n!}{n_1! n_2! \cdots n_r!}$.

Example 20

In how many ways can a permutation of all the letters of the word EXCELLENCE be formed?

Solution

In the word EXCELLENCE, there are 10 letters consisting of four E's, one X, two C's, two L's and one N. So number of ways is

$$ \frac{10!}{4!1!2!2!1!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 1 \cdot (2 \cdot 1) \cdot (2 \cdot 1) \cdot 1} = 10 \cdot 9 \cdot 2 \cdot 7 \cdot 6 \cdot 5 = 37800. $$

5.4.2 The Exclusion Principle

The exclusion principle is a way to count what you are interested in by first counting what you are not interested in. This is often needed for counting where a certain property is prohibited (not allowed).

Count what you are not interested in and subtract it from the total. Consider 5-digit codes containing each of the digits 1, 2, 3, 4, 5 exactly once. We have known that the codes can be formed in 5! ways. Suppose we want to know the number of those codes, not ending in 25 . On contrary, we count the number of the codes ending in 25 . The number of these codes is 3 !, since the last two digits are fixed. Subtract it from the number of all the codes, in which the restriction for the last two digits is neglected. The resulting number $5! - 3! = 114$ is the number of codes not ending in 25.

Example 21

How many permutations are there of the letters of the word PROGRAM, if they do not end in: (a) $2R$'s? (b) MAP?

Solution

In the word PROGRAM, there are 7 letters consisting of one P, two R's, one O, one G, one A and one M, so the number of permutations of the letters is

$$ \frac{7!}{2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!} = 2520. $$

(Hereafter the factor 1! may be omitted in the denominators of such fractions.)

(a) The number of permutations ending in two R's is $5! = 120$, since the two R's are fixed at the end.

The number of permutations not ending in two R's $= 2520 - 120 = 2400$.

(b) The number of permutations ending in MAP is

$$\frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2!}{2!} = 12.$$

since MAP is fixed at the end, and there are two R's in the remaining 4 letters. The number of permutations not ending in two R's $= 2520 - 12 = 2508$.

5.4.3 Counting the Subsets of a Finite Set

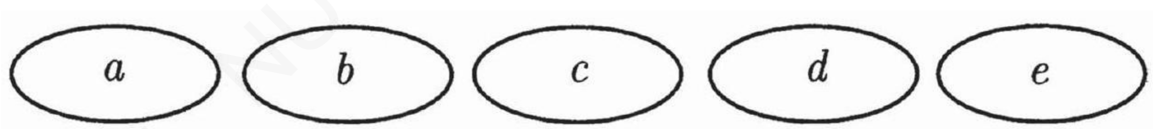
Consider a set $X = \{a, b, c, d, e\}$. A subset containing r elements, with $0 \leq r \leq 5$, can be regarded as a combination of 5 elements taking r at a time. Thus there are ${}^5C_r$ subsets containing r elements. For example, there are ${}^5C_4 = 5$ subsets containing 4 elements, as listed below.

$$\{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}.$$

So, by the addition principle, the number of all the subsets of X is given by

$${}^5C_0 + {}^5C_1 + {}^5C_2 + \cdots + {}^5C_5.$$

On the other hand, to form a subset, we have to decide which elements of X should be included (I) or excluded(E) in that subset. There are two choices (2 ways) for each element of X.



I or E? I or E? I or E? I or E? I or E?

By the multiplication principle, the number ways to form a subset of X is

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5.$$

This means that the set X has 2^5 subsets.

We have demonstrated the two different techniques for counting the subsets of the set X, and noticed that

$${}^5\mathrm{C}_0 + {}^5\mathrm{C}_1 + {}^5\mathrm{C}_2 + \cdots + {}^5\mathrm{C}_5 = 2^5.$$

We can generalize the above result as follows.

- If a finite set contains n elements, then it has 2^n subsets.
- If n is an integer with $n \geq 0$, then ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$.

Example 22

If A is a set containing 9 distinct elements, how many subsets of A contain

(a) at most 2 elements? (b) at least 3 elements?

Solution

(a) The number of permutations containing no element, 1 element and 2 elements are ${}^9C_0$, ${}^9C_1$ and ${}^9C_2$ respectively.So the number of subsets containing at most 2 elements is

$$\begin{aligned} {}^9C_0 + {}^9C_1 + {}^9C_2 &= 1 + 9 + \frac{9 \cdot 8}{2 \cdot 1} \\ &= 1 + 9 + 36 \\ &= 46. \end{aligned}$$

(b) The number of all the subsets of A is $2^9 = 512$.

The number of subsets containing at least 3 elements $= 512 - 46 = 466$.

5.4.4 Miscellaneous Counting Problems

Example 23

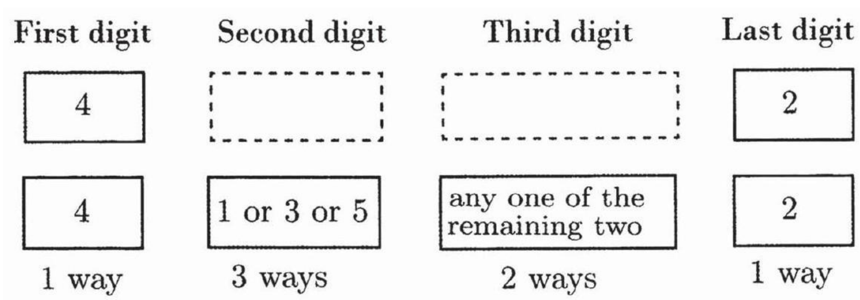
How many 4-digit even numbers, greater than 4000 , can be formed using the digits 1,2,3,4 and 5 , without repeating any digit?

Solution

Since the numbers are greater than 4000 , the first digit must be 4 or 5 .

Case 1. If the first digit is 4 and the last digit is 2 , the numbers can be formed in

$1 \cdot 3 \cdot 2 \cdot 1 = 6$ ways.



Case 2. If the first digit is 5 and the last digit is 2 or 4 , the numbers can be formed in

$1 \cdot 3 \cdot 2 \cdot 2 = 12$ ways.

First digit	Second digit	Third digit	Last digit
5			2 or 4
5	any one of three digits 1 or 3 or (4 or 2)	any one of the remaining two	2 or 4
1 way	3 ways	2 ways	2 ways

So, the number of 4-digit even numbers greater than 4000 is $6 + 12 = 18$.

Example 24

In how many ways can 2 different chemistry books, 4 different mathematics books and 3 different physics books be arranged in a line on a shelf if

(a) 2 chemistry books are to be placed on the left, 4 mathematics books in the middle and 3 physics books on the right?

(b) books of the same subjects are together?

Solution

(a) The number of ways to place 2 chemistry books in the left part is

$$2! = 2 \cdot 1 = 2.$$

The number of ways to place 3 physics books in the right part is

$$3! = 3 \cdot 2 \cdot 1 = 6.$$

The number of ways to place 4 mathematics books in the middle part is

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$$

The required number of ways $= 2 \cdot 24 \cdot 6 = 288$.

(b) The 3 subject wise groups can be placed in $3! = 6$ ways, and in each such arrangement the books can be placed in 288 ways (as in part (a)).

The required number of ways to arrange books of the same subject together

$$= 6 \cdot 288 = 1728.$$

Example 25

How many permutations of the letters S, U, N, D, A, Y are there if the two vowels are placed together?

Solution

If the two vowels U and A unite to form a new single letter, then the mumber of permutations of the letters is $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Again, U and A can be arranged in 2 different ways as UA and AU in each such permutation.

So, the required number of permutations is $120 \cdot 2 = 240$.

Example 26

Find the number of permutations of all the letters of the word INTERNET in such a way that there are exactly 4 letters between the two T's.

Solution

The letters must be arranged in one of the three forms:

$$\mathrm{T} \star\star\star \mathrm{T} \star\star \qquad \star \mathrm{T} \star\star\star\star \mathrm{T} \star \qquad \star\star \mathrm{T} \star\star\star\star \mathrm{T}$$

in which the 6 letters (2 N's, 2 E's, 1I and 1R) other than 2 T's are to be put in star (*) positions. For each such form, the number of ways to put the six letters in star $(\star)$ positions is

$$\frac{6!}{2! \cdot 2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2! \cdot 2 \cdot 1} = 180.$$

So, the required number of permutations is $3 \cdot 180 = 540$.

Exercise 5.4

1. Find the number of permutations of the letters of each of the following words. (i) INFINITY (ii) PINEAPPLE (iii) ACCEPTABLE.

2. Passwords for a mobile payment system are to consist of 6 digits from a set of digits $0,1,2, \ldots, 9$ and assume that there is no other restriction. How many such passwords have repeated digits?

3. A building has 10 windows in front. What is the number of signals that can be given, by having one or more of the windows open?

4. At a burger shop, 3 different types of buns, 7 types of cheeses and 5 types of vegetables are available. Assume that a burger may contain no cheese or no vegetable. In how many ways can different types of burgers be ordered if a person must have a bun, and may have at most 2 types of cheeses and any number of types vegetables.

5. How many different 4-digit codes can be formed using all the digits $0, 1, 2, 3$ if

   (i) there is no restriction?

   (ii) repetition is not allowed?

   (iii) repetition is not allowed, and 0 is either the first or the last digit?

6. Using the letters of the word EQUATION without repetitions, how many 4-letter codes can be formed:

   (i) starting with T and ending with N?

   (ii) starting and ending with a consonant?

   (iii) with vowels only?

   (iv) if it contains 3 consonants?

7. Three brothers and three sisters are lining up to be photographed. How many arrangements are there

   (i) with 3 sisters standing together?

   (ii) if brothers and sisters are in alternating positions?

8. How many permutations of the letters H, E, X, A, G, O, N are there if

   (i) the 3 vowels are placed together?

   (ii) the 3 vowels are not placed together?

   (iii) consonants and vowels do not appear alternately?

9. Find the number of permutations of all the letters of the word PENGUIN in such a way that there are exactly 3 letters between 2 N's.

10. Find the number of permutations of all the letters of the word STRESSLESS in such a way that there are exactly 5 letters between:

    (i) 2 E's. (ii) 2 S's.

Chapter 6
CONIC SECTIONS

In this chapter, we introduce the equations of conic sections such as circles and parabolas. We discuss its graphical representations. We determine and illustrate the translation and rotation of parabolas with their axes.

6.1 Introduction

I. The intersections of a plane with a double right-circular cone are called conics or conic sections.

II. The most important of conic sections are
- Circles
- Ellipses
- Parabolas
- Hyperbolas.

6.2 Circles

Standard form of the equation of a circle $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the centre and r is the radius.

Example 1

Find an equation of the circle of radius 4 centre at $(-1,2)$ in standard form.

Solution

Since, $r=4$, $(h,k) = (-1,2)$, we have an equation of the circle

$(x+1)^2+(y-2)^2=4^2$

$(x+1)^2+(y-2)^2=16.$

Example 2

Find the centre and radius of the equation $(x+1)^2+y^2=1$.

Solution

$(x+1)^2 + y^2 = 1$

$(x+1)^2 + (y-0)^2=1$

centre $=(-1,0)$ and radius $=1$.

Note: The circle $x^2 + y^2=1$ is the unit circle centered at the origin and has radius 1.

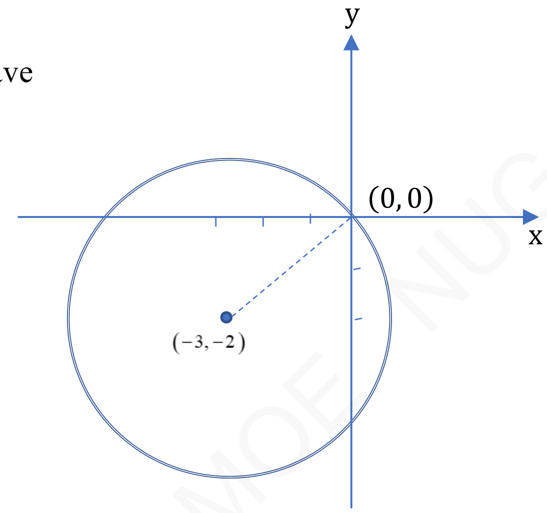
Example 3

Find the standard equation of the circle with centre $(-3, -2)$, that circle passes through the origin.

Solution

Since $(h, k) = (-3, -2)$, $(x_1, y_1) = (0, 0)$, we have

$$r = \sqrt{(x_1 - h)^2 + (y_1 - k)^2}$$
$$= \sqrt{(0 + 3)^2 + (0 + 2)^2}$$
$$= \sqrt{9 + 4}$$
$$= \sqrt{13}.$$



The circle equation is

$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x + 3)^2 + (y + 2)^2 = 13.$$

General form of the equation of a circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ where $(-g, -f)$ is the centre and $\sqrt{g^2 + f^2 - c}$ is the radius.

Example 4

Find the centre and radius of the equation $2x^2 + 2y^2 - 12x + 8y - 24 = 0$.

Solution

Method 1: $2x^2 + 2y^2 - 12x + 8y - 24 = 0$

$$x^2 + y^2 - 6x + 4y - 12 = 0$$
$$x^2 - 6x + y^2 + 4y - 12 = 0$$
$$(x - 3)^2 - 9 + (y + 2)^2 - 4 - 12 = 0$$
$$(x - 3)^2 + (y + 2)^2 = 25.$$

Comparing with $(x - h)^2 + (y - k)^2 = r^2$, we have

centre $= (3, -2)$ and radius $= 5$.

Method 2: $2x^2 + 2y^2 - 12x + 8y - 24 = 0$

$$x^2 + y^2 - 6x + 4y - 12 = 0$$

Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0$,

We get $2g = -6, \quad 2f = 4, \quad c = -12.$

So, $g = -3, \quad f = 2, \quad c = -12.$

centre $= (-g, -f) = (3, -2)$ and

radius $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{9 + 4 + 12} = \sqrt{25} = 5.$

Degenerate Cases of a Circle

Consider the equation

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

with centre $= (-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

(i) If $g^2 + f^2 - c > 0$, the graph is a circle with centre at $(-g, -f)$ and $\sqrt{g^2 + f^2 - c}$ is the radius.

(ii) If $g^2 + f^2 - c = 0$, the graph is a single point $(-g, -f)$.

(iii) If $g^2 + f^2 - c < 0$, the equation has no real solution and consequently no graph.

Example 5

Describe the graph of the equation $x^2 + y^2 - 4x + 2y - 44 = 0$.

Solution

$$x^2 + y^2 - 4x + 2y - 44 = 0.$$

Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0$,

we get

$$2g = -4, 2f = 2, c = -44$$

$$g = -2, f = 1, c = -44.$$

Since $g^2 + f^2 - c = 4 + 1 + 44 = 49 > 0$, we have the graph is a circle with

centre $= (-g, -f) = (2, -1)$ and radius $= \sqrt{g^2 + f^2 - c} = 7.$

Example 6

Describe the graph of the equation $x^2 + y^2 - 4x - 6y + 13 = 0$.

Solution

Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$$2g = -4, \quad 2f = -6, \quad c = 13$$

$$g = -2, \quad f = -3, \quad c = 13$$

Since $g^2 + f^2 - c = 4 + 9 - 13 = 0$, we have the graph is a single point $(-g, -f) = (2, 3)$.

Example 7

Describe the graph of the equation $x^2 + y^2 + 10x + 26 = 0$.

Solution

Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$$2g = 0, \quad 2f = 10, \quad c = 26$$

$$\therefore g = 0, \quad f = 5, \quad c = 26$$

Since $g^2 + f^2 - c = 0 + 25 - 26 = -1 < 0$, we have the given equation has no graph.

Exercise 6.1

1. Find the centre and radius of each circle:
   (a) $x^2 + y^2 = 16$

   (b) $(x - 1)^2 + (y - 4)^2 = 25$

   (c) $(x - 3)^2 + (y + 5)^2 = 49$

   (d) $2x^2 + 2y^2 - 12x + 8y - 24 = 0$

   (e) $x^2 + y^2 + x + y - \frac{1}{2} = 0$

2. Find the standard equation of the circle satisfying the given conditions.

   (a) centre $(3, -2)$, radius $= 4$.

   (b) centre $(2, 5)$, radius $= \sqrt{5}$.

   (c) centre $(-4, 1)$, radius $= 3$.

   (d) centre $(-4, 8)$, circle is tangent to the x-axis.

   (e) centre $(5, 8)$, circle is tangent to the y-axis.

   (f) centre $(-3, -2)$, circle passes through the origin.

   (g) centre $(4, -5)$, circle passes through $(1, 3)$.

3. Determine whether each equation represents a circle, a point or no graph. If the equation represents a circle, find the centre and radius.

   (a) $x^2 + y^2 - 6x + 2y + 9 = 0$.

   (b) $2x^2 + 2y^2 - 12x + 8y - 24 = 0$.

   (c) $x^2 + y^2 - 10x - 2y + 29 = 0$.

   (d) $16x^2 + 16y^2 + 40x + 16y - 7 = 0$.

   (e) $\frac{x^2}{4} + \frac{y^2}{4} = 1$.

   (f) $2x^2 + 2y^2 + 8x + 7 = 0$.

6.3 The Parabola

Definition of a parabola

A parabola is the set of all points (x, y) in the plane that are equidistant from a fixed line and a fixed point which is not on the line.

- Fixed line (directrix)
- Fixed point (focus)
- The line through the focus and perpendicular to the directrix (axis of the parabola)
- The point of intersection of the axis and the parabola(vertex)
- Notation

Focus F

Vertex V

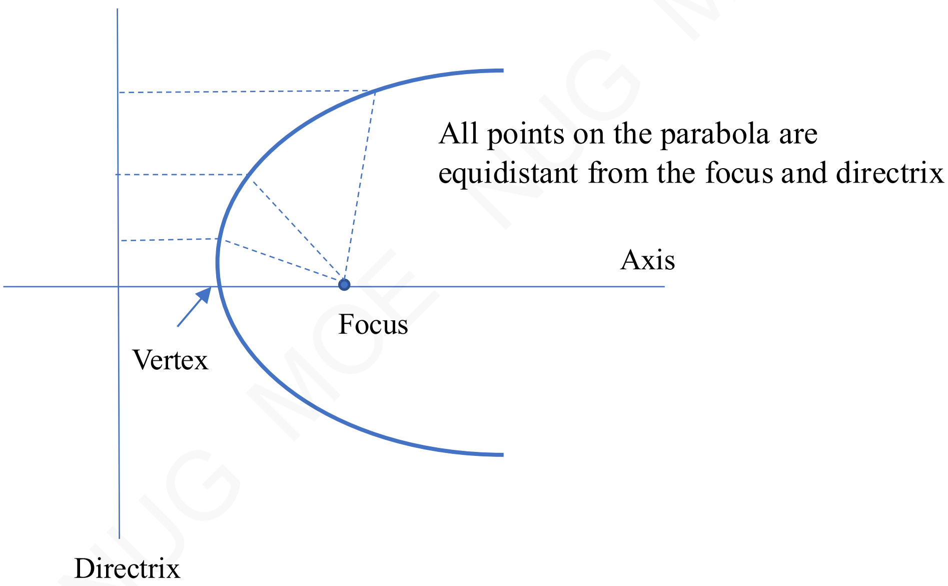


Figure 6.1

Geometric properties of a parabola

- The distance between the focus and the vertex by p.
- The vertex is equidistant from the focus and the directrix.
- The distance between the vertex and the directrix is also p.
- The distance between the focus and the directrix is 2p.
- latus rectum is the line segment passes through the focus and is parallel to the directrix.

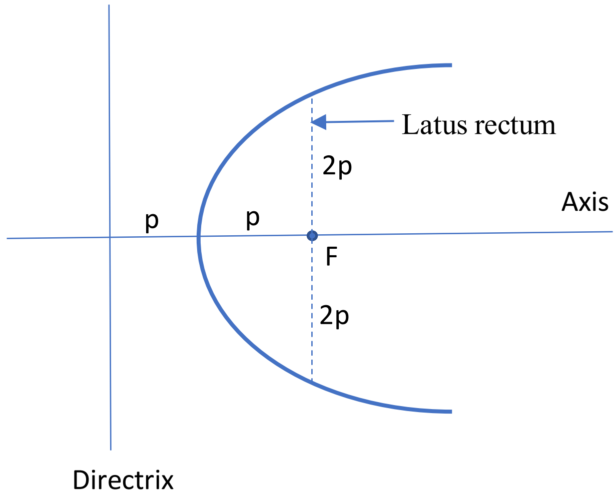


Figure 6.2

Equation of a parabola: Vertex at (0, 0)

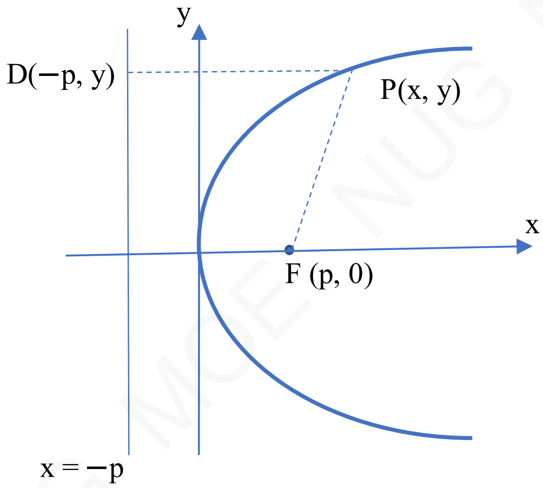


Figure 6.3

In the figure (6.3), PF = PD.

The distance $PF = \sqrt{(x-p)^2 + y^2}$ and $PD = \sqrt{(x+p)^2}$

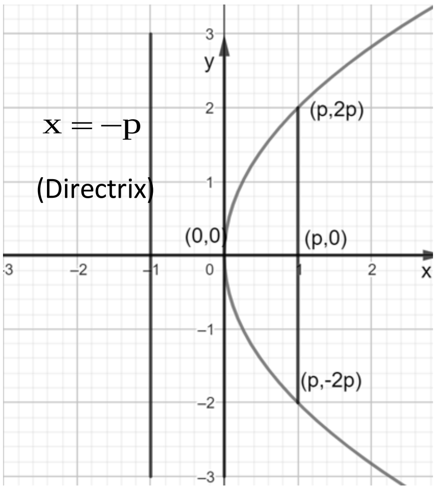
$$\therefore (x-p)^2 + y^2 = (x+p)^2$$

$$\therefore y^2 = 4px.$$

The remaining equations have similar derivations.

Standard Equations of a Parabola

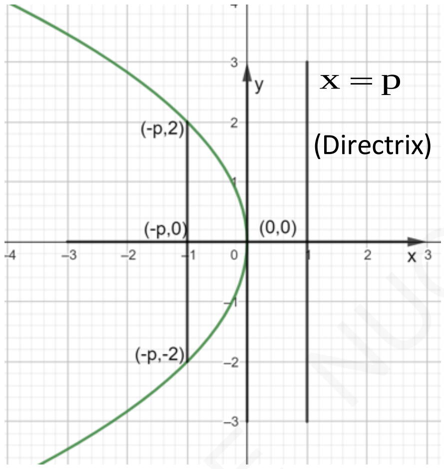
Standard positions of a parabola by the coordinate axes



$y^2 = 4px$

Open to the right

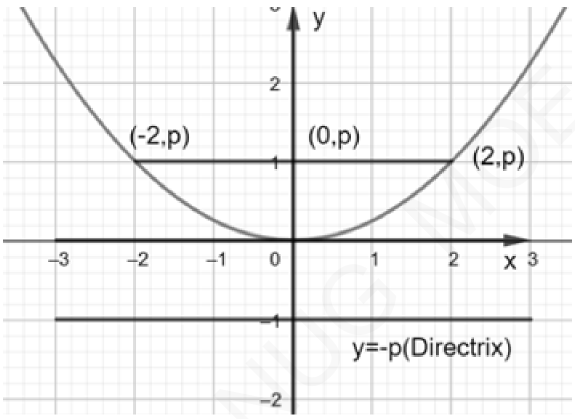
Figure 6.4



$y^2 = -4px$

Open to the left

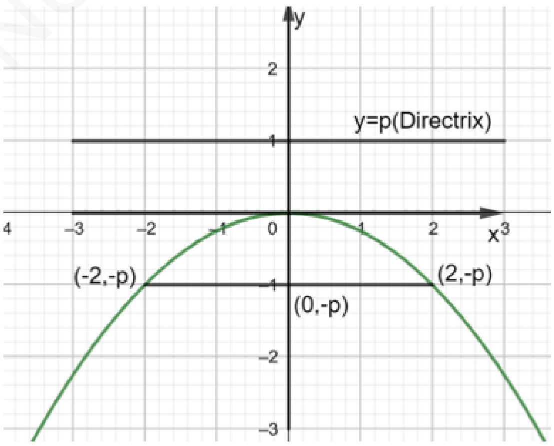
Figure 6.5



$x^2 = 4py$

Opens up

Figure 6.6



$x^2 = -4py$

Opens down

Figure 6.7

Equation of a parabola:

Vertex at (0,0), Focus on Coordinate Axis, p > 0

Equation	Vertex	Focus	Directrix	Axis of Symmetry	Description
$y^2 = 4px$	(0,0)	$(p, 0)$	$x = -p$	Horizontal axis, $y = 0$	Opens to the right
$y^2 = -4px$	(0,0)	$(-p, 0)$	$x = p$	Horizontal axis, $y = 0$	Opens to the left
$x^2 = 4py$	(0,0)	$(0, p)$	$y = -p$	Vertical axis, $x = 0$	Opens up
$x^2 = -4py$	(0,0)	$(0, -p)$	$y = p$	Vertical axis, $x = 0$	Opens down

Table 6.1

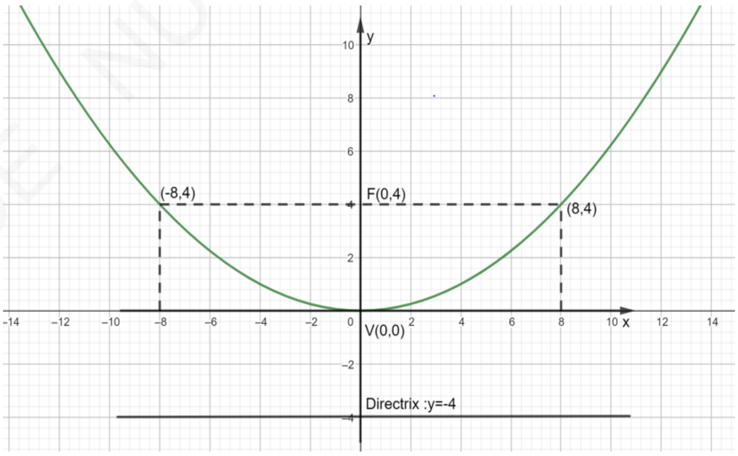
Example 8

Sketch the graphs of the parabolas, showing the vertex, focus, directrix and end points of latus rectum of each equation.

(a) $x^2 = 16y$ (b) $y^2 + 8x = 0$.

Solution

(a) $x^2 = 16y$



Comparing with $x^2 = 4py$, we get

$$4p = 16$$

$$p = 4$$

$$\text{Vertex} = (0, 0)$$

$$Focus = (0, 4)$$

Directrix : $y = -4$

End points of latus rectum are $(8,\ 4)$ and $(-8,\ 4)$.

(b) $y^2 + 8x = 0$

$$y^2 = -8x.$$

Comparing with $x^2 = -4py$, we get

$4p = 8$

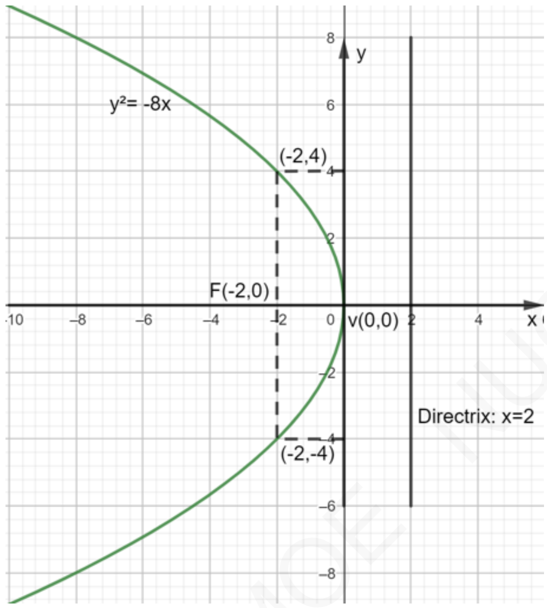
$p = 2$

Vertex $= (0, 0)$

Focus $= (-2, 0)$

Directrix : $x = 2$

End points of latus rectum are $(-2, 4)$ and $(-2, -4)$.



Example 9

Sketch the graphs of the parabolas, showing the vertex, focus, directrix and end points of latus rectum of each equation.

(a) $y^2 = 8x$, (b) $y = -x^2$.

Solution

(a) $y^2 = 8x$

Comparing with $y^2 = 4px$, we get

$4p = 8$

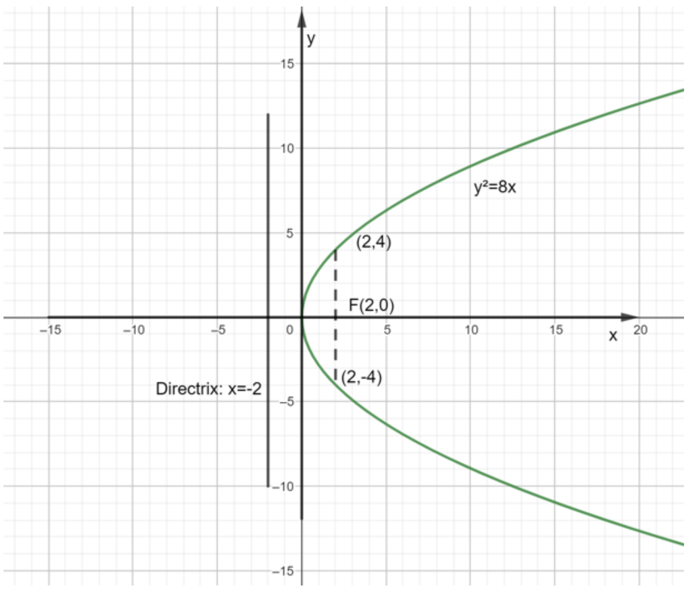
$p = 2$

Vertex $= (0,0)$

Focus $= (2,0)$

Directrix : $x = -2$

End points of latus rectum are $(2,4)$ and $(2, -4)$.



(b) $y = -x^2$

$x^2 = -y$

Comparing with $x^2 = -4py$, we get

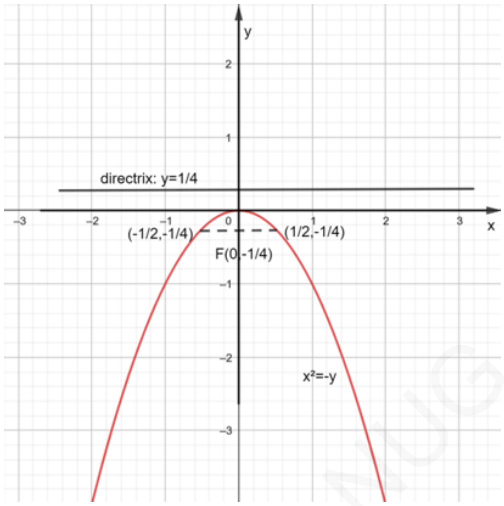
$$4p = 1$$

$$p = \frac{1}{4}$$

$$\text{Vertex} = (0, 0)$$

$$\text{Focus} = \left(0, -\frac{1}{4}\right)$$

$$\text{Directrix}: y = \frac{1}{4}$$



End points of latus rectum are $\left(\frac{1}{2}, -\frac{1}{4}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{4}\right)$.

Example 10

Find an equation for the parabola satisfying the given conditions.

(a) Vertex (0.0); symmetric about the x- axis; passes through (2, 2).

(b) Vertex (0.0); symmetric about the y- axis; passes through (−1, 3).

Solution

(a) Since the parabola is a symmetric about the x-axis, we have

$$y^2 = 4px$$

$$\text{At } (2, 2),\ 4 = 8p$$

$$\therefore p = \frac{1}{2}.$$

∴ The parabola equation is $y^2 = 2x$.

Solution

(b) Since the parabola is a symmetric about the y-axis, we have

$$x^2 = 4py.$$

$$\text{At } (-1, 3),\ 1 = 12p$$

$$\therefore p = \frac{1}{12}.$$

∴ The parabola equation is $x^2 = \frac{1}{3}y$.

Exercise 6.2

1. In exercise a to f, (i) sketch the parabola, (ii) show the focus, vertex, directrix and end points of latus rectum.
   (a) $y^2 = -6x$.
   (b) $x + y^2 = 0$.
   (c) $y = x^2$.
   (d) $y = 3x^2$.
   (e) $y = -3x^2$.
   (f) $x^2 + 6y = 0$.

2. In Exercise a to f, find an equation for the parabola satisfying the given conditions.
   (a) Vertex (0, 0), focus (3, 0).
   (b) Vertex (0, 0), focus $(0, -4)$.
   (c) Vertex (0, 0) directrix $x = 7$.
   (d) Vertex (0, 0) directrix $y = \frac{1}{2}$
   (e) Focus $(0, -3)$. directrix $y = 3$.
   (f) Focus (6, 0) directrix $x = -6$.

Translation of Axes

- Translation the axes of an xy-coordinate system to obtain a new $x'y'$-coordinate system whose origin O'.

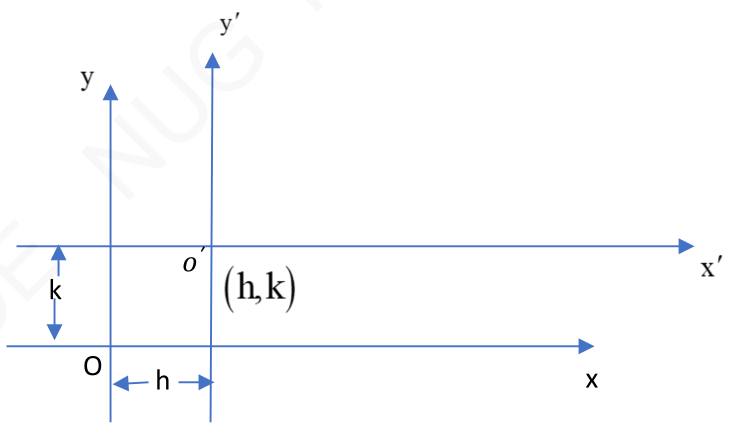


Figure 6.8

- In the figure (6.9), a point P in the plane has both (x, y) coordinates and (x', y') coordinates.

- The translation equations,

$$x' = x - h, \qquad y' = y - k$$
$$x = x' + h, \qquad y = y' + k$$

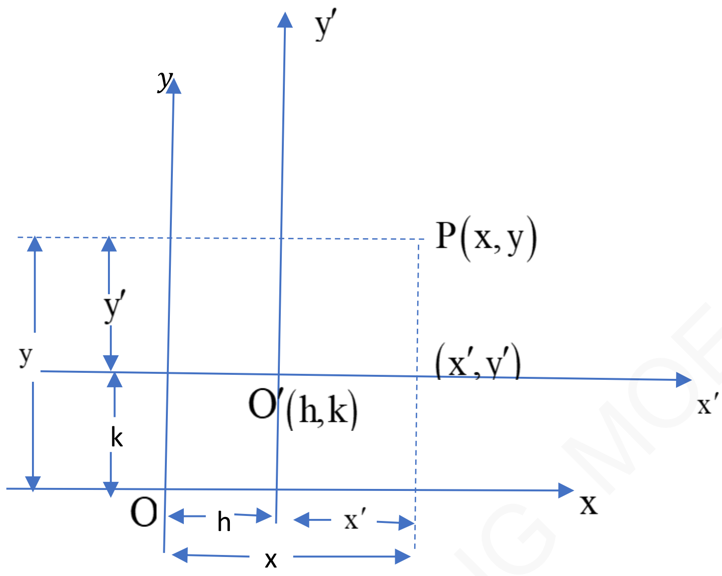


Figure 6.9

Example 11

If the new origin is at $(h,\ k) = (4,\ -1)$ and the xy-coordinates of a point P are (2, 5), find the $x'y'$-coordinates of P.

Solution

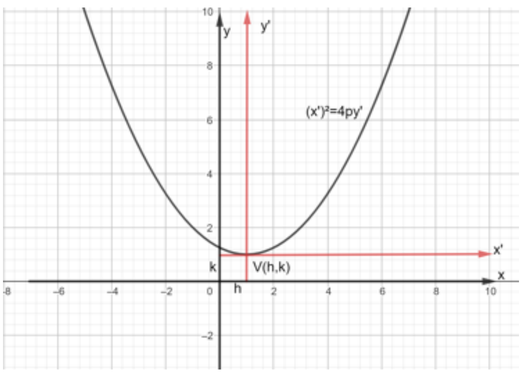
$$(x,\ y) = (2,\ 5),\ (h,\ k) = (4, -1)$$
$$x' = x - h = 2 - 4 = -2$$
$$y' = y - k = 5 + 1 = 6.$$

$\therefore x'y'$-coordinates of P is $(x',\ y') = (-2,\ 6)$.

Translated Parabolas

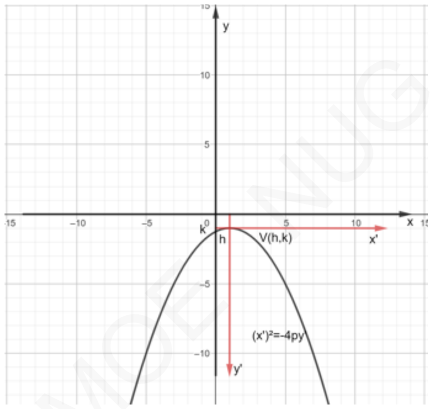
If $x'y'$-coordinates, the equation of the parabola

$(x')^2 = 4py'$, $(x')^2 = -4py'$.



$(x')^2 = 4py'$

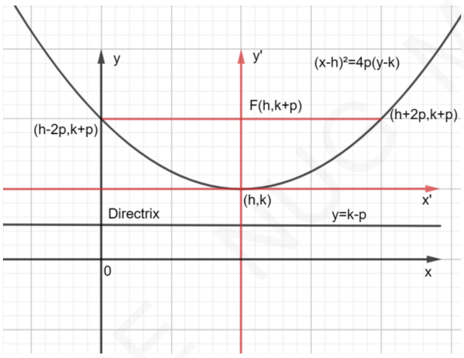
Figure 6.10



$(x')^2 = -4py'$

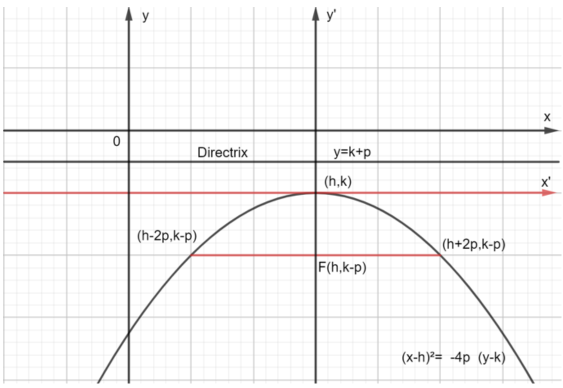
Figure 6.11

Parabola with Vertex (h, k) and Axis Parallel to y-axis



$(x-h)^2 = 4p(y-k)$

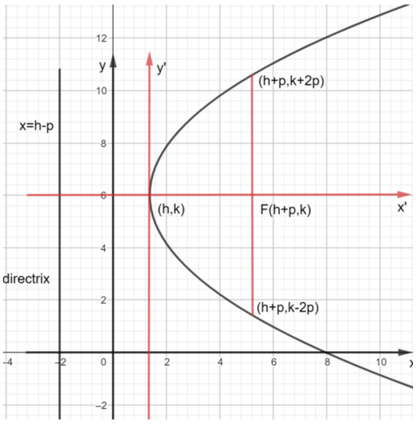
Figure 6.12



$(x-h)^2 = -4p(y-k)$

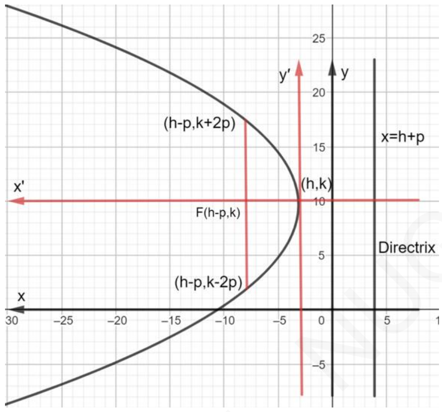
Figure 6.13

Parabola with Vertex (h, k) and Axis Parallel to x-axis



$$(y-k)^2 = 4p(x-h)$$

Figure 6.14



$$(y-k)^2 = -4p(x-h)$$

Figure 6.15

Equations of a Parabola:

Vertex at (h, k), parallel to coordinate axis, $p > 0$

Equation	Vertex	Focus	Directrix	Axis of symmetry	Description
$(y-k)^2 = 4p(x-h)$	(h,k)	$(h+p,k)$	$x = h-p$	Horizontal axis, y=k	Opens to the right
$(y-k)^2 = -4p(x-h)$	(h,k)	$(h-p,k)$	$x = h+p$	Horizontal axis, y=k	Opens to the left
$(x-h)^2 = 4p(y-k)$	(h,k)	$(h,k+p)$	$y = k-p$	Vertical axis x=h	Opens up
$(x-h)^2 = -4p(y-k)$	(h,k)	$(h,k-p)$	$y = k+p$	Vertical axis, x=h	Opens down

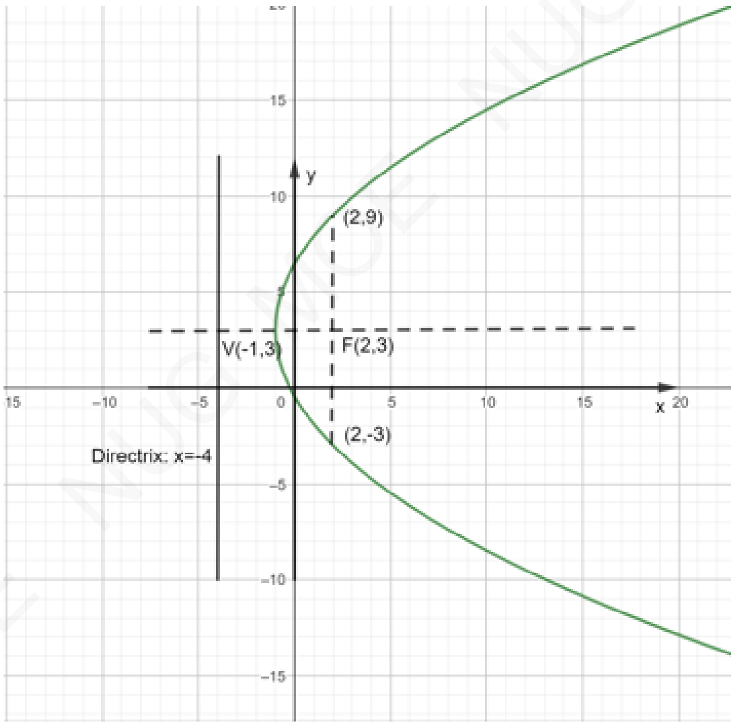
Table 6.2

Example 12

Show that the curve $y^2 - 12x - 6y - 3 = 0$ is a parabola. Sketch the graph, showing the vertex, focus, directrix and end points of latus rectum.

Solution

$$y^2 - 12x - 6y - 3 = 0$$
$$y^2 - 6y = 12x + 3$$
$$(y-3)^2 - 9 = 12x + 3$$
$$(y-3)^2 = 12x + 3 + 9$$
$$(y-3)^2 = 12x + 12$$
$$(y-3)^2 = 12(x+1).$$



Comparing with $(y-k)^2 = 4p(x-h)$

$k = 3, \ h = -1, \ 4p = 12$

$p = 3.$

Vertex: $V(h, \ k) = (-1, 3)$

Focus: $F(h+p, k) = (2, \ 3)$

Directrix: $x = h - p = -1 - 3 = -4.$

End point of latus rectum are

$(h + p, \ k + 2p) = (2, 9)$ and $(h + p, \ k - 2p) = (2, \ -3).$

Example 13

$$(x+5) + (y-1)^2 = 0$$

(a) Sketch the parabola.

(b) Show the focus , vertex, directrix and end points of latus rectum.

Solution

$$(x+5)+(y-1)^2=0$$

$$(y-1)^2=-(x+5).$$

Comparing with $(y-k)^2 = -4p(x-h)$,

$$\therefore h = -5, \quad k = 1, \quad 4p = 1$$

$$p = \frac{1}{4} = 0.25.$$

Vertex: $V(h,\ k) = (-5,\ 1)$

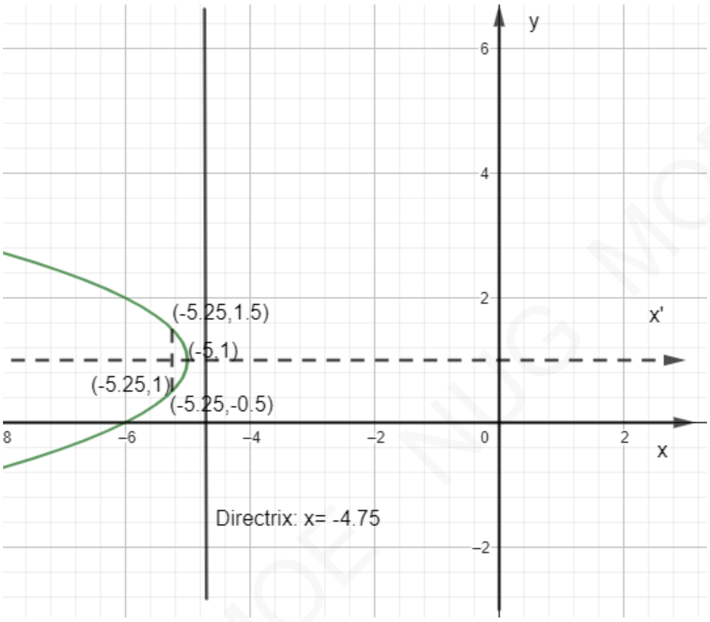
Focus: $F = (h-p, k) = (-5.25,\ 1)$

Directrix: $x = h + p = -5 + 0.25 = -4.75$

End points of latus rectum are

$(h-p, k+2p) = (-5.25, 1.5)$ and

$(h-p, k-2p) = (-5.25, 0.5)$.



Example 14

Write the given parabola equation $x^2 - 2x + 8y - 23 = 0$ in standard form. Sketch its graph showing the vertex, focus, directrix and end points of latus rectum.

Solution

$$x^2 - 2x + 8y - 23 = 0$$

$$x^2 - 2x = -8y + 23$$

$$(x-1)^2 - 1 = -8y + 23$$

$$(x-1)^2 = -8y + 23 + 1$$

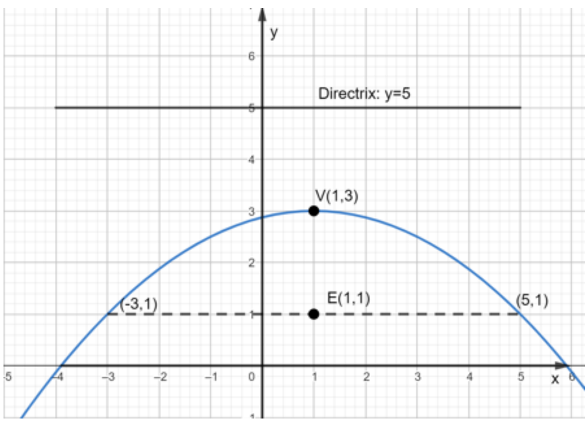
$$(x-1)^2 = -8y + 24$$

$$(x-1)^2 = -8(y-3).$$

Comparing with $(x-h)^2 = -4p(y-k)$

$$\therefore h = 1, \quad k = 3, \quad 4p = 8$$

$$p = 2.$$



Vertex: $V = (h,\ k) = (1, 3)$

Focus: $F = (h,\ k-p) = (1,\ 1)$

Directrix: $y = k + p = 5$

End points of latus rectum are $(h-2p,\ k-p) = (-3,\ 1)$ and $(h+2p,\ k-p) = (5, 1)$.

Example 15

Find an equation for the parabola with vertex (5, 1) and Focus (5, 3).

Solution

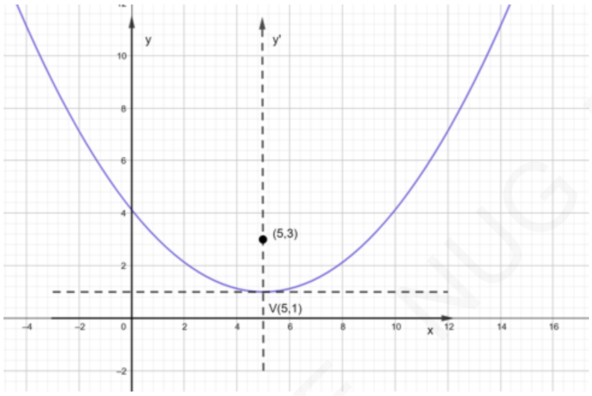
Vertex: $V = (h, k) = (5, 1)$

$\therefore h = 5, k = 1.$

Focus: $F = (h, k + p) = (5,3)$

$k + p = 3$

$p = 2.$



The required parabola equation is

$(x - h)^2 = 4p(y - k)$

$(x - 5)^2 = 8(y - 1).$

Example 16

Find an equation for the parabola with focus $(-1, 4)$, directrix $x = 5$.

Solution

Focus: $F = (-1, 4)$

Directrix: $x = 5$

Vertex: $V = (h, k) = \left(\frac{-1+5}{2}, \frac{4+4}{2}\right)$

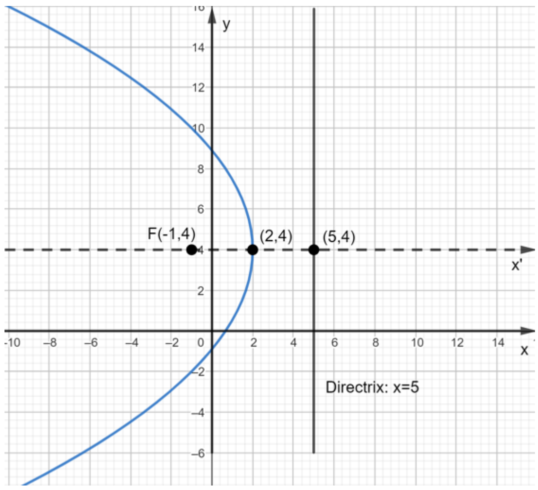
$(h, k) = (2,4)$

$h = 2, \ k = 4.$

Focus: $F = (h - p, k) = (-1,4)$

$h - p = -1$

$p = 3.$



The required parabola equation is

$$(y - k)^2 = -4p(x - h)$$

$$(y - 4)^2 = -12(x - 2).$$

Example 17

Find an equation for the parabola with axis $x = 0$ passes through $(2, -1)$and $(-4, 5)$.

Solution

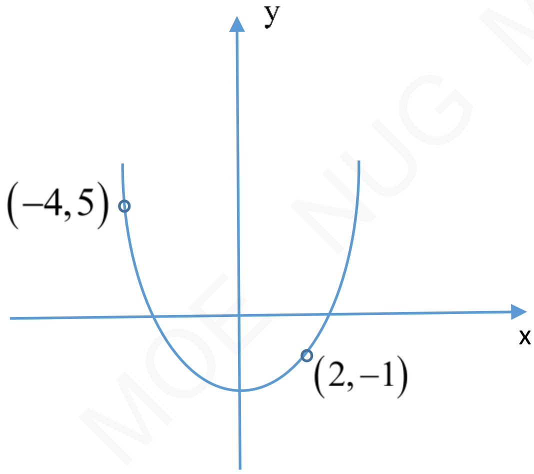
Axis $x = 0$ passes through $(2, -1)$ and $(-4, 5)$.

Vertex $V = (h, k) = (0, k)$

The standard equation of parabola is

$$(x - h)^2 = 4p(y - k)$$

$$x^2 = 4p(y - k).$$



At $(2, -1)$, $4 = 4p(-1 - k)$

$$4 = -4p - 4pk - - - (1)$$

At $(-4, 5)$, $16 = 4p(5 - k)$

$$16 = 20p - 4pk - - - (2)$$

Equation (1) and (2),

$$12 = 24p$$

$$\therefore p = \frac{1}{2}.$$

Substituting $p = \frac{1}{2}$ in equation (1),

$$4 = -2 - 2k$$

$$\therefore k = -3.$$

The required parabola equation is

$$x^2 = 2(y + 3).$$

Exercise 6.3

1. In Exercise a to e, (i) sketch the parabola, (ii) show the focus, vertex, directrix and end point of latus rectum:

(a) $x^2 + 4x + 4y = 0$.

(b) $(x - 1)^2 + 8(y + 1) = 0$.

(c) $y^2 + 6y + 8x + 25 = 0$.

(d) $x^2 + 4x + 6y - 2 = 0$.

(e) $y^2 + x + y = 0$.

2. In Exercise a to e, find an equation for the parabola satisfying the given conditions

(a) Vertex $(4, -5)$, focus $(1, -5)$.

(b) Vertex $(1, 1)$, directrix $y = -2$.

(c) Vertex $(2, -1)$, directrix $x = -1$.

(d) Focus $(6, 0)$, directrix $x = -6$.

(e) Axis $y = 0$ passes through $(3, 2)$ and $(2, -3)$.

6.4 Rotation of Axes

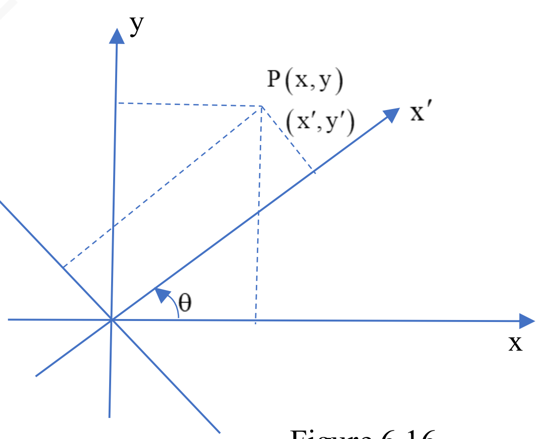


Figure 6.16

The axes of an xy-coordinate system have been rotated about the origin through an angle θ to produce a new $x'y'$-coordinate system. In the figure (6.16), each point P in the plane has coordinates $(x', \; y')$ as well as coordinates $(x, \; y)$.

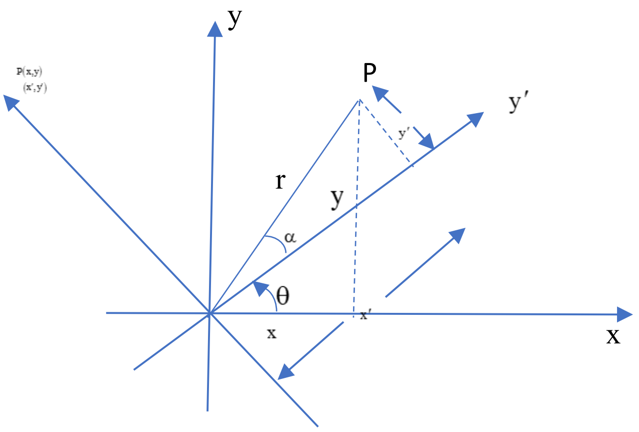


Figure 6.17

Let r be the distance from the common origin to the point P, and let α be the angle shown in figure (6.17).

$$x = r\cos(\theta + \alpha)\,, y = r\sin(\theta + \alpha) \text{ and}$$

$$x' = r\cos\alpha\,, y' = r\sin\alpha.$$

Using familiar trigonometric identities,

$$x = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$$

$$y = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha.$$

The rotation equations:

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta.$$

If the rotation equations are solved for x' and y' in terms of x and y, then

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta.$$

Example 18

Find the new coordinates of the point (1, 2) if the coordinate axes are rotated through an angle of $\theta = 30°$.

Solution

Using the rotation equations,

$$x = 1, \quad y = 2, \quad \cos\theta = \cos 30° = \frac{\sqrt{3}}{2}, \text{ and}$$

$\sin\theta = \sin 30° = \frac{1}{2}$, we obtain

$$x' = x\cos\theta + y\sin\theta$$

$$= 1\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2} + 1.$$

$$y' = -x\sin\theta + y\cos\theta$$

$$= -1\left(\frac{1}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right) = -\frac{1}{2} + \sqrt{3}.$$

$\therefore$ the new coordinates are $\left(\frac{\sqrt{3}}{2} + 1, -\frac{1}{2} + \sqrt{3}\right)$.

Eliminating the $x'y'$-Term

The general equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ with $B \neq 0$,

we substitute $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$

in the general equation, then $A'(x')^2 + B'x'y' + C'(y')^2 + D'(x') + E'(y') + F' = 0$,

where

$$A' = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta$$

$$B' = B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta$$

$$C' = A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta$$

$$D' = D\cos\theta + E\sin\theta$$

$$E' = -D\sin\theta + E\cos\theta$$

$$F' = F.$$

Eliminate the **x′y′**-term, choose θ such that $B' = 0$.

$$B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta = 0$$

$$B(\cos 2\theta) + (C - A)\sin 2\theta = 0$$

$$B(\cos 2\theta) = (A - C)\sin 2\theta$$

$$\frac{\cos 2\theta}{\sin 2\theta} = \frac{A-C}{B}.$$

$$\cot 2\theta = \frac{A-C}{B}.$$

The Discriminant

Theorem

Consider a second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

(a) If $B^2 - 4AC < 0$, the equation represents an ellipse, a circle, a point, or else has no graph.

(b) If $B^2 - 4AC > 0$, the equation represents a hyperbola or a pair of intersecting lines.

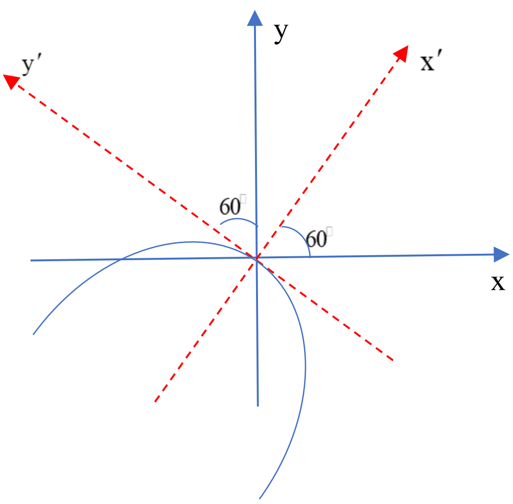
(c) If $B^2 - 4AC = 0$, the equation represents a parabola, a line, a pair of parallel lines, or else has no graph.

Example 19

Rotate the coordinate axes to remove the xy- term of equation

$3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$

in x′y′-coordinate system. Then sketch the graph.



Solution

$3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$

Comparing with $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

$A = 3$, $B = -2\sqrt{3}$, $C = 1$.

Checked by

$$B^2 - 4AC = 12 - 12 = 0$$

The given equation represents a parabola.

Again, $\cot 2\theta = \frac{A-C}{B} = \frac{3-1}{-2\sqrt{3}} = -\frac{1}{\sqrt{3}}$

$$2\theta = 120°, \qquad \theta = 60°.$$

$$\cos\theta = \cos 60° = \frac{1}{2}, \qquad \sin\theta = \sin 60° = \frac{\sqrt{3}}{2}.$$

Since, $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$, we get

$$x = \frac{1}{2}x' - \frac{\sqrt{3}}{2}y', \quad y = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y'.$$

Substitute in given equation,

$$3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$$

$$\left(\sqrt{3}x\right)^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$$

$$\left(\sqrt{3}x - y\right)^2 + 2\left(x + \sqrt{3}y\right) = 0$$

$$\left[\sqrt{3}\left(\frac{1}{2}x' - \frac{\sqrt{3}}{2}y'\right) - \left(\frac{\sqrt{3}}{2}x' + \frac{1}{2}y'\right)\right]^2 + 2\left[\frac{1}{2}x' - \frac{\sqrt{3}}{2}y' + \sqrt{3}\left(\frac{\sqrt{3}}{2}x' + \frac{1}{2}y'\right)\right] = 0$$

$$(-2y')^2 + 2(2x') = 0$$

$$4y'^2 + 4x' = 0$$

$$y'^2 = -x'.$$

The parabola opens to the left with vertex at $(0,\ 0)$.

Example 20

Rotate the coordinate axes to remove the xy-term of the equation.

$$16x^2 - 24xy + 9y^2 + 100x - 200y + 100 = 0$$

in $x'y'$-coordinate system. Then sketch the graph.

Solution

$$16x^2 - 24xy + 9y^2 + 100x - 200y + 100 = 0$$

Comparing with

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$A = 16, \ B = -24, \ C = 9$$

Checked by

$$B^2 - 4AC = (-24)^2 - 4 \times 16 \times 9 = 0$$

The given equation represents a parabola.

Again,

$$\cot 2\theta = \frac{A-C}{B} = \frac{16-9}{-24} = -\frac{7}{24}$$

Using trigonometric identities,

$$\frac{1}{\cot^2 2\theta} + 1 = \frac{1}{\cos^2 2\theta}$$

$$\left(-\frac{24}{7}\right)^2 + 1 = \frac{1}{\cos^2 2\theta}$$

$$\cos^2 2\theta = \frac{49}{625}$$

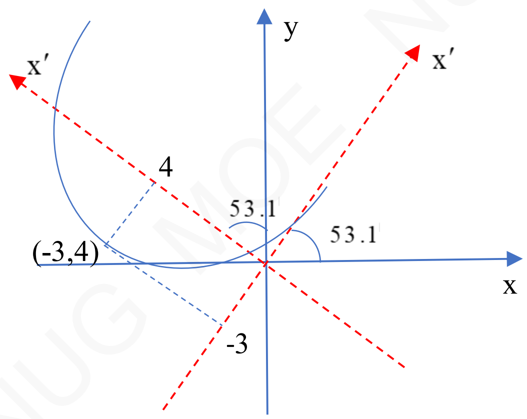
$$\therefore 2\cos^2\theta - 1 = -\frac{7}{25}$$

$$2\cos^2\theta = \frac{18}{25}$$

$$\cos^2\theta = \frac{9}{25}$$

$$\cos\theta = \frac{3}{5}$$

$$\theta = 53.1°$$



$$\text{and } \cos^2\theta + \sin^2\theta = 1$$

$$\left(\frac{3}{5}\right)^2 + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{16}{25}$$

$$\therefore \sin\theta = \frac{4}{5}.$$

Since $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$, we get

$$x = \frac{3}{5}x' - \frac{4}{5}y', \quad y = \frac{4}{5}x' + \frac{3}{5}y'.$$

Substitute in given equation,

$$16x^2 - 24xy + 9y^2 + 100x - 200y + 100 = 0$$

$$(4x - 3y)^2 + 100(x - 2y) + 100 = 0$$

$$\left(\frac{12}{5}x' - \frac{16}{5}y' - \frac{12}{5}x' - \frac{9}{5}y'\right)^2 + 100\left(\frac{3}{5}x' - \frac{4}{5}y' - \frac{8}{5}x' - \frac{6}{5}y'\right) + 100 = 0$$

$$(-5y')^2 + 100(-x' - 2y') + 100 = 0$$

$$25y'^2 - 100x' - 200y' + 100 = 0$$

$$y'^2 - 4x' - 8y' + 4 = 0$$

$$y'^2 - 8y' = 4x' - 4$$

$$(y' - 4)^2 = 4(x' + 3).$$

Opens to the right with vertex $(-3,\ 4)$.

Example 21

Rotate the coordinate axes to remove the xy- term of the equation

$$x^2 - 2xy + y^2 - 20x - 20y + 100 = 0$$

in $x'y'$-coordinate system. Then sketch the graph.

Solution

$$x^2 - 2xy + y^2 - 20x - 20y + 100 = 0$$

Comparing with

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$A = 1, \quad B = -2, \quad C = 1.$$

Checked by

$$B^2 - 4AC = (-2)^2 - 4 \times 1 \times 1 = 0$$

The given equation represents a parabola.

Again,

$$\cot 2\theta = \frac{A-C}{B} = \frac{1-1}{-2} = 0$$

$$2\theta = 90°$$

$$\theta = 45°$$

$$\cos\theta = \cos 45° = \frac{\sqrt{2}}{2}, \quad \sin\theta = \sin 45° = \frac{\sqrt{2}}{2},$$

Since $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$, we get

$$x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y', \qquad y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'.$$

Substitute in given equation,

$$x^2 - 2xy + y^2 - 20x - 20y + 100 = 0$$

$$(x-y)^2 - 20(x+y) + 100 = 0$$

$$\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' - \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)^2 - 20\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' + \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) + 100 = 0$$

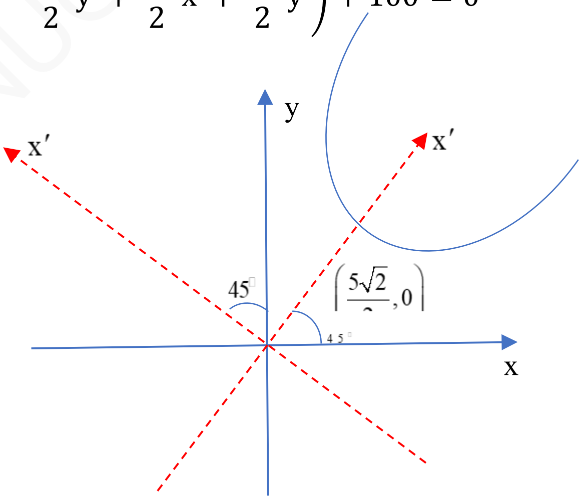
$$\left(-\sqrt{2}y'\right)^2 - 20\left(\sqrt{2}x'\right) + 100 = 0$$

$$2y'^2 - 20\sqrt{2}x' + 100 = 0$$

$$2y'^2 = 20\sqrt{2}x' - 100$$

$$y'^2 = 10\sqrt{2}x' - 50$$

$$y'^2 = 10\sqrt{2}\left(x' - \frac{5\sqrt{2}}{2}\right)$$



Opens to the right with vertex $\left(\frac{5\sqrt{2}}{2}, 0\right)$.

Exercise 6.4

Rotate the coordinate axes to remove the xy- term. Then sketch the graph.

1. $x^2 + 2\sqrt{3}xy + 3y^2 + 2\sqrt{3}x - 2y = 0.$
2. $9x^2 - 24xy + 16y^2 - 80x - 60y + 100 = 0.$
3. $16x^2 + 24xy + 9y^2 - 130x + 90y = 0$
4. $3x^2 + 2\sqrt{3}xy + y^2 - 8x + 8\sqrt{3}y = 0.$

Chapter 7

TRIGONOMETRIC FUNCTIONS

In this chapter, we start the graph of sine function, periodic function and the amplitude of a periodic function.Then, the graph of cosine, tangent and other trigonometric functions are shown. After that, the inverse of those functions and their graphs come. Finally, we introduce the differentiation of trigonometric functions.

7.1 Graphs of Sine Functions

Periodic Function

If f (x) = f (x + p) where p is a positive real number, the function f is called a periodic function. If p is the smallest such number, then p is called the period of function f.

Since sin x = sin (x+ 2π), then y = sin x is a periodic function with period 2π.

Amplitude of a periodic function

The amplitude of a periodic function is the half the difference between the maximum and minimum values.

So, amplitude of y = sin x = $\frac{1}{2}$(1–(– 1)) = 1.

Graph of the Sine Function y = sin x

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π
sin x	0	–1	0	1	0	–1	0	1	0

Table 7.1

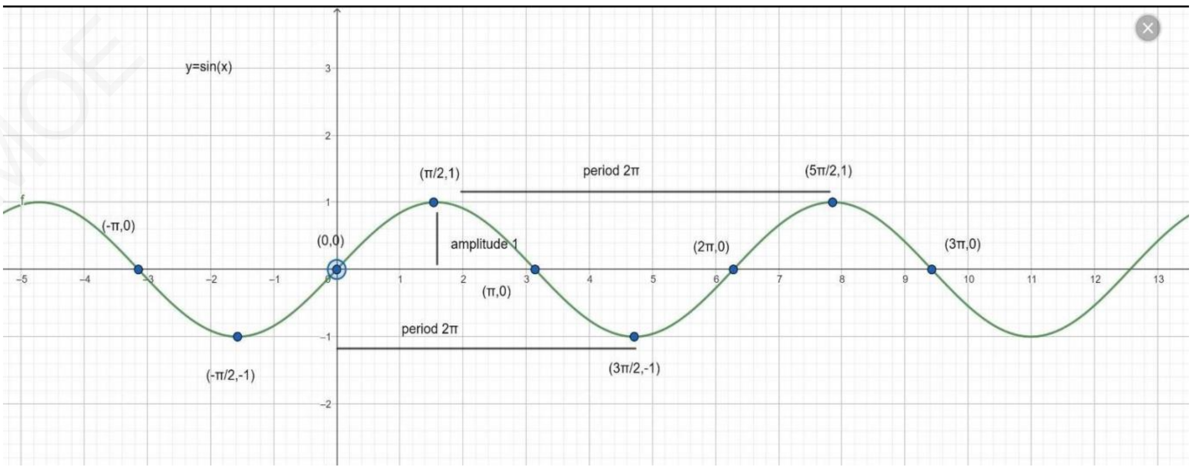


Figure 7.1

Domain : The set $\mathbb{R}$ of all real numbers

Range : $\{y| -1 \le y \le 1\}$

Period : 2π

Amplitude : 1.

Five key points:

x-intercepts : $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$,

maximum point: $(\frac{\pi}{2}, 1)$,

minimum point : $(\frac{3\pi}{2}, -1)$.

Graph of the Sine Function y = a sin bx, a > 0, b > 0

In the following example we will see how to obtain the graph of $y = a \sin bx$, where $a > 0$ and $b > 0$, from the graph of $y = \sin x$.

Example 1

From the graph of $y = \sin x$, draw step-by-step transformation graphs to get the graph of $y = 2\sin \frac{\pi}{2}x$.

Solution

<u>Method 1</u>

$$y = \sin x \xrightarrow[\text{scale factor 2}]{\text{vertical scaling}} y = 2 \sin x$$

$(x, y) \to (x, 2y)$

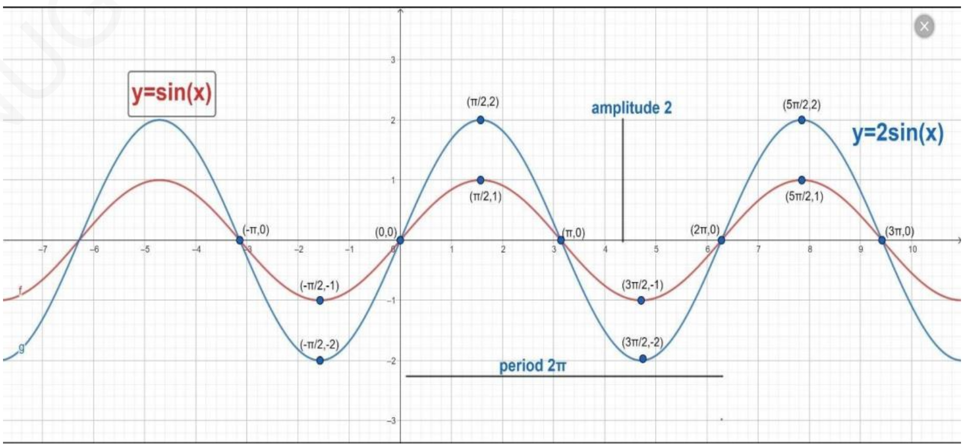
$(0, 0) \to (0, 0)$

$(\frac{\pi}{2}, 1) \to (\frac{\pi}{2}, 2)$

$(\pi, 0) \to (\pi, 0)$

$(\frac{3\pi}{2}, -1) \to (\frac{3\pi}{2}, -2)$

$(2\pi, 0) \to (2\pi, 0)$



$$y = \sin x \xrightarrow[\text{scale factor 2}]{\text{vertical scaling}} y = 2 \sin x \xrightarrow[\text{scale factor 2}]{\text{horizonal scaling}} y = 2\sin \frac{\pi}{2}x$$

$(x, y) \rightarrow (x, 2y) \rightarrow (\frac{2x}{\pi}, 2y)$

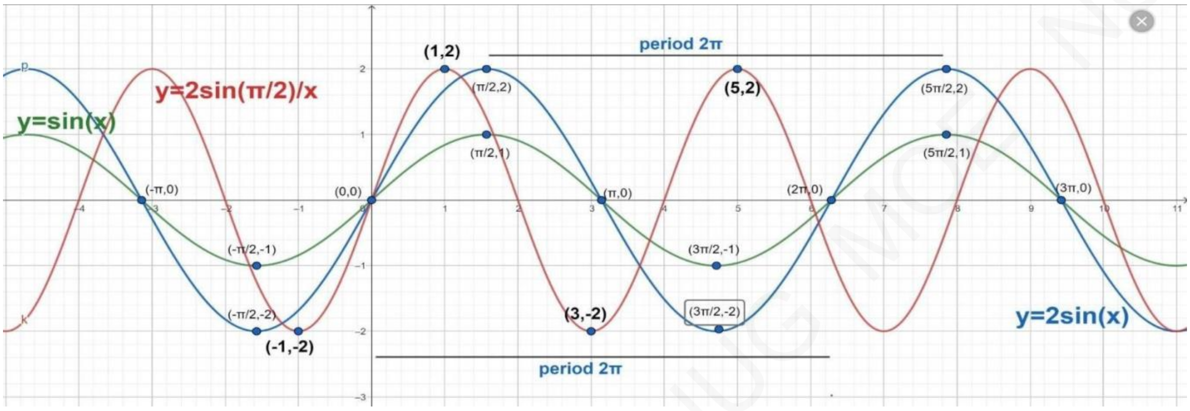
$(0, 0) \rightarrow (0, 0) \rightarrow (0, 0)$

$(\frac{\pi}{2}, 1) \rightarrow (\frac{\pi}{2}, 2) \rightarrow (1, 2)$

$(\pi, 0) \rightarrow (\pi, 0) \rightarrow (2, 0)$

$(\frac{3\pi}{2}, -1) \rightarrow (\frac{3\pi}{2}, -2) \rightarrow (3, -2)$

$(2\pi, 0) \rightarrow (2\pi, 0) \rightarrow (4, 0)$.



Domain = ℝ, Range = $\{y| -2 \le y \le 2\}$, Period = 4, Amplitude = 2.

Five key points:

x-intercepts : (0, 0), (2, 0), (4, 0),

maximum point: (1, 2),

minimum point : (3, −2) .

Method 2

$$y = \sin x \xrightarrow[\text{scale factor } \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}]{\text{horizontal scaling}} y = \sin \frac{\pi}{2}x$$

$(x, y) \rightarrow (\frac{2x}{\pi}, y)$

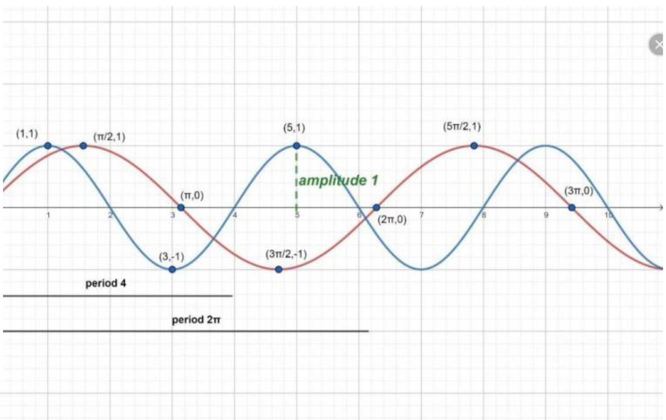
$(0, 0) \rightarrow (0, 0)$

$(\frac{\pi}{2}, 1) \rightarrow (1, 1)$

$(\pi, 0) \rightarrow (2, 0)$

$(\frac{3\pi}{2}, -1) \rightarrow (3, -1)$

$(2\pi, 0) \rightarrow (4, 0)$



$$y = \sin x \xrightarrow[\text{scale factor } \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}]{\text{horizontal scaling}} = \sin\frac{\pi}{2}x \xrightarrow[\text{scale factor 2}]{\text{vertical scaling}} y = 2\sin\frac{\pi}{2}x$$

$(x, y) \rightarrow (\frac{2x}{\pi}, y) \rightarrow (\frac{2x}{\pi}, 2y)$

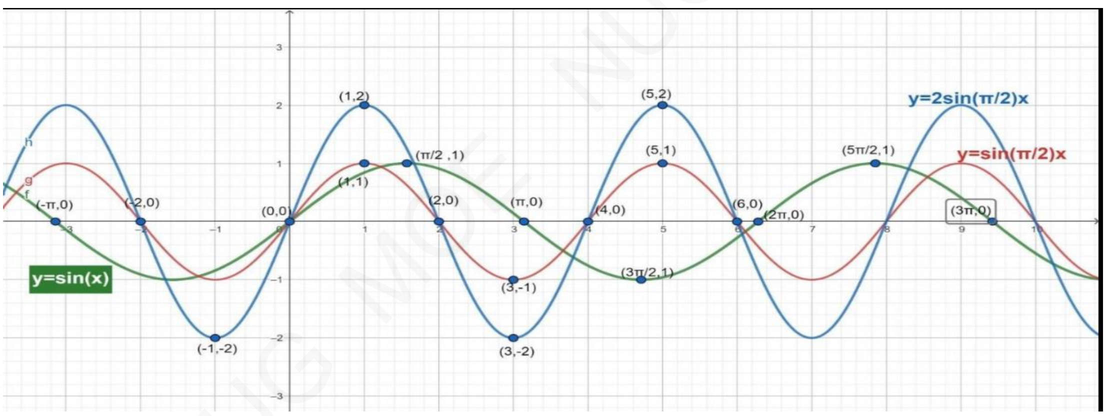
$(0, 0) \rightarrow (0, 0) \rightarrow (0, 0)$

$(\frac{\pi}{2}, 1) \rightarrow (1, 1) \rightarrow (1, 2)$

$(\pi, 0) \rightarrow (2, 0) \rightarrow (2, 0)$

$(\frac{3\pi}{2}, -1) \rightarrow (3, -1) \rightarrow (3, -2)$

$(2\pi, 0) \rightarrow (4, 0) \rightarrow (4, 0)$.



Domain = $\mathbb{R}$

Range = $\{y| -2 \le y \le 2\}$

Period = 4

Amplitude = 2.

Five key points:

x-intercepts : (0, 0), (2, 0), (4, 0),

maximum point: (1, 2),

minimum point : (3, –2).

From the graph of y = sin x, the graph of y = a sin bx, a > 0, b > 0, can be obtained as

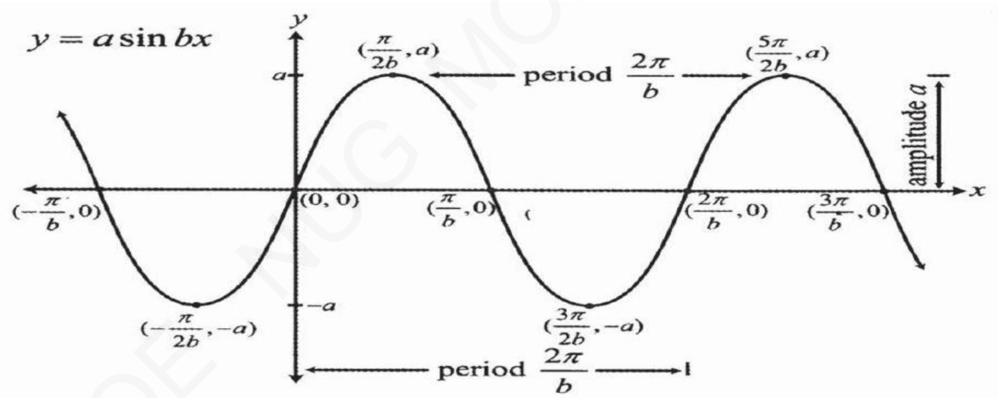
$$y = \sin x \xrightarrow[\text{scale factor } 2]{\text{vertical scaling}} y = 2\sin x \xrightarrow[\text{scale factor } \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}]{\text{horizontal scaling}} y = 2\sin \frac{\pi}{2}x .$$

(or)

$$y = \sin x \xrightarrow[\text{scale factor } \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}]{\text{horizontal scaling}} y = \sin \frac{\pi}{2}x \xrightarrow[\text{scale factor } 2]{\text{vertical scaling}} y = 2\sin \frac{\pi}{2}x.$$

$$y = \sin x \;\rightarrow\; y = a \sin bx$$
$$(x, y) \;\rightarrow\; (\frac{x}{b}, ay)$$

x	$-\frac{\pi}{b}$	$-\frac{\pi}{2b}$	0	$\frac{\pi}{2b}$	$\frac{\pi}{b}$	$\frac{3\pi}{2b}$	$\frac{2\pi}{b}$	$\frac{5\pi}{2b}$	$\frac{3\pi}{b}$
a sin bx	0	–a	0	a	0	–a	0	a	0



Domain : ℝ

Range : {y| –a ≤ y ≤ a}

Period : $\frac{2\pi}{b}$

Amplitude : a

Five key points:

x- intercepts : (0, 0), $(\frac{\pi}{b}, 0)$, $(\frac{2\pi}{b}, 0)$

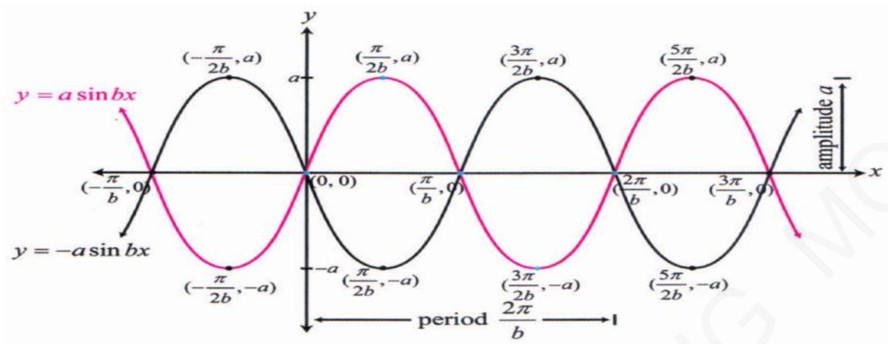
Maximum point : $(\frac{\pi}{2b}, a)$,

Minimum point : $(\frac{3\pi}{2b}, -a)$.

Graph of the Sine Function y = – a sin bx, a > 0, b > 0

x	$-\frac{\pi}{b}$	$-\frac{\pi}{2b}$	0	$\frac{\pi}{2b}$	$\frac{\pi}{b}$	$\frac{3\pi}{2b}$	$\frac{2\pi}{b}$	$\frac{5\pi}{2b}$	$\frac{3\pi}{b}$
–a sin bx	0	a	0	–a	0	a	0	–a	0

Table 7.2



Domain : ℝ

Range : $\{y \mid -a \le y \le a\}$

Period : $\frac{2\pi}{b}$

Amplitude : a

Figure 7.2

Five key points:

x- intercepts : (0, 0), $(\frac{\pi}{b}, 0)$, $(\frac{2\pi}{b}, 0)$

Maximum point : $(\frac{3\pi}{2b}, a)$

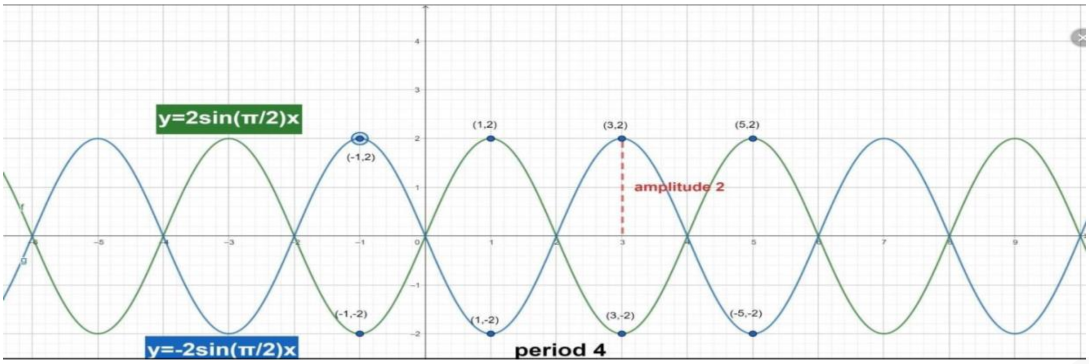
Minimum point : $(\frac{\pi}{2b}, -a)$.

Example 2

Draw the graph of $y = 2\sin\frac{\pi}{2}x$ and $y = -2\sin\frac{\pi}{2}x$.

Solution

x	–2	–1	0	1	2	3	4	5	6
$y = 2\sin\frac{\pi}{2}x$	0	–2	0	2	0	–2	0	2	0
$y = -2\sin\frac{\pi}{2}x$	0	2	0	–2	0	2	0	–2	0



Five key points:

x- intercepts: (0, 0),(2, 0), (4,0)

For $y = 2\sin\frac{\pi}{2}x$,

maximum point : (1, 2)

minimum point : (3, –2).

For $y = -2\sin\frac{\pi}{2}x$,

maximum point : (3, 2)

minimum point : (1, –2)

Domain = $\mathbb{R}$

Range = $\{y \mid -2 \le y \le 2\}$

Period = 4

Amplitude = 2 .

Graph of the Sine Function y = a sin b(x–h) + k

$$y = a\sin bx \xrightarrow[\text{vertical translation k units}]{\text{horizontal translation h units}} y = a\sin b(x-h) + k$$

$$y = \sin x \rightarrow y = a\sin bx \rightarrow y = a\sin b(x-h) + k$$
$$(x, y) \rightarrow (\tfrac{x}{b}, ay) \rightarrow (\tfrac{x}{b} + h, ay + k)$$

Key points:

On the midline y = k

* $(0, 0) \rightarrow (0, 0) \rightarrow (h, k)$
* $(\pi, 0) \rightarrow (\frac{\pi}{b}, 0) \rightarrow (\frac{\pi}{b} + h, k)$
* $(2\pi, 0) \rightarrow (\frac{2\pi}{b}, 0) \rightarrow (\frac{2\pi}{b} + h, k)$

Maximum and minimum points

* $(\frac{\pi}{2}, 1) \rightarrow (\frac{\pi}{2b}, a) \rightarrow (\frac{\pi}{2b} + h, a+k)$
* $(\frac{3\pi}{2}, -1) \rightarrow (\frac{3\pi}{2b}, -a) \rightarrow (\frac{3\pi}{2b} + h, -a+k)$

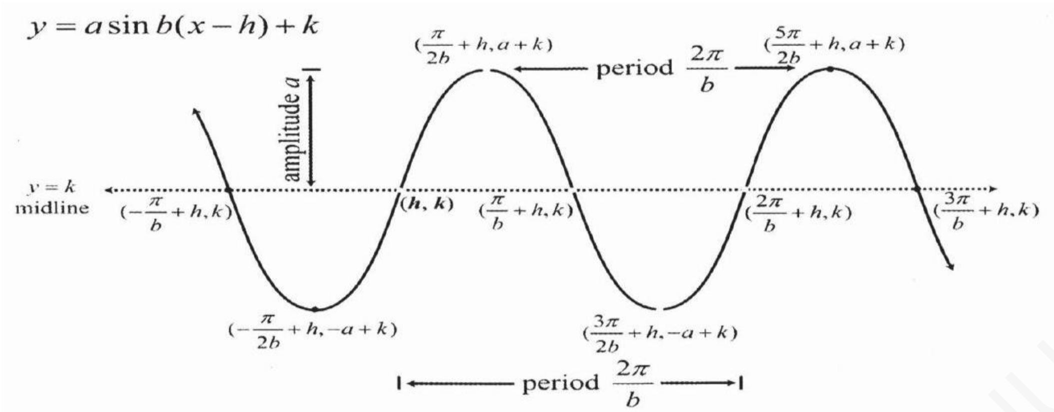


Figure 7.3

Example 3

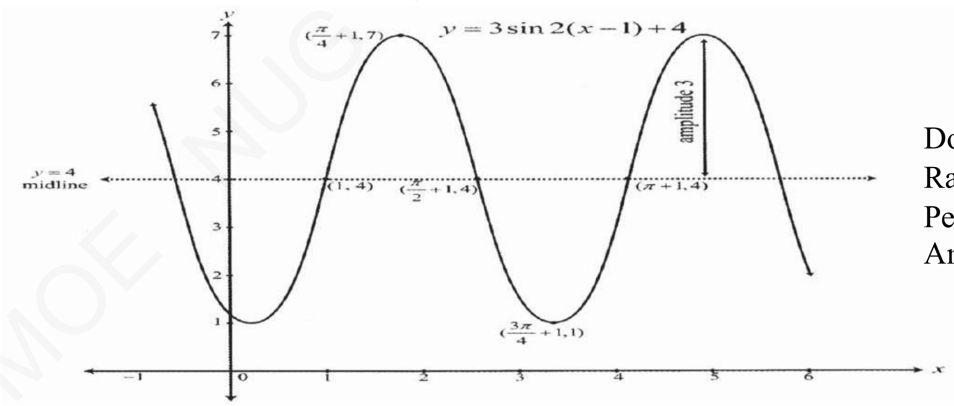
Draw the graph of y = 3 sin 2(x– 1) + 4.

Solution

a = 3, b= 2, h = 1, k = 4

Midline y = 4, amplitude = 3,

x	$-\frac{\pi}{4}+1$	1	$\frac{\pi}{4}+1$	$\frac{\pi}{2}+1$	$\frac{3\pi}{4}+1$	$\pi+1$	$\frac{5\pi}{4}+1$	$\frac{3\pi}{2}+1$
y = 3 sin 2(x– 1) + 4	1	4	7	4	1	4	7	4



Domain = ℝ
Range = {y| 1 ≤ y ≤ 7}
Period = π
Amplitude = 3

Five key points:

x- intercepts : (1 4), ($\frac{\pi}{2}$ +1, 4), (π+1, 4)

Maximum point : ($\frac{\pi}{4}$ +1, 7)

Minimum point : ($\frac{3\pi}{4}$ +1, 1) .

Example 4

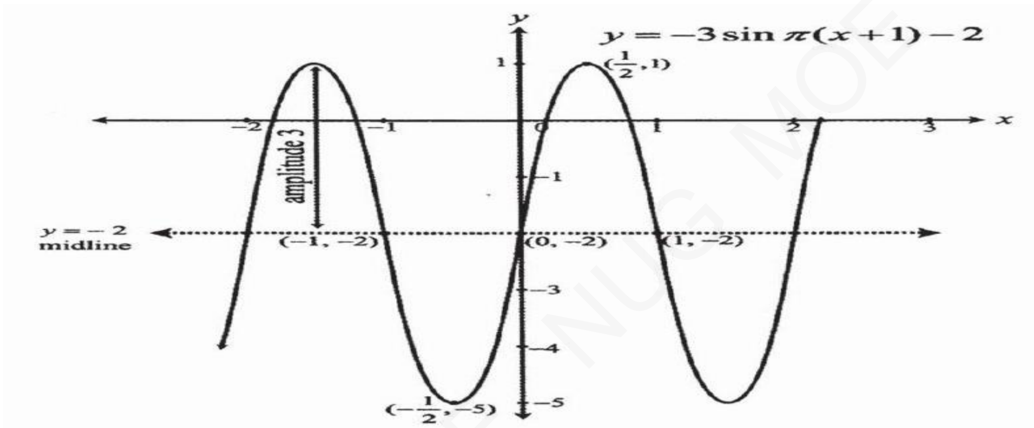
Draw the graph of $y = -3 \sin \pi(x + 1) - 2$.

Solution

$a = 3$, $b = \pi$, $h = -1$, $k = -2$

Midline: $y = -2$, Amplitude = 3, Period = 2.

x	–1	–0.5	0	0.5	1	1.5	2	2.5
$y = -3 \sin \pi(x+1) - 2$	–2	–5	–2	1	–2	–5	–2	1



Domain = $\mathbb{R}$

Range = $\{y \mid -5 \le y \le 1\}$.

Five key points:

x- intercepts : $(-1, -2)$, $(0, -2)$, $(1, -2)$

Minimum point : $(-0.5, -5)$

Maximum point : $(0.5, 1)$.

Exercise 7.1

1. From the graph of y = sin x, draw step-by-step transformation graphs to get the graph of $y = -3\sin\frac{\pi}{3}x$.
2. Draw the graph of (a) $y = \frac{1}{2}\sin x$, (b) $y = \sin 4x$.
3. Draw the graph of (a) $y = 2\sin\frac{\pi}{4}x$, (b) $y = -\sin \pi x$.
4. Draw the graph of (a) $y = 2\sin\frac{\pi}{3}(x-2) + 1$, (b) $y = -2\sin\frac{1}{2}(x+1) + 2$.
5. Show that y = a sin bx is an odd function.

7.2 Graphs of Cosine Functions

Graph of Cosine Function y = cos x

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π
y = cos x	–1	0	1	0	–1	0	1	0	–1

Table 7.3

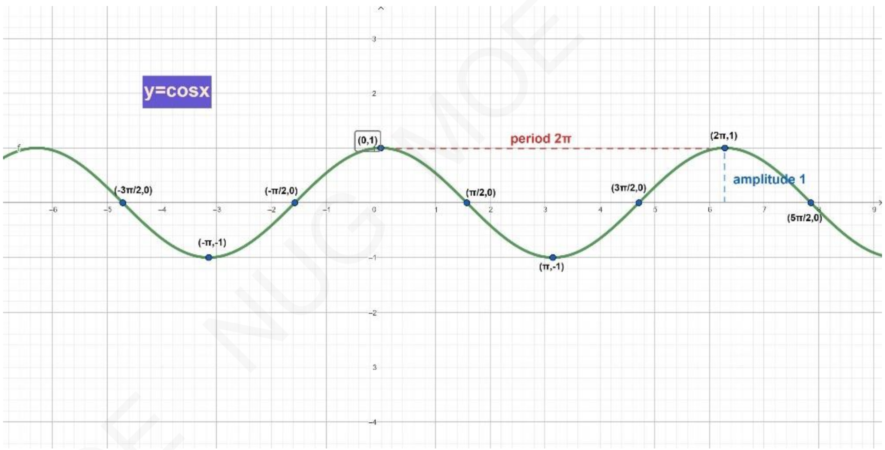


Figure 7.4

Domain = $\mathbb{R}$
Range = $\{y \mid -1 \le y \le 1\}$
Period = 2π
Amplitude = 1
Five key points:
x- intercepts: $(\frac{\pi}{2}, 0), (\frac{3\pi}{2}, 0)$
Maximum point : $(0, 1), (2\pi, 1)$
Minimum point : $(\pi, -1)$

Graph of Cosine Function y = a cos bx, a > 0, b > 0

From the graph of y = cos x, the graph of y = a cos bx can be obtained as

y = cos x —(vertical scaling, scale factor a; horizontal scaling, scale factor $\frac{1}{b}$)→ y = a cos bx

$(x, y) \to (\frac{x}{b}, ay)$

x	$-\frac{\pi}{b}$	$-\frac{\pi}{2b}$	0	$\frac{\pi}{2b}$	$\frac{\pi}{b}$	$\frac{3\pi}{2b}$	$\frac{\pi}{2b}$	$\frac{5\pi}{2b}$	$\frac{3\pi}{b}$
y = a cos bx	–a	0	a	0	–a	0	1	0	–1

Table 7.4

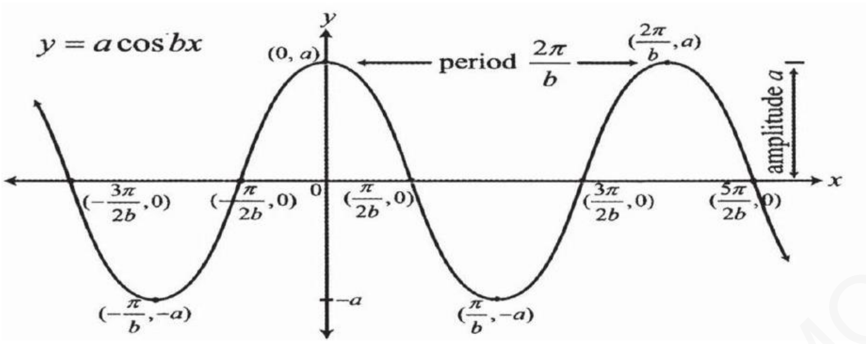


Figure 7.5

Domain = ℝ
Range = {y| – a ≤ y ≤ a}
Period $= \frac{2\pi}{b}$
Amplitude = a
Five key points:
x- intercepts : $(\frac{\pi}{2b}, 0), (\frac{3\pi}{2b}, 0)$
Maximum point : (0, a), $(\frac{2\pi}{b}, a)$
Minimum point : $(\frac{\pi}{b}, -a)$

Graph of Cosine Function y = – a cos bx, a > 0, b > 0

y = a cos bx $\xrightarrow{\text{Reflection on the x-axis}}$ y = – a cos bx

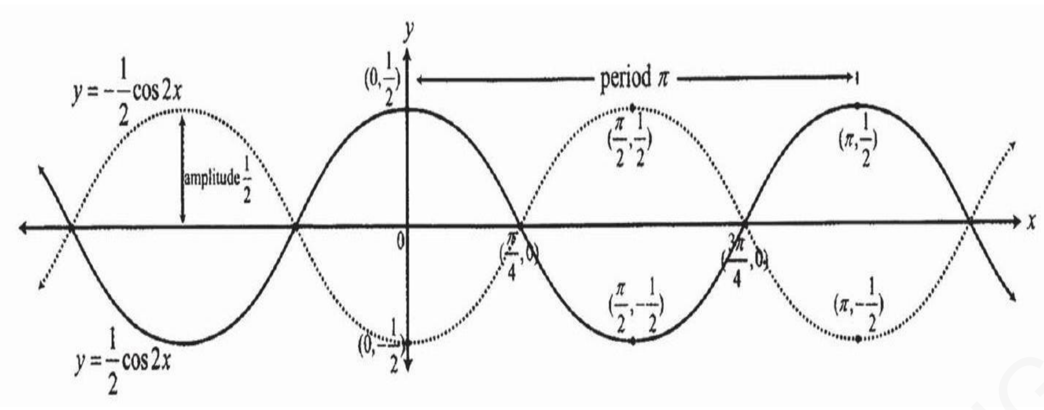
*Note that y = a cos (–b) x = a cos bx, so no need to consider b < 0.

Example 5

Draw the graphs of $y = \frac{1}{2}\cos 2x$ and $y = -\frac{1}{2}\cos 2x$.

Solution

x	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$
$y = \frac{1}{2}\cos 2x$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0
$y = -\frac{1}{2}\cos 2x$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0



Graph of Cosine Function y = a cos b(x –h) + k, a > 0, b > 0

$y = a\cos bx \xrightarrow[\text{vertical translation k units}]{\text{horizontal translation h units}} = a\cos b(x-h) + k$

$y = \cos x \;\rightarrow\; y = a\cos bx \rightarrow\; y = a\cos b(x-h) + k$

$(x, y) \rightarrow (\frac{x}{b}, ay) \rightarrow (\frac{x}{b} + h, ay + k)$

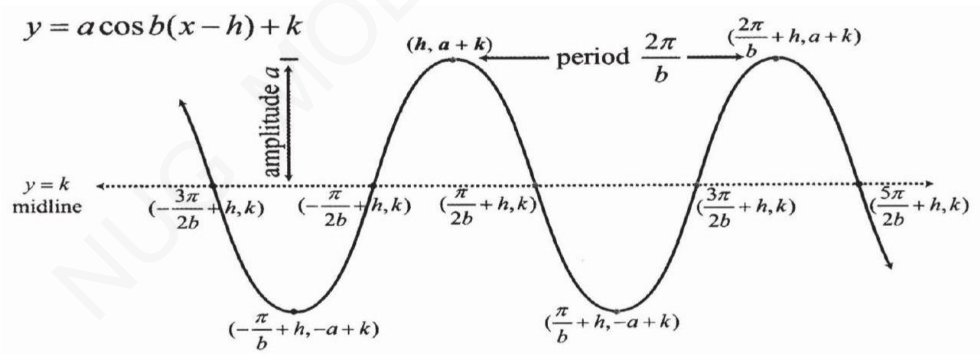


Figure 7.6

Key points:

points on the number line : $(\frac{\pi}{2b} + h, k)$, $(\frac{3\pi}{2b} + h, k)$

Maximum point : $(\frac{2\pi}{b} + h, a + k)$

Minimum point : $(\frac{\pi}{b} + h, -a + k)$

Mid-line: y = k

Period: $\frac{2\pi}{b}$

Amplitude: a.

Example 6

Draw the graphs of $y = 2\cos\frac{\pi}{3}(x+1) + 3$ and $y = -2\cos\frac{\pi}{3}(x+1) + 3$.

Solution

$a = 2,\ b = \frac{\pi}{3},\ h = -1,\ k = 3.$

Midline: $y = 3$, Amplitude: 2, Period: $\frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{3}} = 6$,

Five key points for $y = 2\cos\frac{\pi}{3}(x+1) + 3$:

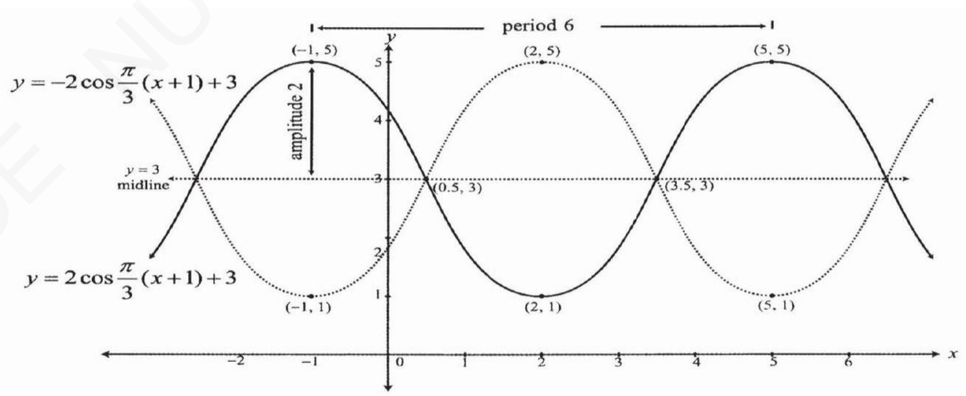
x- intercepts: $(\frac{\pi}{2b} + h, k) = (0.5, 3)$, $(\frac{3\pi}{2b} + h, k) = (3.5, 3)$,

Maximum point: $(h, a+k) = (5, 5)$, $(\frac{2\pi}{b} + h, a+k) = (-1, 5)$,

Minimum point: $(\frac{\pi}{b} + h, -a+k) = (2,1)$.

$y = 2\cos\frac{\pi}{3}(x+1) + 3$ $\xrightarrow{\text{Reflection on the x-axis}}$ $y = -2\cos\frac{\pi}{3}(x+1) + 3$.

x	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$y = 2\cos\frac{\pi}{3}(x+1) + 3$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$
$y = -2\cos\frac{\pi}{3}(x+1) + 3$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$



Exercise 7.2

1. From the graph of y = cos x, draw step-by-step transformation graphs to get the graph of $y = -3\cos\frac{\pi}{2}x$.

2. Draw the graph of (a) $y = \frac{1}{2}\cos x$, (b) $y = \cos 4x$.

3. Draw the graphs of (a) $y = 2\cos\frac{\pi}{4}x$, (b) $y = -\cos \pi x$.

4. Draw the graphs of (a) $y = 2\cos \pi (x - 2) + 1$, (b) $y = -2\cos\frac{1}{2}(x + 1) + 2$.

5. Show that y = a cos bx is an even function.

7.3 Graphs of other Trigonometric Functions

Graph of y = tan x

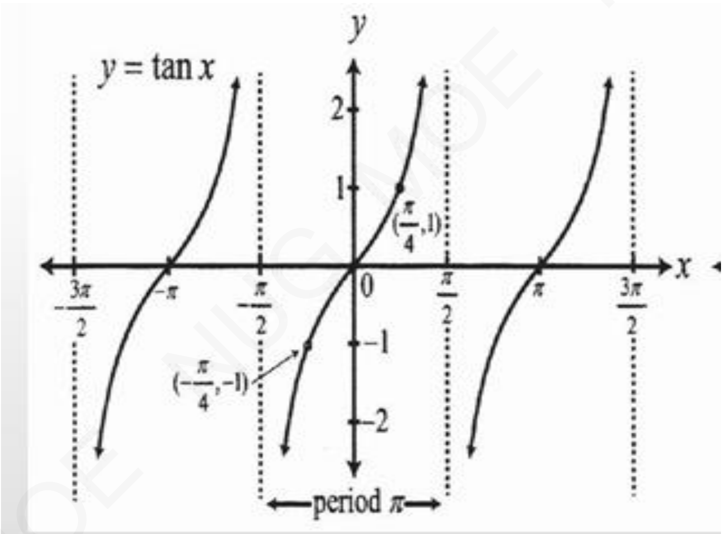


Figure 7.7

Domain : $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

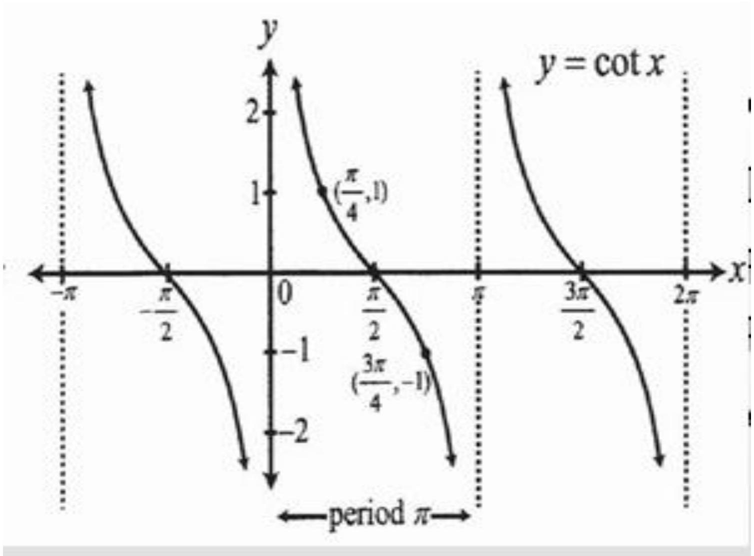
Range : $\mathbb{R}$

Period : π

x- intercepts : $x = 0, \pm\pi, \pm 2\pi, \ldots$

Asymptotes : $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Graph of y = cot x



Domain : x ≠ 0, ± π, ± 2π, …
Range : $\mathbb{R}$
Period : π
x- intercepts : x = ± $\frac{\pi}{2}$, ± $\frac{3\pi}{2}$, …
Asymptotes : x = 0, ± π, ± 2π, …

Figure 7.8

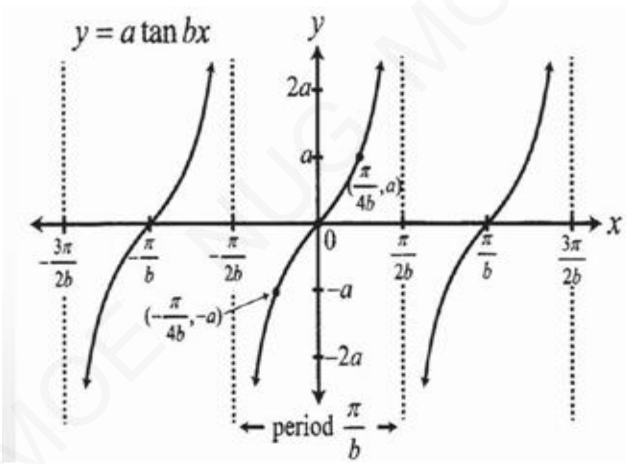
Example 7

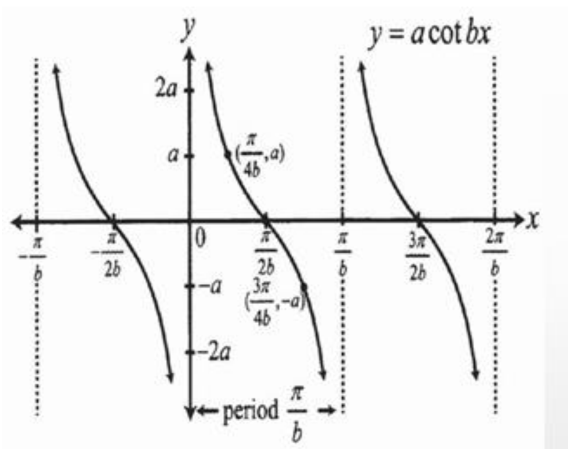
Draw the graphs of y = a tan bx and y = a cot bx for a, b > 0.

Solution:

y = tan x → y = a tan bx, y = cot x → y = a cot bx

(x, y) → ($\frac{x}{b}$, ay).





x- intercepts: x = 0, ± $\frac{\pi}{b}$, ± $\frac{2\pi}{b}$, ...
Domain: x ≠ ± $\frac{\pi}{2b}$, ± $\frac{3\pi}{2b}$, …
Range : $\mathbb{R}$
Period : $\frac{\pi}{b}$
Asymptotes: x = ± $\frac{\pi}{2b}$, ± $\frac{3\pi}{2b}$, …

x- intercepts: x = ± $\frac{\pi}{2b}$, ± $\frac{3\pi}{2b}$, …
Domain: x ≠ 0, ± $\frac{\pi}{b}$, ± $\frac{2\pi}{b}$, …
Range : $\mathbb{R}$
Period : $\frac{\pi}{b}$
Asymptotes: x = 0, ± $\frac{\pi}{b}$, ± $\frac{2\pi}{b}$, …

Graphs of y = sec x and y = csc x

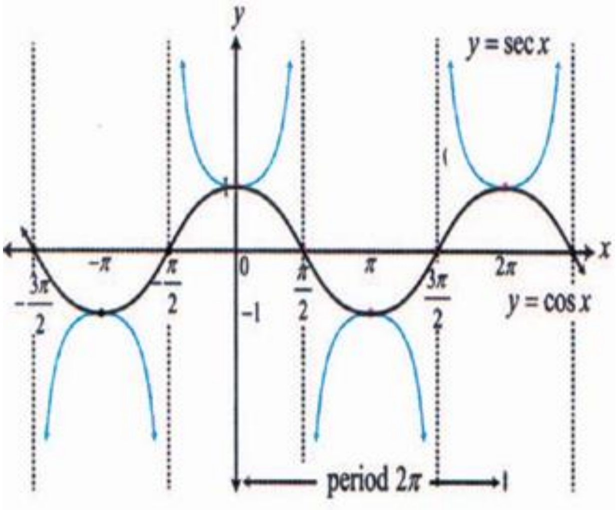


Figure 7.9

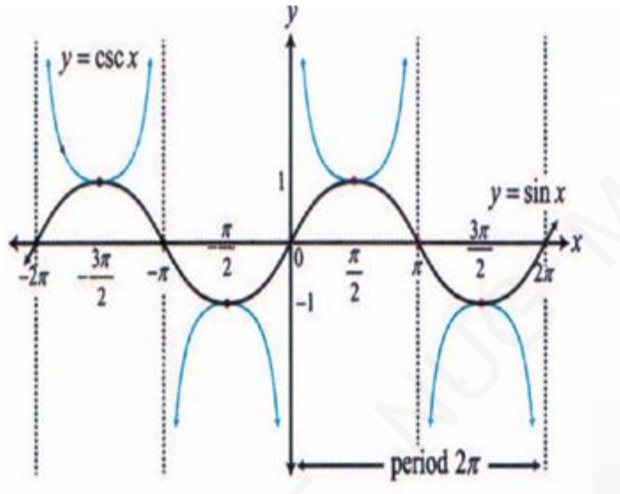


Figure 7.10

Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range : $\{y|\ y \leq -1 \text{ (or) } y \geq 1\}$
Period : 2π
Asymptotes: $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range : $\{y|\ y \leq -1 \text{ (or) } y \geq 1\}$
Period : 2π
Asymptotes: $x = 0, \pm\pi, \pm 2\pi, \ldots$

Note that transform y = sec x to y = a sec bx and y = csc x to y = a csc bx with $(x, y) \rightarrow (\frac{x}{b}, ay)$, we get the graphs of y = a sec bx and y = a csc bx.

Exercise 7.3

1. Draw the graph of (a) $y = \frac{1}{2}\tan \pi x$, (b) $y = 2\tan\frac{1}{2}x$.

2. Draw the graph of (a) $y = 2\cot\frac{\pi}{3}x$, (b) $y = 3\cot 2x$.

3. Draw the graph of (a) $y = 2\sec \pi x$, (b) $y = 3\csc\frac{1}{2}x$.

4. Show that y = a tan bx and y = a csc bx are odd functions.

5. Show that y = a sec bx is an even function.

7.4 Inverse of Trigonometric Functions

Trigonometric functions are not one-to-one functions, we need to restrict the domain of each function to be a one-to-one function to get the inverse trigonometric functions.
The following graphs show that y = sin x, y = cos x, and y = tan x are one-to-one in the restricted domains.

$y = \sin x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ $y = \cos x,\ 0 \le x \le \pi$ $y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

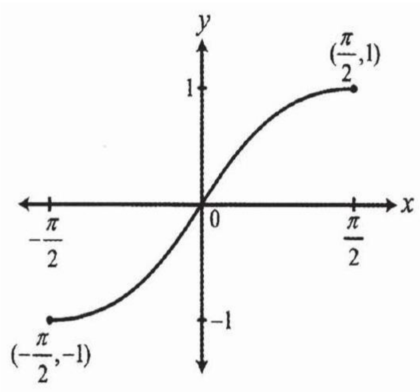


Figure 7.11

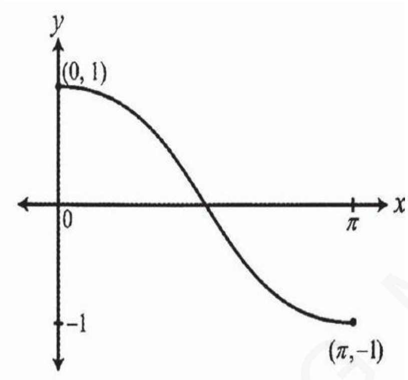


Figure 7.12

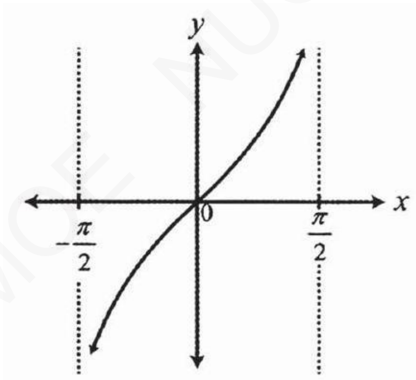


Figure 7.13

In these restricted domains, we can define the inverse functions $y = \sin^{-1}x$, $y = \cos^{-1}x$, and $y = \tan^{-1}x$.

$y = \sin^{-1} x,\ -1 \le x \le 1$ $y = \cos^{-1} x,\ -1 \le x \le 1$ $y = \tan^{-1} x,\ x \in \mathbb{R}$

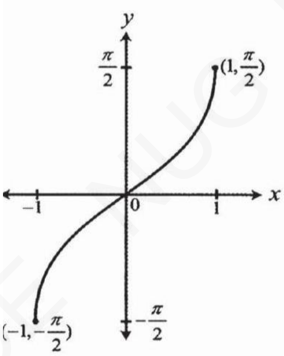


Figure 7.14

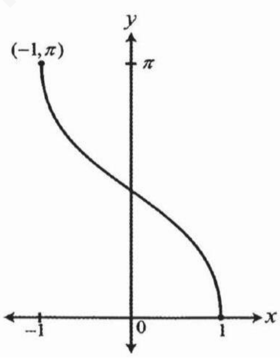


Figure 7.15

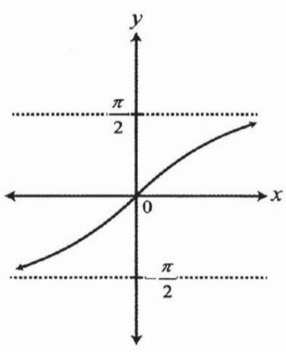


Figure 7.16

$y = \sin^{-1}x$	if $\sin y = x$	for	$-1 \le x \le 1$,	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1}x$	if $\cos y = x$	for	$-1 \le x \le 1$,	$0 \le y \le \pi$
$y = \tan^{-1}x$	if $\tan y = x$	for	$x \in \mathbb{R}$,	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

$y = \cot^{-1}x,\ x \in \mathbb{R}$

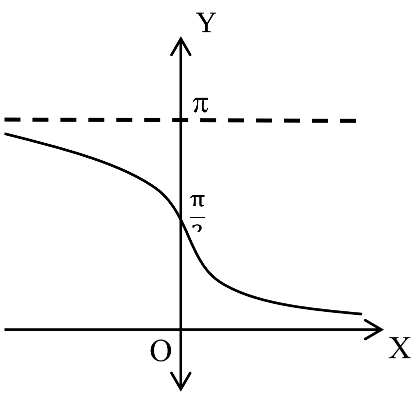


Figure 7.17

$y = \sec^{-1}x,\ |x| \geq 1$

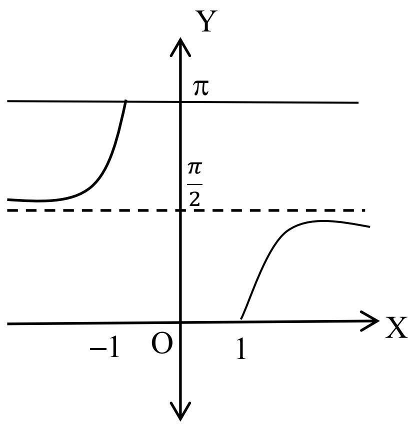


Figure 7.18

$y = \csc^{-1}x,\ |x| \geq 1$

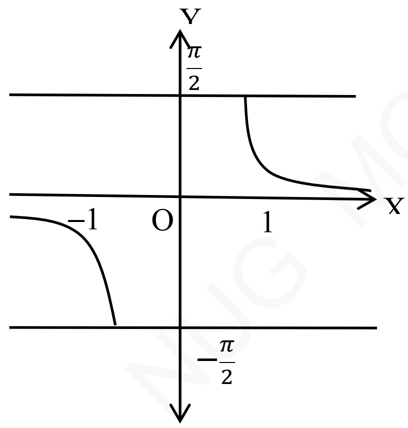


Figure 7.19

We can define the other trigonometric functions in the same manner.

$y = \cot^{-1}x$ if $\cot y = x$ for $x \in \mathbb{R}$, $0 < y < \pi$.

$y = \sec^{-1}x$ if $\sec y = x$ for $|x| \geq 1$, $0 \leq y \leq \pi$, $y \neq \frac{\pi}{2}$.

$y = \csc^{-1}x$ if $\csc y = x$ for $|x| \geq 1$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$.

Example 8

Evaluate (a) $\sin^{-1}\frac{1}{2}$ (b) $\cos^{-1}(-\frac{1}{2})$ (c) $\tan^{-1}(-\sqrt{3})$ (d) $\sin^{-1}(\sin\frac{5\pi}{6})$.

Solution

(a) Let $y = \sin^{-1}\frac{1}{2}$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\sin y = \frac{1}{2}$,

$y = \frac{\pi}{6}$

$\therefore\ \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

(b) Let $y = \cos^{-1}(-\frac{1}{2})$, $0 \leq y \leq \pi$

$\cos y = -\frac{1}{2}$,

$y = \frac{2\pi}{3}$

$\therefore\ \cos^{-1}\frac{1}{2} = \frac{2\pi}{3}$.

(c) $\tan^{-1}(-\sqrt{3})$

Let $y = \tan^{-1}(-\sqrt{3})$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\tan y = -\sqrt{3}$

$y = -\frac{\pi}{3}$

$\therefore \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$.

(d) $\sin^{-1}(\sin\frac{5\pi}{6})$

Let $y = \sin^{-1}(\sin\frac{5\pi}{6})$

$= \sin^{-1}(\frac{1}{2})$, $\quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$= \frac{\pi}{6}$.

From $y = \sin^{-1}x$ if $\sin y = x$, we have $\sin(\sin^{-1}x) = x$ for domain $-1 \leq x \leq 1$ of $y = \sin^{-1}x$, but in example 8(d), $\sin^{-1}(\sin\frac{5\pi}{6}) \neq \frac{5\pi}{6}$, one can see that $\sin^{-1}(\sin x) = x$ is not true in general.

Since $\sin(-\sin^{-1}x) = -\sin(\sin^{-1}x) = -x$, we have

$$\sin^{-1}(-x) = -\sin^{-1}x$$

So, $y = \sin^{-1}x$ is an odd function.

From $y = \cos^{-1}x$, if $\cos y = x$, we have $\cos(\cos^{-1}x) = -x$ for the domain $-1 \leq x \leq 1$ of $y = \cos^{-1}x$. Since $\cos(\pi - \cos^{-1}x) = -\cos(\cos^{-1}x) = -x$,

$$\pi - \cos^{-1}x = \cos^{-1}(-x) \quad \text{(or)} \quad \cos^{-1}x + \cos^{-1}(-x) = \pi$$

Exercise 7.4

1. Evaluate (a) csc $(\sin^{-1}0.3)$ (b) $\cot(\tan^{-1}5)$ (c) $\sec(\cos^{-1}(-0.75))$.

2. Evaluate (a) $\sin^{-1}\frac{\sqrt{3}}{2}$ (b) $\cos^{-1}\frac{\sqrt{2}}{2}$ (c) $\tan^{-1}\frac{\sqrt{3}}{2}$.

3. Evaluate (a) $\sin(\cos^{-1}\frac{\sqrt{3}}{2})$ (b) $\tan(\sin^{-1}\frac{\sqrt{3}}{2})$ (c) csc $(\tan^{-1}(-\frac{\sqrt{3}}{2}))$.

4. Evaluate (a) $\sin^{-1}(\sin\frac{2\pi}{3})$ (b) $\cos^{-1}(\cos(-\frac{1}{2}))$ (c) $\tan^{-1}(\tan(-\sqrt{3}))$.

5. Prove that $\sin^{-1}x + \cos^{-1} x = \frac{\pi}{2}$.

7.5 Differentiation of Trigonometric Functions

Before we study the differentiation of trigonometric functions, we first evaluate an important limit,

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Proof:

Consider the unit circle O,

OA = OB = 1, $\angle$ AOB = x radians.

Obviously, BD < arc AB < AC

BD = OB $\cdot$ sin x = 1$\cdot$ sin x = sin x

AC = OA tan x = 1 tan x = tan x

Arc AB = OB $\cdot$ x = x

$$\sin x < x < \tan x$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$1 > \frac{\sin x}{x} > \cos x.$$

When $x \to 0$, $\cos x \to 1$.

Since $\lim_{x\to 0} 1 = 1$, $\lim_{x\to 0} \cos x = 1$, $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

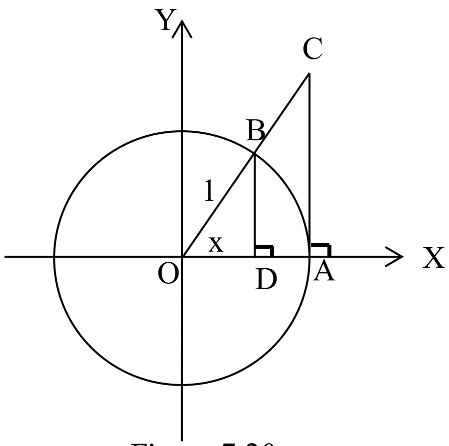


Figure 7.20

Derivative of sin x

Let y = sin x

$$y + \delta y = \sin(x + \delta x) \quad \frac{2\cos\left(x+\frac{\delta x}{2}\right)\cdot \sin\frac{\delta x}{2}}{\delta x}$$

$$\therefore \quad \delta y = \sin(x+\delta x) - \sin x$$

$$= 2\cos\left(x+\frac{\delta x}{2}\right)\cdot \sin\frac{\delta x}{2} \qquad \left(\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}\right)$$

$$\frac{dy}{dx} = \lim_{x\to 0}\frac{\delta y}{\delta x}$$

$$= \lim_{x\to 0}\frac{2\cos\left(x+\frac{\delta x}{2}\right)\cdot \sin\frac{\delta x}{2}}{\delta x}$$

$$= \lim_{x\to 0}\left[\cos\left(x+\frac{\delta x}{2}\right)\cdot\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right]$$

$$= \cos x \cdot 1 \qquad \left(\text{when } \delta x \to 0, \frac{\delta x}{2}\to 0\right)$$

$$\frac{d}{dx}\sin x = \cos x.$$

Derivative of cos x

Since $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sin\left(\frac{\pi}{2}-x\right)$$

$$= \cos\left(\frac{\pi}{2}-x\right)(-1)$$

$$= -\sin x.$$

Derivative of tan x

Since $\tan x = \frac{\sin x}{\cos x}$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$

$$= \frac{\cos x\frac{d}{dx}\sin x - \sin x\frac{d}{dx}\cos x}{\cos^2 x} \qquad \text{(quotient formula)}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} \qquad (\because \cos^2 x + \sin^2 x = 1)$$

$$= \sec^2 x \qquad \left(\because \sec x = \frac{1}{\cos x}\right).$$

Formulae for derivatives of trigonometric functions

1	$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\sin u = \cos u\cdot\frac{du}{dx}$
2	$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cos u = -\sin u\cdot\frac{du}{dx}$
3	$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\tan u = \sec^2 u\cdot\frac{du}{dx}$
4	$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$	$\frac{d}{dx}\cot u = -\operatorname{cosec}^2 u\cdot\frac{du}{dx}$
5	$\frac{d}{dx}\sec x = \sec x\cdot\tan x$	$\frac{d}{dx}\sec u = \sec u.\tan u\cdot\frac{du}{dx}$
6	$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x\cdot\cot x$	$\frac{d}{dx}\operatorname{cosec} u = -\operatorname{cosec} u\cdot\cot u\,\frac{du}{dx}$

Table 7.5

Example 9

Differentiate the following with respect to x.

(a) $\sin 5x$ (b) $\cos(7x^2-2)$ (c) $\tan(6x+7)$

(d) $5\sec(3x+1)$ (e) $\frac{\cot(1-2x)}{3}$ (f) $-2\operatorname{cosec} 3x$

Solution

(a) $\frac{d}{dx}\sin 5x = \cos 5x\cdot\frac{d}{dx}(5x) = \cos 5x(5) = 5\cos 5x$.

(b) $\frac{d}{dx}(\cos(7x^2-2)) = -\sin(7x^2-2)\cdot\frac{d}{dx}(7x^2-2) = -\sin(7x^2-2)\cdot(14x) = -14x\sin(7x^2-2)$.

(c) $\frac{d}{dx}\tan(6x+7) = \sec^2(6x+7)\cdot\frac{d}{dx}(6x+7) = \sec^2(6x+7)\ (6) = 6\sec^2(6x+7)$.

(d) $\frac{d}{dx}5\sec(3x+1) = 5\sec(3x+1)\cdot\tan(3x+1)\cdot\frac{d}{dx}(3x+1)$

$= 5\sec(3x+1)\cdot\tan(3x+1)(3)$

$= 15\sec(3x+1)\cdot\tan(3x+1)$.

(e) $\frac{d}{dx}\frac{\cot(1-2x)}{3} = \frac{1}{3}(-\operatorname{cosec}^2(1-2x))\frac{d}{dx}(1-2x)$

$= \frac{1}{3}(-\operatorname{cosec}^2(1-2x))(-2)$

$= \frac{2}{3}\operatorname{cosec}^2(1-2x)$.

(f) $\frac{d}{dx}(-2 \text{ cosec } 3x) = -2(-\text{cosec } 3x)(\cot 3x)\frac{d}{dx}(3x)$

$= -2(-\text{cosec } 3x)(\cot 3x)(3)$

$= 6(\text{cosec } 3x)(\cot 3x).$

Example 10

Find $\frac{dy}{dx}$.

(i) $y = -\sin^2 x$ (ii) $y = \cos\sqrt{x}$ (iii) $y = \tan^2(x^2)$

(iv) $y = \sin 2x - x\cos x$ (v) $y = \sin x \cdot \cos^2 x$ (vi) $y = \frac{x}{\tan x}$

(vii) $y = \sqrt{x + \sin x}$.

Solution

(i) $y = -\sin^2 x$

$\frac{dy}{dx} = -2(\sin x)(\cos x).$

(ii) $y = \cos\sqrt{x}$

$\frac{dy}{dx} = -\sin\sqrt{x}\left(\frac{1}{2} \cdot x^{\frac{-1}{2}}\right)$

$= \frac{-1}{2\sqrt{x}}\sin\sqrt{x}$.

(iii) $y = \tan^2(x^2)$

$\frac{dy}{dx} = 2\tan(x^2)\sec^2(x^2)(2x)$

$= 4x\tan(x^2)\sec^2(x^2).$

(iv) $y = \sin 2x - x\cos x$

$\frac{dy}{dx} = \cos 2x(2) - [x(-\sin x) + \cos x]$

$= 2\cos 2x + x\sin x - x\cos x.$

(v) $y = \sin x \cdot \cos^2 x$

$\frac{dy}{dx} = \sin x[2\cos x(-\sin x)] + \cos^2 x(\cos x)$

$= -2\sin^2 x(\cos x) + \cos^3 x.$

(vi) $y = \frac{x}{\tan x}$

$$\frac{dy}{dx} = \frac{\tan x(1) - x\,(\sec^2 x)}{(\tan x)^2}$$

$$= \frac{\tan x - x\,(\sec^2 x)}{\tan^2 x}.$$

(vii) $y = \sqrt{x + \sin x}$

$$\frac{dy}{dx} = \frac{1}{2}\,(x + \sin x)^{\frac{-1}{2}}(\,1 + \cos x)$$

$$= \frac{1 + \cos x}{2\sqrt{x + \sin x}}.$$

Example 11

Given that $x + \sin y = \cos(xy)$, find $\frac{dy}{dx}$.

Solution

$$x + \sin y = \cos(xy)$$

Differentiate both sides with respect to x,

$$1 + \cos y\,(\frac{dy}{dx}) = -\sin(xy)(\,x\frac{dy}{dx} + y)$$

$$1 + \cos y\,(\frac{dy}{dx}) = -\,x\sin(xy)\frac{dy}{dx} - y\sin(xy)$$

$$(\cos y + x\sin(xy))(\frac{dy}{dx}) = -\,y\sin(xy) - 1$$

$$\frac{dy}{dx} = -\frac{y\sin(xy) + 1}{\cos y + x\sin(xy)}.$$

Example 12

Given that $y = x\sin x$, find $\frac{d^2y}{dx^2}$.

Solution

$$y = x\sin x$$

$$\frac{dy}{dx} = x\cos x + \sin x$$

$$\frac{d^2y}{dx^2} = [x(-\sin x) + \cos x] + \cos x$$

$$= 2\cos x - x\sin x.$$

Exercise 7.5

1. Differentiate the following functions with respect x.

(i) $\sin(2x+3)$ (ii) $\cos\frac{3}{x}$ (iii) $x^3\cos 2x$ (iv) $\cos 7x+\sin 3x$

(v) $\sin x \cdot \cos 2x$ (vi) $\cos^2(5x)$ (vii) $\tan^3\sqrt{x}$ (viii) $\sin(\cos x)$

(ix) $\frac{\sin x}{1+\tan x}$ (x) $\sqrt{\sin x+\cos x}$

2. Find $\frac{dy}{dx}$.

(i) $y = \sin(1-x^2)$ (ii) $y = 2\pi x + 2\cos\pi x$ (iii) $y = \sin^2 x\cos 3x$

(iv) $y = x^2\sin(\frac{1}{x})$ (v) $3x^2 + 2\sin y = y^2$ (vi) $\sin x \cdot \cos y = 2y$

3. Given that $y = \cos^2 x$, prove that $\frac{d^2y}{dx^2} + 4y = 2$.

4. Given that $y = \frac{1}{3}\cos^3 x - \cos x$, prove that $\frac{dy}{dx} = \sin^3 x$.

Given that $y = \frac{1}{3}\cos^3 x - \cos x$, prove that $\frac{dy}{dx} = \sin^3 x$.

Chapter 8

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In this chapter, we will study logarithmic functions, exponential functions and the characteristics of their graphs. This chapter also shows how to differentiate logarithmic and exponential functions.

8.1 Logarithmic Functions

A logarithm base b of a positive number x satisfies the following definition.

For $x > 0, b > 0, b \neq 1$,

$$y = \log_b(x) \text{ is equivalent to } b^y = x.$$

The **domain** of $y = \log_b(x)$ is $(0, \infty) = \{x/x > 0\}$

The **range** of $y = \log_b(x)$ is $(-\infty, \infty) = \{y/-\infty < y < \infty\} = \mathbb{R}$

The Graphs of $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ are as follows.

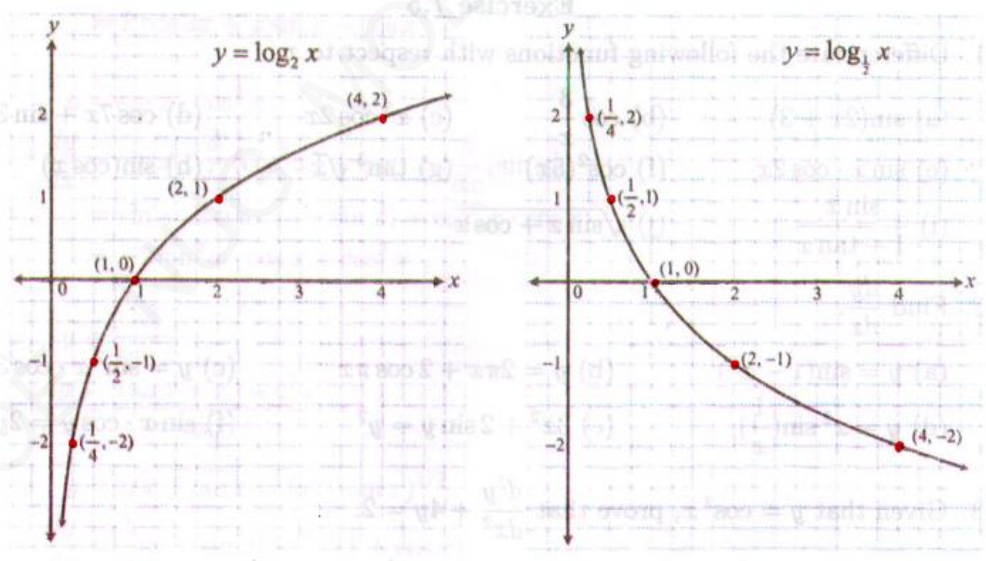


Figure 8.1

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$ and the vertical asymptote is $x = 0$.

Characteristics of the Graph of the Logarithmic Function $\mathbf{f(x) = \log_b(x)}$

For any real number x and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- x-intercept: $(1, 0)$ and key point $(b, 1)$
- y-intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$

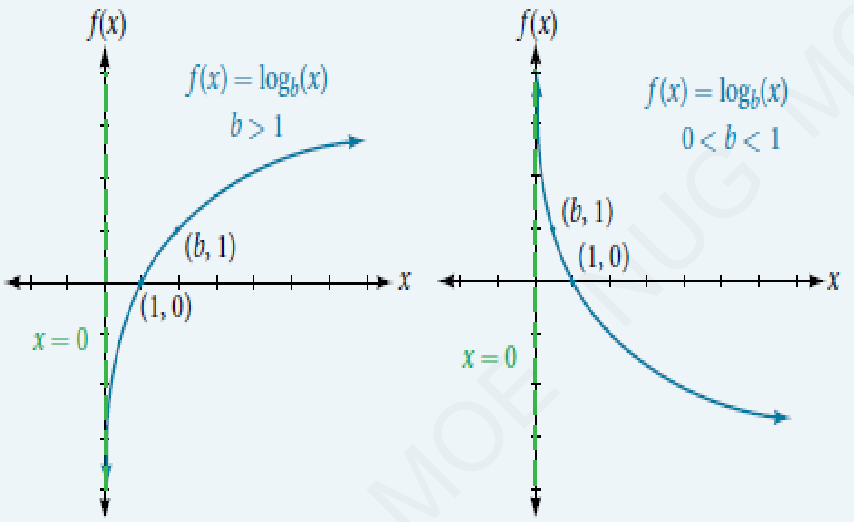
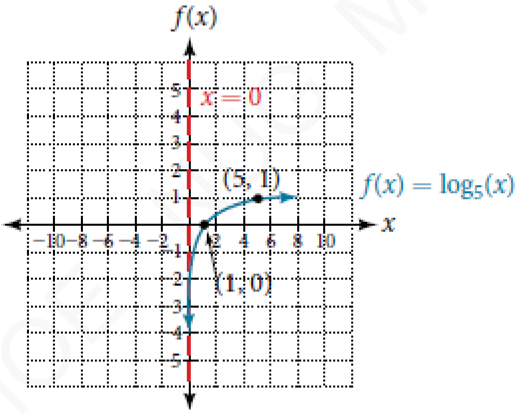


Figure 8.2

For Example, graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.
Since $b = 5 > 1$, the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0$, and the right tail will increase slowly without bound.

The x-intercept is $(1, 0)$. The key point $(5, 1)$ is on the graph.



The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Graphing Transformations of Logarithmic Functions

We can shift, stretch, compress, and reflect the parent function $f(x) = \log_b(x)$ without loss of shape.

Graphing a Horizontal Shift of $y = \log_b(x)$

When a constant c is added to the input of the parent function $f(x) = \log_b(x)$, the result is a horizontal shift c units in the opposite direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure 8.3.

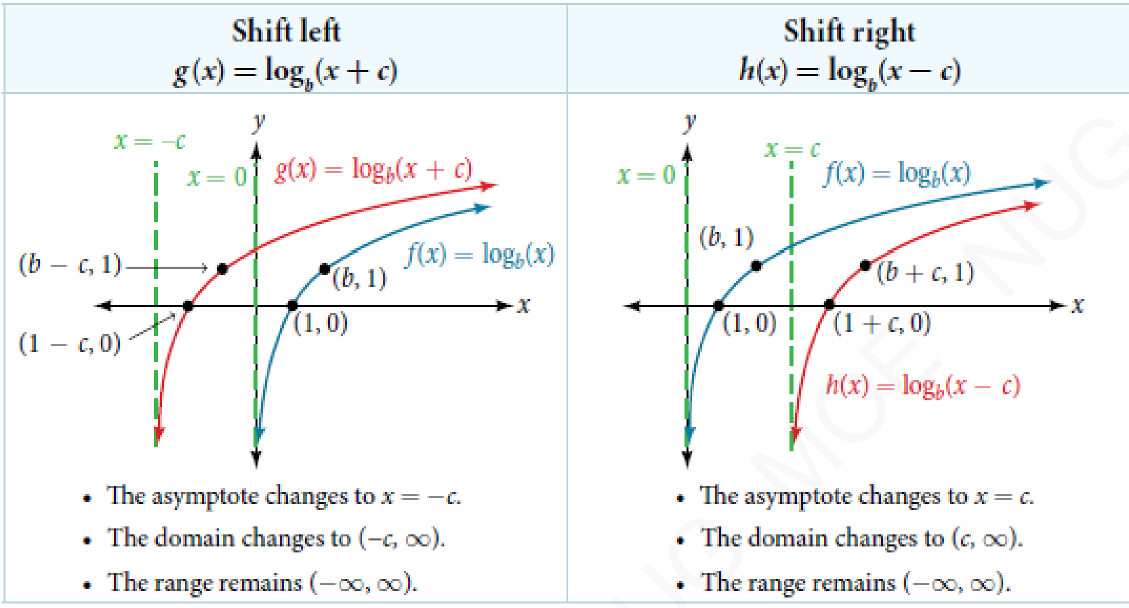


Figure 8.3

Graphing a Vertical Shift of $y = \log_b(x)$

When a constant d is added to the parent function $f(x) = \log_b(x)$, the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$. See Figure 8.4.

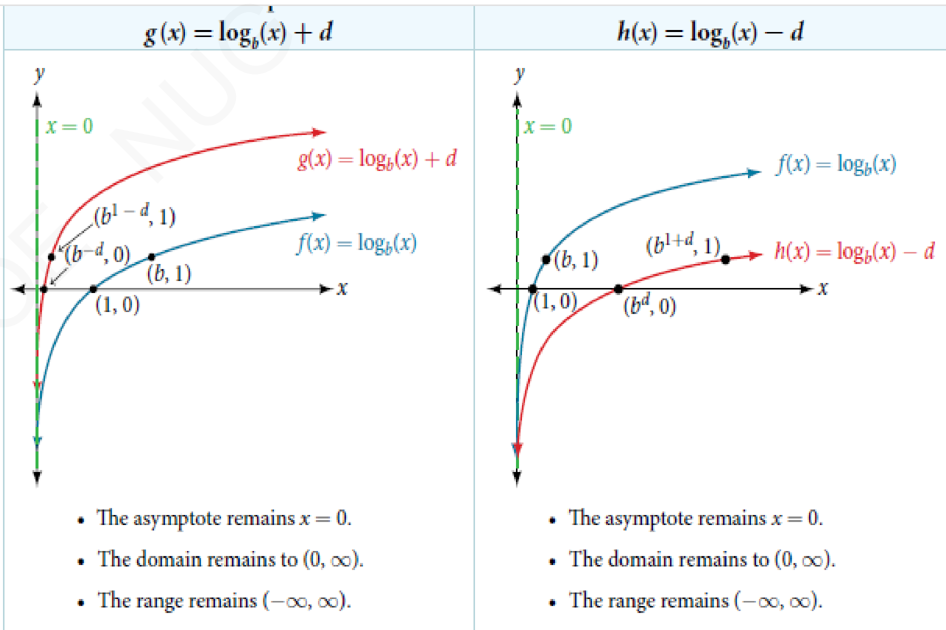


Figure 8.4

Graphing Stretches and Compressions of $y = \log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by a constant $a > 0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a > 1$ and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the vertical stretch, $g(x) = a\log_b(x)$ and the vertical compression, $h(x) = \frac{1}{a}\log_b(x)$. See Figure 8.5.

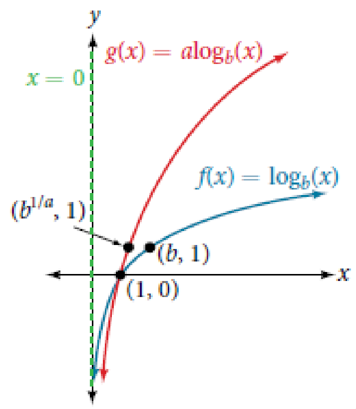
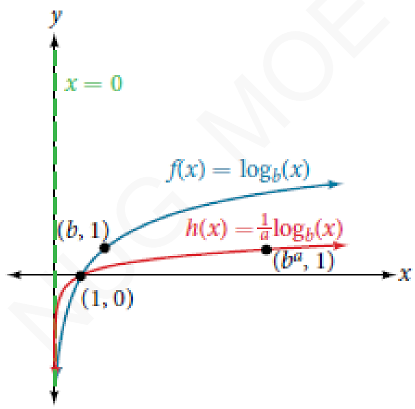
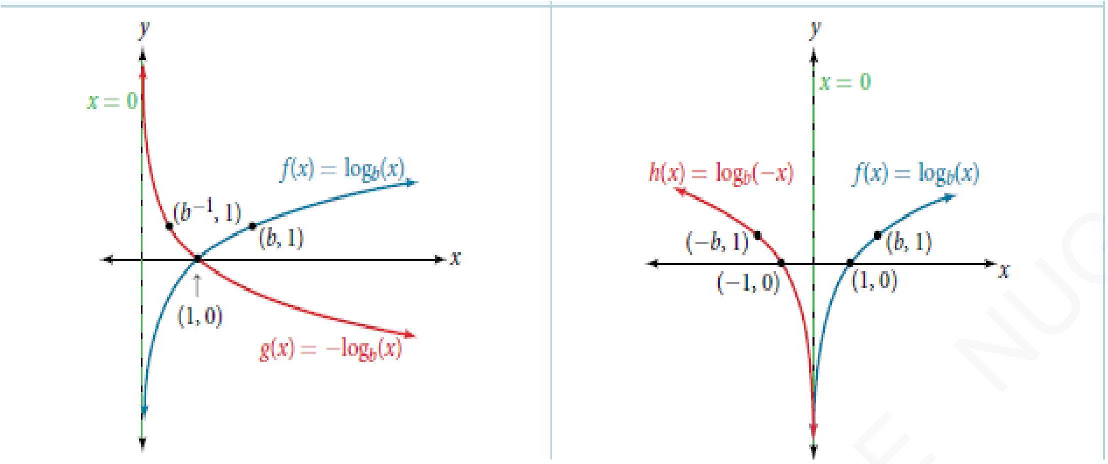
Vertical Stretch $g(x) = a\log_b(x),\ a > 1$	Vertical Compression $h(x) = \frac{1}{a}\log_b(x),\ a > 1$
	
• The asymptote remains $x = 0$. • The x-intercept remains $(1, 0)$. • The domain remains $(0, \infty)$. • The range remains $(-\infty, \infty)$.	• The asymptote remains $x = 0$. • The x-intercept remains $(1, 0)$. • The domain remains $(0, \infty)$. • The range remains $(-\infty, \infty)$.

Figure 8.5

Graphing Reflections of $y = \log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by -1, the result is a reflection about the x-axis. When the input is multiplied by -1, the result is a reflection about the y-axis. To visualize reflections, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the x-axis, $g(x) = -\log_b(x)$ and the reflection about the y-axis, $h(x) = \log_b(-x)$. See Figure 8.6.

Reflection about the x-axis $g(x) = -\log_b(x), b > 1$	Reflection about the y-axis $h(x) = \log_b(-x), b > 1$



- Reflected function is decreasing as x moves from 0 to ∞.
- The asymptote remains $x = 0$.
- The domain remains $(0, \infty)$
- The range remains $(-\infty, \infty)$.
- The x-intercept remains $(1, 0)$.
- The key point changes to $(b^{-1}, 1)$.

- Reflected function is decreasing as x moves from $-\infty$ to 0.
- The asymptote remains $x = 0$.
- The domain changes to $(-\infty, 0)$.
- The range remains $(-\infty, \infty)$.
- The x-intercept changes to $(-1, 0)$.
- The key point changes to $(-b, 1)$.

Figure 8.6

Summarizing Transformations of the Logarithmic Function

Now that we have worked with each type of transformation for the logarithmic function, we can summarize each in the following table to arrive at the general equation for transformation of the logarithmic function.

Transformations of the Parent Function $y = \log_b(x)$	
Transformations	**Form**
Shift • Horizontally c units to the left • Vertically d units up	$y = \log_b(x + c) + d$
Stretch and Compress • Stretch if $\lvert a \rvert > 1$ • Compression if $\lvert a \rvert < 1$	$y = a\log_b(x)$
Reflection about the x-axis	$y = -\log_b(x) = \log_{\frac{1}{b}}(x)$
Reflection about the y-axis	$y = \log_b(-x)$
General equation for all transformations	$y = a\log_b(x + c) + d$

Example 1

From the graph of $y = \log_2 x$, draw step-by-step transformation graphs to get the graph of (a) $y = \log_2(x-1) + 2$ (b) $y = \log_{\frac{1}{2}}(x+2) - 1$.

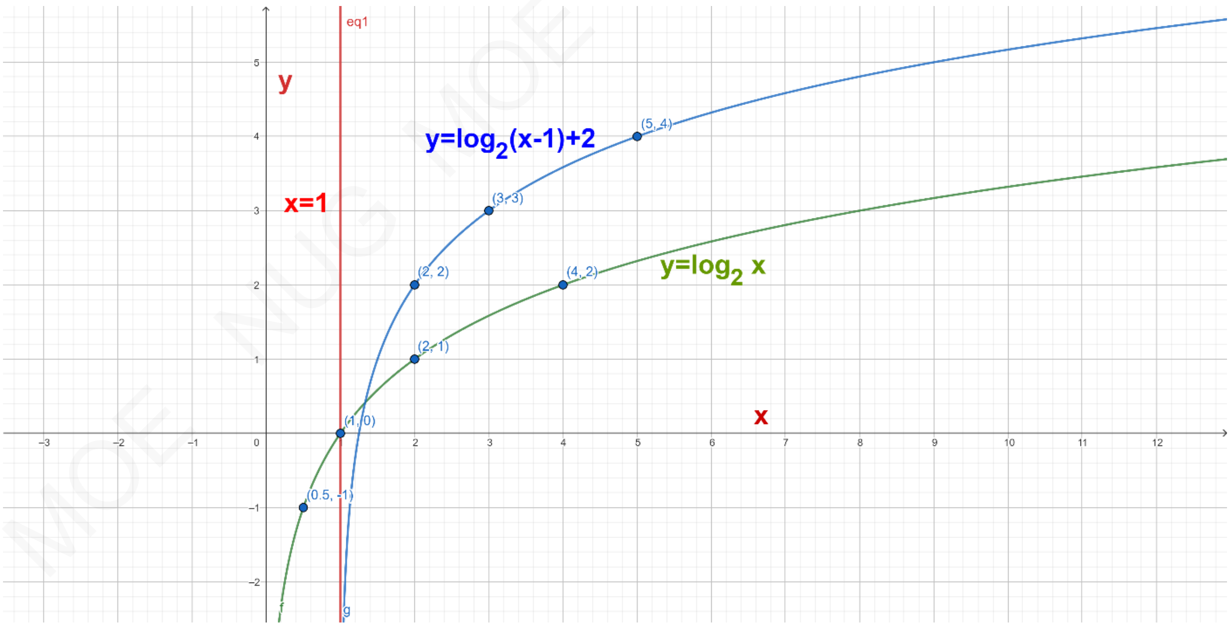
Solution

(a) $y = \log_2 x \xrightarrow[\text{Vertically 2 units up}]{\text{Horizontally 1unit to the right}} y = \log_2(x-1) + 2$.

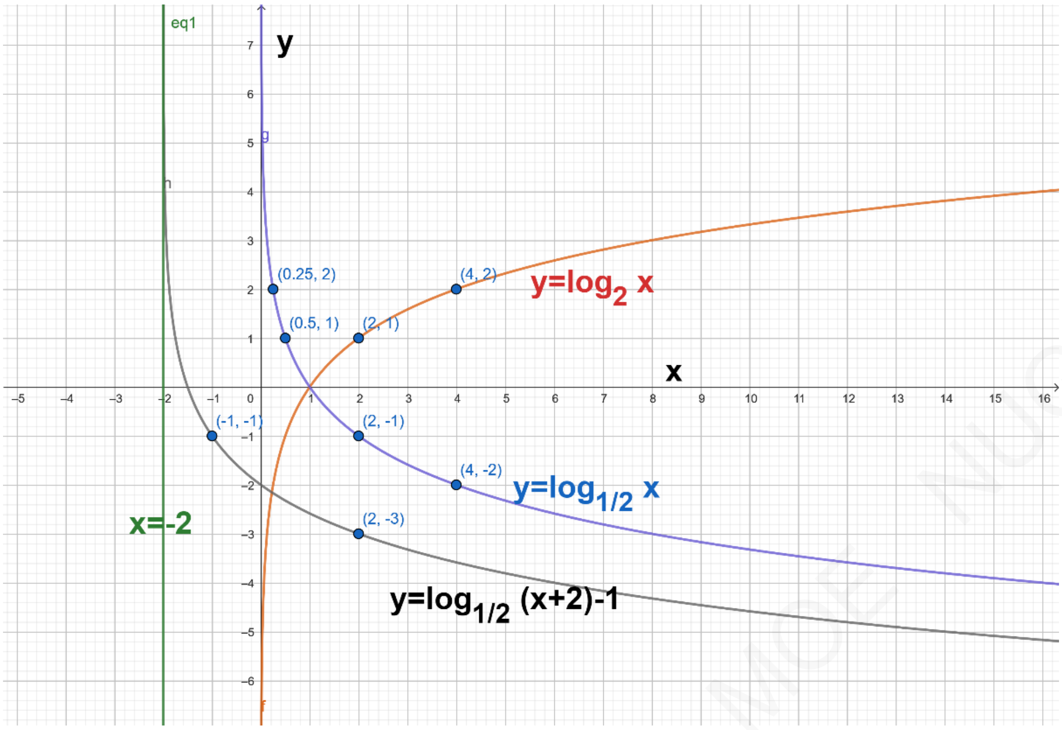
(b) $\log_{\frac{1}{2}}(x) = -\log_2(x)$

$$y = \log_2 x \xrightarrow[\text{the x- axis}]{\text{Reflection about}} y = \log_{\frac{1}{2}} x \xrightarrow[\text{Vertically 1 unit down}]{\text{Horizontally 2 units to the left}} y = \log_2(x+2) - 1.$$

x	1/4	1/2	1	2	3	4	…
$\mathbf{y = \log_2 x}$	-2	-1	0	1	1.58	2	…



Note: Asymptote x = 1



Note: Asymptote $x = -2$

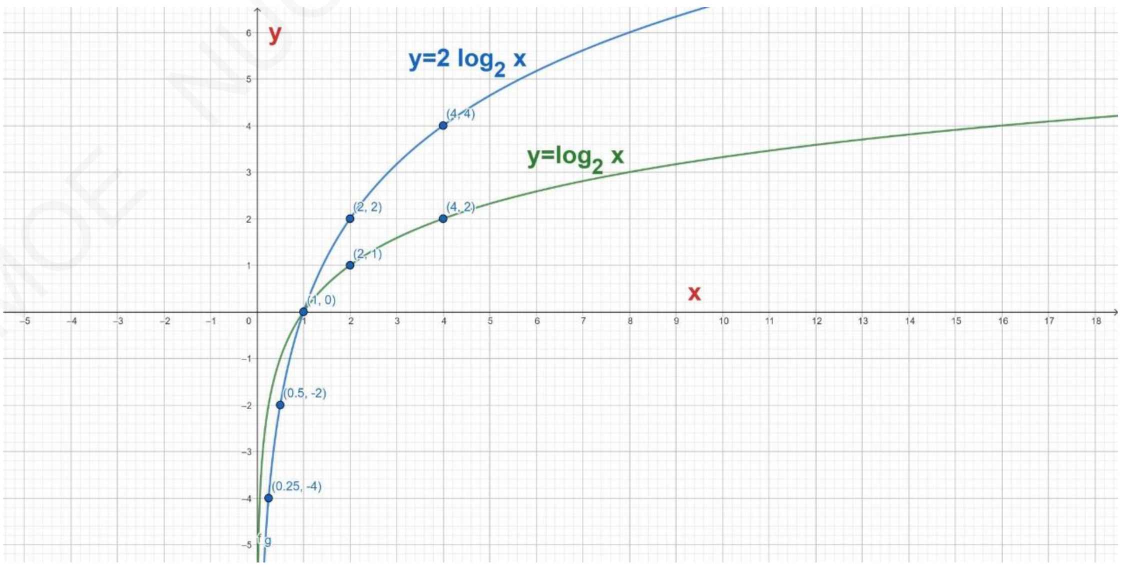
Example 2

Draw the graphs $y = 2\log_2 x$ and $y = \log_2 x^2$.

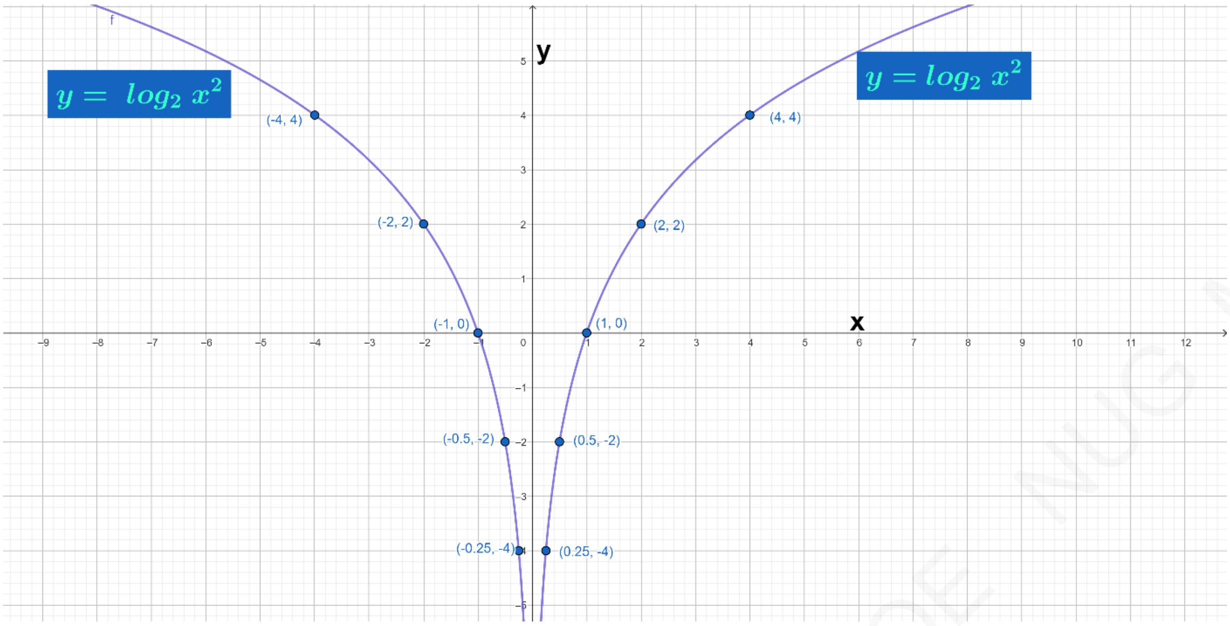
Solution

$y = \log_2 x$ $\xrightarrow{\text{Vertical Stretch 2 units}}$ $y = 2\log_2 x$

Domain of $y = \log_2 x^2$ is $\mathbf{R} \setminus \{0\}$ and $y = \log_2 x^2$ is an even function.



Note: Asymptote y-axis $(x = 0)$



Note: Asymptote y-axis (x = 0)

Exercise 8.1

1. Draw the graphs of

   (a) $y = \log_2(x-2)+1$ (b) $y = \log_2(x+1)-2$

   (c) $y = \log_{\frac{1}{2}}(x+1)+2$ (d) $y = \log_{\frac{1}{2}}(x+1)+2.$

2. Draw the graphs of (a) $y = \log_{\frac{1}{2}} x^2$ (b) $y = 2\log_{\frac{1}{2}} x.$

3. Draw the graphs of (a) $y = \log_2(-x)$ (b) $y = \log_{\frac{1}{2}}(-x).$

4. Draw the graphs of (a) $y = \log_2|x|$ (b) $y = \log_{\frac{1}{2}}|x|.$

8.2 Differentiation of Logarithmic Functions

Derivative of $y = \log_b x$

Let $y = \log_b x$.

Let δx be the small increment in x and δy be the corresponding small increment in y.

Then $y + \delta y = \log_b(x + \delta x)$

$$\delta y = \log_b(x + \delta x) - \log_b x.$$

$$\delta y = \log_b\left(\frac{x + \delta x}{x}\right), \qquad x > 0$$

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\log_b\left(\frac{x + \delta x}{x}\right) = \log_b\left(\frac{x + \delta x}{x}\right)^{1/\delta x}$$

$$\frac{\delta y}{\delta x} = \log_b\left(1 + \frac{\delta x}{x}\right)^{1/\delta x}.$$

Let $\frac{\delta x}{x} = t$. Then $\frac{1}{\delta x} = \frac{1}{xt}$.

$$\therefore \frac{\delta y}{\delta x} = \log_b(1 + t)^{1/xt} = \frac{1}{x}\log_b(1 + t)^{1/t}$$

When $\delta x \to 0, t \to 0$. Then

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \frac{1}{x}\lim_{t \to 0}\log_b(1 + t)^{1/t}$$

$$\frac{dy}{dx} = \frac{1}{x}\log_b\left(\lim_{t \to 0}(1 + t)^{1/t}\right).$$

The following table shows

$$\lim_{t \to 0}(1 + t)^{1/t} = ?$$

t	$1/t$	$(1+t)^{1/t}$
1	1	2
0.5	2	2.25
0.25	4	2.4414
0.10	10	2.5837
0.01	100	2.7048
0.001	1000	2.7169
0.0001	10000	2.7181
0.00001	100000	2.7183
0.000001	1000000	2.7183

Table 8.1

The above table shows that $(1+t)^{1/t} \to 2.7183$ as , $t \to 0$.

This limit value is denoted by e which is called the **exponential number**. In fact, e is an irrational number.

Therefore $\frac{dy}{dx} = \frac{1}{x}\log_b e$.

$$\therefore \frac{d}{dx}(\log_b x) = \frac{1}{x}\log_b e.$$

If $u(x) > 0$ is a function of x,

$$\frac{d}{dx}\log_b u(x) = \frac{1}{u(x)}\log_b e \,.\, \frac{d}{dx}u(x).$$

Logarithm of base e is called **natural or Napierian Logarithm** and denote $\log_e x = \ln x$.

Since $\frac{d}{dx}(\log_b x) = \frac{1}{x}\log_b e$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}\log_e e$$

$$\therefore \frac{d}{dx}\ln x = \frac{1}{x},\ (x > 0).$$

In general, $\frac{d}{dx}\ln u(x) = \frac{1}{u(x)} \cdot \frac{d}{dx}u(x)$.

Example 3

Differentiate the following functions with respect to x.

(a) $\log_{10} x^3$ (b) $\log_2 x^3$ (c) $\ln x^3$ (d) $\ln\sqrt{x^2+5}$

(e) $\ln \sin 2x$ (f) $\ln x \cdot \log_{10} x$ (g) $\ln \frac{x}{\sqrt{x^2+2}}$ (h) $\frac{x^2}{\log_{10} x}$

Solution

(a) $\frac{d}{dx}\log_{10} x^3 = \frac{1}{x^3} \cdot \log_{10} e \cdot \frac{d}{dx}x^3$

$= \frac{1}{x^3} \cdot \log_{10} e \cdot 3x^2$

$= \frac{3}{x} \cdot \log_{10} e.$

(b) $\frac{d}{dx}\log_2 x^3 = \frac{1}{x^3}\cdot\log_2 e\cdot\frac{d}{dx}x^3$

$= \frac{1}{x^3}\cdot\log_2 e\cdot 3x^2$

$= \frac{3}{x}\cdot\log_2 e.$

(c) $\frac{d}{dx}\ln x^3 = \frac{1}{x^3}\cdot\frac{d}{dx}x^3 = \frac{1}{x^3}\cdot 3x^2 = \frac{3}{x}.$

(d) $\frac{d}{dx}\ln\sqrt{x^2+5} = \frac{1}{\sqrt{x^2+5}}\cdot\frac{d}{dx}\sqrt{x^2+5} = \frac{1}{\sqrt{x^2+5}}\cdot\frac{1}{2\sqrt{x^2+5}}\cdot 2x = \frac{x}{x^2+5}.$

(e) $\frac{d}{dx}\ln\sin 2x = \frac{1}{\sin 2x}\cdot\frac{d}{dx}\sin 2x = \frac{1}{\sin 2x}\cdot 2\cos 2x = 2\cot 2x.$

(f) $\frac{d}{dx}\ln x\cdot\log_{10}x = \ln x\cdot\frac{d}{dx}\log_{10}x + \log_{10}x\cdot\frac{d}{dx}\ln x$

$= \ln x\cdot\frac{1}{x}\cdot\log_{10}e + \log_{10}x\cdot\frac{1}{x}.$

(g) $\frac{d}{dx}\ln\frac{x}{\sqrt{x^2+2}} = \frac{\sqrt{x^2+2}}{x}\cdot\frac{d}{dx}\frac{x}{\sqrt{x^2+2}}$

$= \frac{\sqrt{x^2+2}}{x}\cdot\frac{\sqrt{x^2+2}\cdot\frac{dx}{dx} - x\cdot\frac{d}{dx}\sqrt{x^2+2}}{x^2+2}$

$= \frac{\sqrt{x^2+2}}{x}\cdot\frac{\sqrt{x^2+2} - x\cdot\frac{1}{2}\frac{1}{\sqrt{x^2+2}}\cdot 2x}{x^2+2}$

$= \frac{\sqrt{x^2+2}}{x}\cdot\frac{\sqrt{x^2+2} - \frac{x^2}{\sqrt{x^2+2}}}{x^2+2}$

$= \frac{\sqrt{x^2+2}}{x}\cdot\frac{\frac{x^2+2-x^2}{\sqrt{x^2+2}}}{x^2+2} = \frac{2}{x(x^2+2)}.$

(h) $$\frac{d}{dx}\frac{x^2}{\log_{10}x} = \frac{\log_{10} x \cdot \frac{d}{dx}x^2 - x^2 \cdot \frac{d}{dx}\log_{10} x}{(\log_{10} x)^2}$$

$$= \frac{\log_{10} x \cdot 2x - x^2 \cdot \frac{1}{x}\log_{10} e}{(\log_{10} x)^2}$$

$$= \frac{2x\log_{10} x - x\log_{10} e}{(\log_{10} x)^2}.$$

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process is called **logarithmic differentiation**.

Example 4

Find $\frac{dy}{dx}$ if $y = \sqrt{(x^2 + 1)(x-1)^2}$.

Solution

$$y = \sqrt{(x^2 + 1)(x-1)^2}$$

$$\ln y = \ln\sqrt{(x^2 + 1)(x-1)^2}$$

$$= \frac{1}{2}\ln(x^2 + 1)(x-1)^2$$

$$= \frac{1}{2}(\ln(x^2 + 1) + \ln(x-1)^2)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\{\frac{1}{2}(\ln(x^2 + 1) + \ln(x-1)^2)\}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}(\frac{1}{x^2 + 1}\cdot 2x + \frac{1}{(x-1)^2}\cdot 2(x-1)\cdot 1)$$

$$= \frac{1}{2}(\frac{2x}{x^2+1} + \frac{2}{x-1}) = \frac{x}{x^2+1} + \frac{1}{x-1}$$

$$\frac{dy}{dx} = y\,(\frac{x}{x^2 + 1} + \frac{1}{x-1})$$

$$\frac{dy}{dx} = \sqrt{(x^2 + 1)(x-1)^2}\,(\frac{x}{x^2+1} + \frac{1}{x-1}).$$

Example 5 Find $\frac{dy}{dx}$ if $y = \frac{x \sin x}{\sqrt{\sec x}}, 0 < x < \frac{\pi}{2}$.

Solution

$$y = \frac{x \sin x}{\sqrt{\sec x}}, \qquad 0 < x < \frac{\pi}{2}$$

$$\ln y = \ln\left(\frac{x \sin x}{\sqrt{\sec x}}\right)$$

$$\ln y = \ln x + \ln \sin x - \frac{1}{2}\ln \sec x$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(\ln x + \ln \sin x - \frac{1}{2}\ln \sec x\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\sin x}\cdot \cos x - \frac{1}{2}\,\frac{1}{\sec x}\cdot \sec x \cdot \tan x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \cot x - \frac{1}{2}\tan x$$

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \cot x - \frac{1}{2}\tan x\right)$$

$$\frac{dy}{dx} = \frac{x \sin x}{\sqrt{\sec x}}\left(\frac{1}{x} + \cot x - \frac{1}{2}\tan x\right).$$

Exercise 8.2

1. Differentiate the following functions with respect to x.

(a) $\ln(2x^2 + 3)$ (b) $\ln|x|$ (c) $x^2 \log_2 x$

(d) $\sin 3x \cdot \log_{10}(x+1)$ (e) $\ln\sqrt{5x-4}$ (f) $\ln(\ln x)$

(Note: The absolute value function $|x| = \sqrt{x^2}$ is not differentiable at $x = 0$.)

2. Use logarithmic differentiation to find the derivative of y with respect to x.

(a) $y = \sqrt{x(x+1)}$ (b) $y = x(x+1)(x+2)$

(c) $y = \frac{x\sqrt{x^2+1}}{(x+1)^{\frac{2}{3}}}$

8.3 Exponential Functions

A function of the form $f(x) = b^x$, where base $b > 0, b \neq 1$ is called an **exponential function** of x. All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$.

So an **exponential function never assumes the value 0**.

For $b > 1$, $f(x) = b^x$ is an **exponential growth** function.

For $0 < b < 1$, $f(x) = b^x$ is an **exponential decay** function.

Graphing the exponential growth function $f(x) = 2^x$

Before we begin graphing, it is helpful to observe the behavior of exponential growth. Draw the table of values for a function $\mathbf{f(x) = 2^x}$. Observe how the output values in table change as the input increases by 1.

x	…	−3	−2	−1	0	1	2	3	…
$f(x) = 2^x$	…	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	…

Table 8.2

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

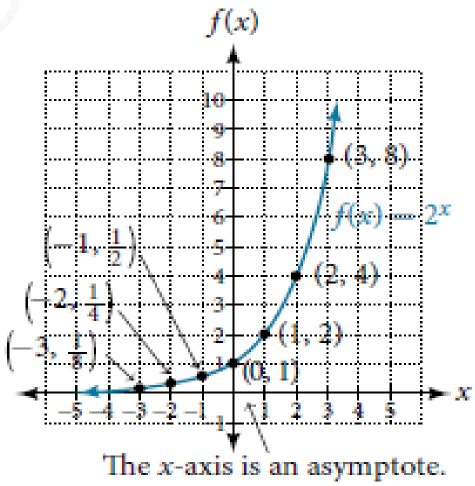


Figure 8.7

Notice that the graph gets close to the x-axis, but never touches it.

Graphing the exponential decay function $f(x) = (\frac{1}{4})^x = (0.25)^x$

x	⋯	−3	−2	−1	0	1	2	3	⋯
$f(x) = (\frac{1}{4})^x$	⋯	64	16	4	1	0.25	0.625	0.015625	⋯

Table 8.3

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values increase without bound.

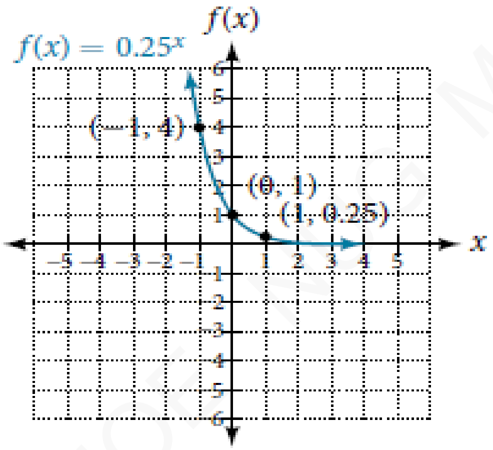
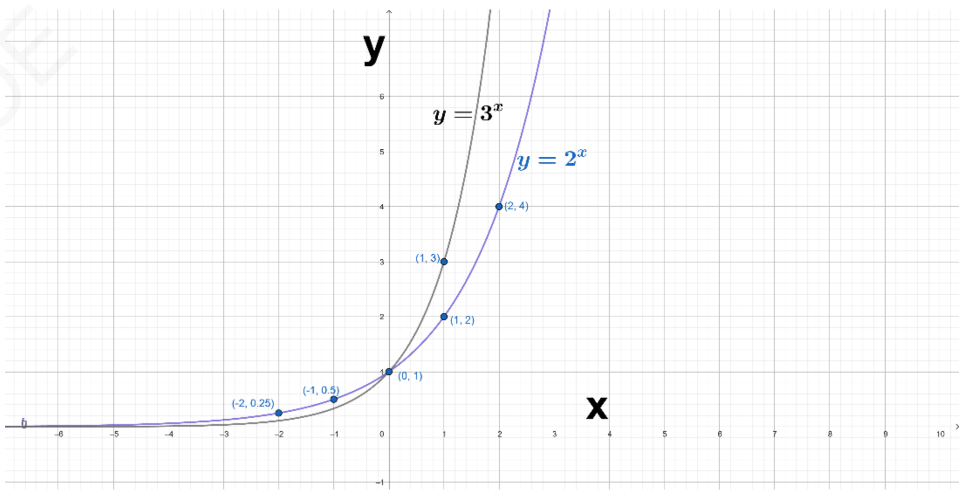


Figure 8.8

Plot the y-intercept, (0, 1), along with two other points. We can use (−1, 4) and (1, 0.25).

The Graphs of $y = 2^x$, $y = 3^x$, $y = 2^{-x}$ and $y = 3^{-x}$ are as follows:



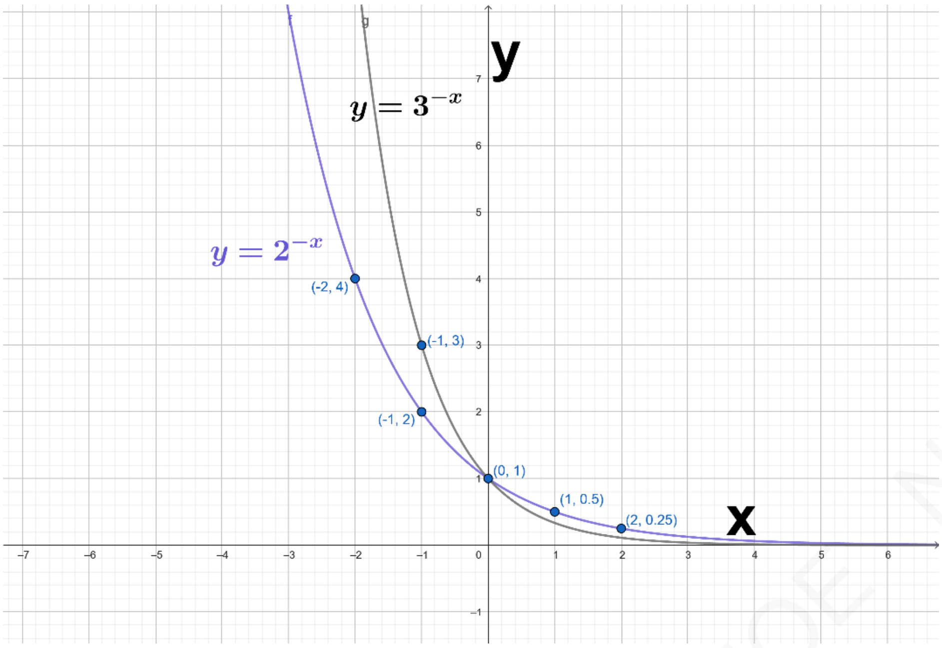


Figure 8.9

$y = 2^x$, $y = 3^x$ are exponential growth functions.

$y = 2^{-x} = (\frac{1}{2})^x$ and $y = 3^{-x} = (\frac{1}{3})^x$ are exponential decay functions.

Note: $y = \log_b(x)$ and $y = b^x$ are inverse of each other. Their graphs are reflections of each other across the line y = x.

x	-3	-2	-1	0	1	2	3
$y = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2 y = x$	-3	-2	-1	0	1	2	3
$\mathbf{f(x) = 2^x}$	$(-3, \frac{1}{8})$	$(-2, \frac{1}{4})$	$(-1, \frac{1}{2})$	(0, 1)	(1, 2)	(2, 4)	(3, 8)
$\mathbf{g(x) = \log_2 x}$	$(\frac{1}{8}, -3)$	$(\frac{1}{4}, -2)$	$(\frac{1}{2}, -1)$	(1, 0)	(2, 1)	(4, 2)	(8, 3)

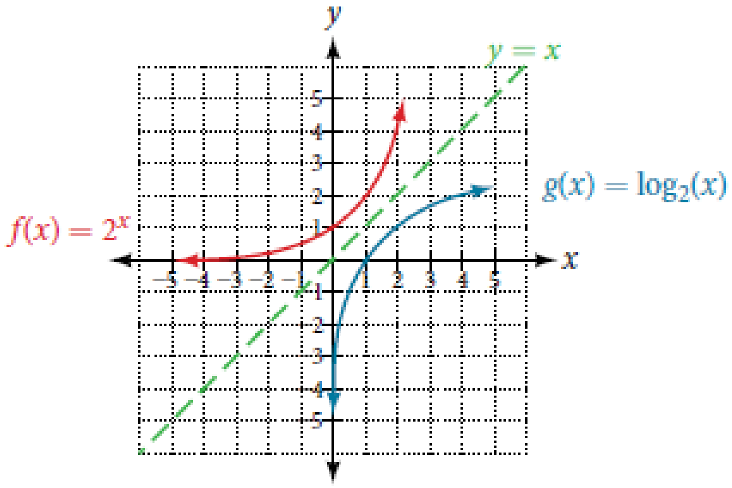


Figure 8.10

Notice that the graphs of $f(x) = 2^x$ and $g(x) = \log_2 x$ are reflections about the line $y = x$.

- $f(x) = 2^x$ has a y-intercept at (0, 1) and $g(x) = \log_2 x$ has an x-intercept at (1, 0).
- The domain of $f(x)$ = the range of $g(x) = (-\infty, \infty)$.
- The range of $f(x)$ = the domain of $g(x) = (0, \infty)$.

characteristics of the graph of the parent function $f(x) = b^x$

An exponential function with the form $f(x) = b^x$, $b > 0$, $b \neq 1$, has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x-intercept: none
- y-intercept: (0, 1)
- increasing if $b > 1$
- decreasing if $b < 1$

Figure compares the graphs of exponential growth and decay functions.

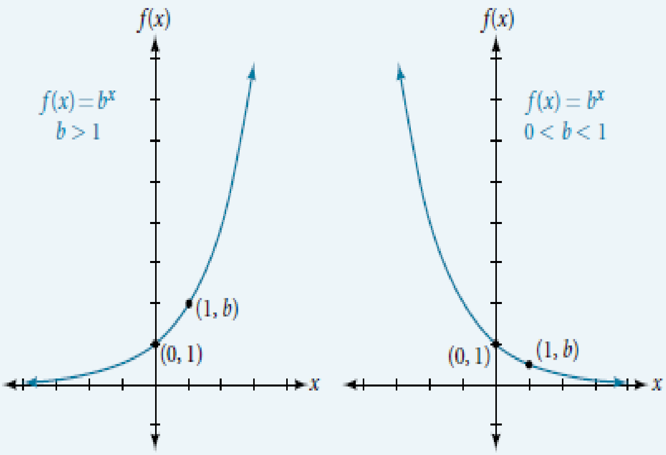


Figure 8.11

Graphing Transformations of Exponential Functions

We can apply the four types of transformations (shifts, reflections, stretches and compressions) to the parent function $f(x) = b^x$ without loss of shape.

Graphing a Horizontal Shift

Addition of a constant c to the input of the parent function $f(x) = b^x$ gives a horizontal shift c units in the opposite direction of the sign. For example, if $f(x) = 2^x$, we can graph two horizontal shifts alongside it, using c = 3: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. Both horizontal shifts are shown in Figure 8.12.

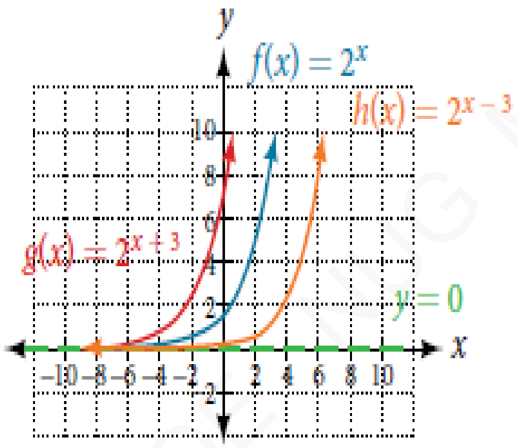


Figure 8.12

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$, remains unchanged.
- The asymptote, $y = 0$, remains unchanged.
- The y-intercept shifts such that:

When the function is shifted left 3 units to $g(x) = 2^{x+3}$, the y-intercept becomes (0, 8). This is because $2^{x+3} = (8)2^x$, so the initial value of the function is 8.

When the function is shifted right 3 units to $h(x) = 2^{x-3}$, the y-intercept becomes $(0, \frac{1}{8})$.

Again, see that $2^{x-3} = (\frac{1}{8})2^x$, so the initial value of the function is $\frac{1}{8}$.

Graphing a Vertical Shift

Addition of a constant d to the parent function $f(x) = b^x$ gives a vertical shift d units in the same direction as the sign. For example, if $f(x) = 2^x$, we can graph two vertical shifts alongside it, using d = 3: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in Figure 8.13.

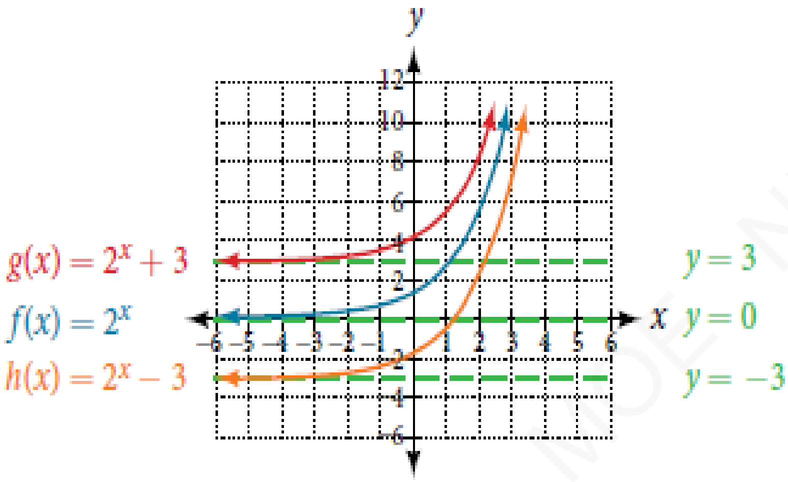


Figure 8.13

Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:

  The y-intercept shifts up 3 units to (0, 4).

  The asymptote shifts up 3 units to y = 3.

  The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$:

  The y-intercept shifts down 3 units to (0, −2).

  The asymptote also shifts down 3 units to y = −3.

  The range becomes $(-3, \infty)$.

Graphing a Stretch or Compression

A stretch or compression occurs when we multiply the parent function $f(x) = b^x$ by a constant $|a| > 0$. For example, if the parent function $f(x) = 2^x$, we can graph the stretch, using a = 3, to get $g(x) = 3(2)^x$ and the compression, using $a = \frac{1}{3}$, to get $h(x) = \frac{1}{3}(2)^x$ as shown in Figure 8.14.

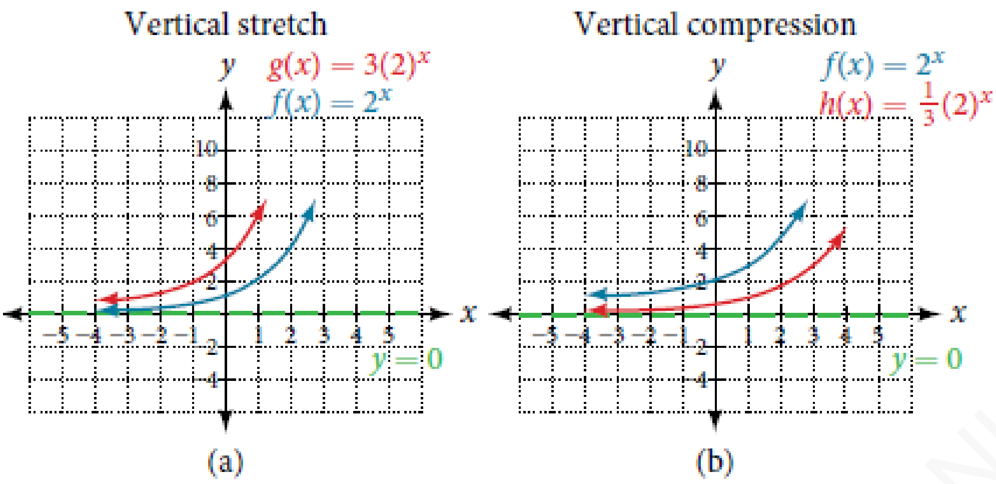


Figure 8.14

(a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3.

(b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.

Graphing Reflections

When we multiply the parent function $f(x) = b^x$ by -1, we get a reflection about the x-axis. When we multiply the input by -1, we get a reflection about the y-axis. For example, if the parent function $f(x) = 2^x$, we can graph the two reflections alongside it. The reflection about the x-axis, $g(x) = -2^x$, is shown on the left side of Figure 8.15, and the reflection about the y-axis $h(x) = 2^{-x}$, is shown on the right side of Figure 8.15.

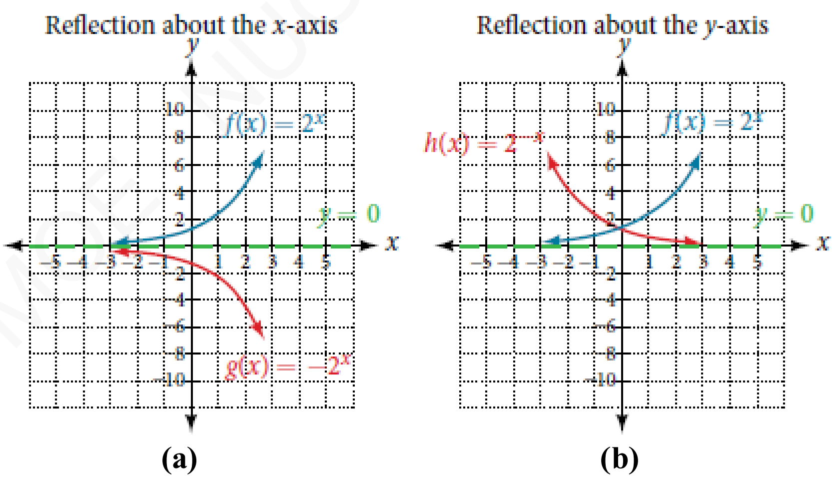


Figure 8.15

(a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis.

(b) $g(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the y-axis.

Summarizing Transformations of the Exponential Function

Now that we have worked with each type of transformation for the exponential function, we can summarize them in the following table to arrive at the general equation for transformation of the exponential function.

Transformations of the Parent Function $y = b^x$	
Transformation	**Form**
Shift • Horizontally c units to the left • Vertically d units up	$y = b^{x+c} + d$
Stretch and Compress • Stretch if $\|a\| > 1$ • Compress if $0 < \|a\| < 1$	$y = ab^x$
Reflection about the x-axis	$y = -b^x$
Reflection about the y-axis	$y = b^{-x} = (\frac{1}{b})^x$
General equation for all transformations	$y = ab^{x+c} + d$

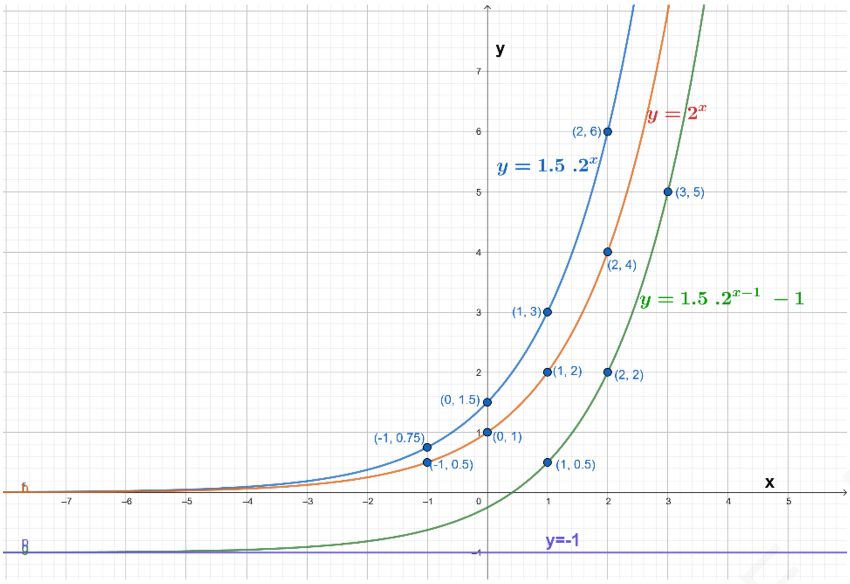
Example 6

Draw the graphs of $y = 1.5 \cdot 2^{x-1} - 1$ from $y = 2^x$, and $y = -2^{-(x+1)} + 2$ from $y = -2^{-x}$.

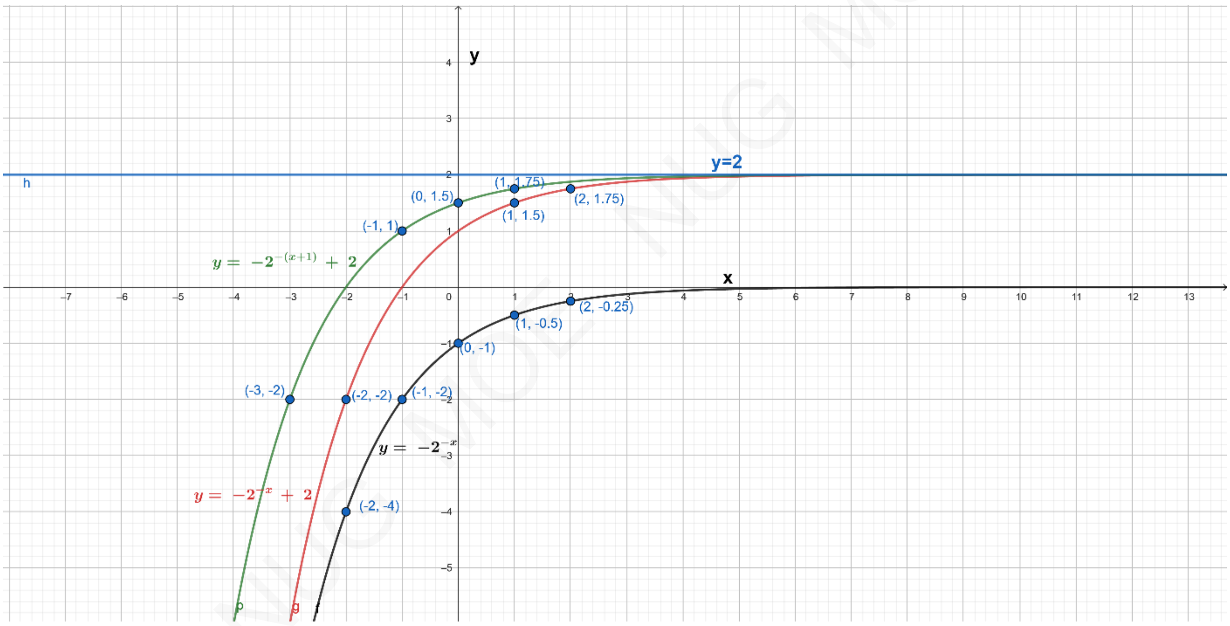
Solution

$y = 2^x$ $\xrightarrow[\text{scale factor 1.5}]{\text{Vertical stretch}}$ $y = 1.5 \cdot 2^x$ $\xrightarrow[\text{vertically 1 unit down}]{\text{Horizontally 1 unit to the right}}$ $y = 1.5 \cdot 2^{x-1} - 1$.

$y = -2^{-x}$ $\xrightarrow{\text{vertically 2 units up}}$ $y = -2^{-x} + 2$ $\xrightarrow[\text{to the left}]{\text{Horizontally 1 unit}}$ $y = -2^{-(x+1)} + 2$.



Note: Asymptote y = − 1



Note: Asymptote y = 2

Example 7

Points (0,1) and (1, b) are on the graph of $y = b^x$. Find the corresponding points on the graphs of $y = ab^x$ and $y = ab^{x-h} + k$. What is the asymptote of $y = ab^{x-h} + k$? Find the range of $y = ab^{x-h} + k$ if $a > 0$ and $a < 0$.

Solution

$y = b^x$ → (Vertical scaling, scale factor a) → $y = ab^x$ → (Horizontally h units to the right, vertically k units up) → $y = ab^{x-h} + k$

$(0,1)$ $\xrightarrow[\text{scale factor a}]{\text{Vertical scaling}}$ $(0, a)$ $\xrightarrow[\text{vertically k units up}]{\text{Horizontally h units to the right}}$ $(h, a + k)$

$(1, b)$ $\xrightarrow[\text{scale factor a}]{\text{Vertical scaling}}$ $(1, ab)$ $\xrightarrow[\text{vertically k units up}]{\text{Horizontally h units to the right}}$ $(1 + h, ab + k)$.

The asymptote of $y = ab^{x-h} + k$ is $y = k$.

If $a > 0$, then the range of $y = ab^{x-h} + k$ is (k, ∞) or $\{y/y > k\}$.

If $a < 0$, then the range of $y = ab^{x-h} + k$ is $(-\infty, k)$ or $\{y/y < k\}$.

Exercise 8.3

1. Draw the graph of (a) $y = 2 \cdot 3^{x+1} - 2$ (b) $y = -2^{x-1} + 3$.

2. Draw the graph of (a) $y = 2^{|x|}$ (b) $y = 2^{-|x|}$.

3. Find the y-intercept, asymptote and the range of
   (a) $y = 3e^{x-1} + 2$ (b) $y = -2e^{-x+1} + 3$.

8.4 Differentiation of Exponential Functions

Derivative of $y = b^x, b > 0, b \neq 1$

$y = b^x, b > 0, b \neq 1$

$\therefore x = \log_b y$

Differentiate both sides with respect to x.

$$1 = \frac{1}{y}\log_b e \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = y \cdot \frac{1}{\log_b e} = b^x \log_e b = b^x \ln b\,.$$

Therefore $\frac{d}{dx}b^x = b^x \ln b$

Since $\ln e = 1$, we have $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}e^{u(x)} = e^{u(x)} . \frac{d}{dx}u(x)$.

In general $\frac{d}{dx}b^{u(x)} = b^{u(x)} \ln b . \frac{d}{dx}u(x)$, $b > 0$.

Example 8

Differentiate the following functions with respect to x.

(a) e^{3x} (b) e^{1-x^2} (c) $e^{\sin x}$ (d) x^2e^{3x}

(e) $e^{2x}\sin 3x$ (f) $(e^x+e^{-x})^2$ (g) $\frac{3e^{2x}}{1-2x}$

Solution

(a) $\frac{d}{dx}e^{3x} = e^{3x}\frac{d}{dx}(3x) = 3e^{3x}$.

(b) $\frac{d}{dx}e^{1-x^2} = e^{1-x^2}\frac{d}{dx}(1-x^2) = e^{1-x^2}(-2x)$.

(c) $\frac{d}{dx}e^{\sin x} = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x}\cos x$.

(d) $\frac{d}{dx}(x^2e^{3x}) = x^2\cdot\frac{d}{dx}e^{3x} + e^{3x}.\frac{d}{dx}x^2 = x^2\cdot e^{3x}\cdot 3 + e^{3x}.\ 2x.$

(e) $\frac{d}{dx}(e^{2x}\sin 3x) = e^{2x}\cdot\frac{d}{dx}\sin 3x + \sin 3x\cdot\frac{d}{dx}e^{2x}$

$= e^{2x}\cdot\sin 3x\cdot 3 + \sin 3x\cdot e^{2x}\cdot 2.$

(f) $\frac{d}{dx}(e^x+e^{-x})^3 = 3(e^x+e^{-x})^2\frac{d}{dx}(e^x+e^{-x})$

$= 3(e^x+e^{-x})^2.(e^x-e^{-x}).$

(g) $\frac{d}{dx}\left(\frac{3e^{2x}}{1-2x}\right) = 3\,\frac{(1-2x)\cdot\frac{d}{dx}e^{2x} - e^{2x}\cdot\frac{d}{dx}(1-2x)}{(1-2x)^2}$

$= 3\,\frac{(1-2x)\cdot 2e^{2x} - e^{2x}\cdot(-2)}{(1-2x)^2}$

$= 3\,\frac{(2-4x)e^{2x}+2e^{2x}}{(1-2x)^2}$

$= \frac{3(4-4x)e^{2x}}{(1-2x)^2}$.

Example 9

Find $\frac{dy}{dx}$.

(a) $y = e^x\ln x$ (b) $y = \log_{10}e^{x^2}$ (c) $y = \log_3(\sin x + e^x)$

(d) $xe^y + \ln(xy) = \sin x$

Solution

(a) $y = e^x \ln x$

$\frac{dy}{dx} = e^x \cdot \frac{d}{dx}\ln x + \ln x \cdot \frac{d}{dx}e^x = e^x \cdot \frac{1}{x} + \ln x \cdot e^x.$

(b) $y = \log_{10} e^{x^2} = x^2 \log_{10} e$

$\frac{dy}{dx} = \log_{10} e \cdot \frac{d}{dx}x^2 = \log_{10} e \cdot 2x.$

(c) $y = \log_3(\sin x + e^x)$

$$\frac{dy}{dx} = \frac{1}{\sin x + e^x} \cdot \log_3 e \cdot \frac{d}{dx}(\sin x + e^x)$$
$$= \frac{1}{\sin x + e^x} \cdot \log_3 e \cdot (\cos x + e^x).$$

(d) $xe^y + \ln(xy) = \sin x$

Differentiate both sides with respect to x.

$$x \cdot e^y \cdot \frac{dy}{dx} + e^y + \frac{1}{xy}\left(x\frac{dy}{dx} + y\right) = \cos x$$
$$x \cdot e^y \cdot \frac{dy}{dx} + e^y + \frac{1}{y}\frac{dy}{dx} + \frac{1}{x} = \cos x$$
$$\left(xe^y + \frac{1}{y}\right)\frac{dy}{dx} = \cos x - e^y - \frac{1}{x}$$
$$\frac{dy}{dx} = \frac{\cos x - e^y - \frac{1}{x}}{xe^y + \frac{1}{y}}.$$

Example 10

Differentiate $y = x^x,\ x > 0.$

Solution

$y = x^x = e^{\ln x^x} = e^{x \ln x}$

$$\frac{dy}{dx} = \frac{d}{dx}e^{x \ln x}$$
$$= e^{x \ln x}\frac{d}{dx}(x \ln x) = e^{x \ln x}\left(x \cdot \frac{1}{x} + \ln x \cdot 1\right) = x^x(1 + \ln x).$$

Exercise 8.4

1. Differentiate the following functions with respect to x.

(a) $(5+3x)e^{-2x}$ (b) $3^x x^3$ (c) $2^x \log_2(x)$

(d) $10^x \log_{10}(x+1)$ (e) $\frac{x^2+\tan 3x}{e^x}$ (f) $x \ln y + e^{xy} = 2$

2. Given that $y = e^{3x}\sin 2x$, prove that $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$.

3. Use logarithmic differentiation to find the derivative of y with respect x.

(a) $y = (\sqrt{x})^x$ (b) $y = (x)^{\cos x}$ (c) $y = (\ln x)^{\ln x}$

Chapter 9
APPLICATION OF DIFFERENTIATION

In this chapter, we will learn how the derivative can be used to find the equations of tangent and normal to a curve at a point and how the differentials can be used to compute linear approximations. We will also study how the first derivative and second derivative can be apply to find extreme values of functions, determine the shape of curves, and analyze the behavior of functions.

9.1 Tangent Line and Normal Line

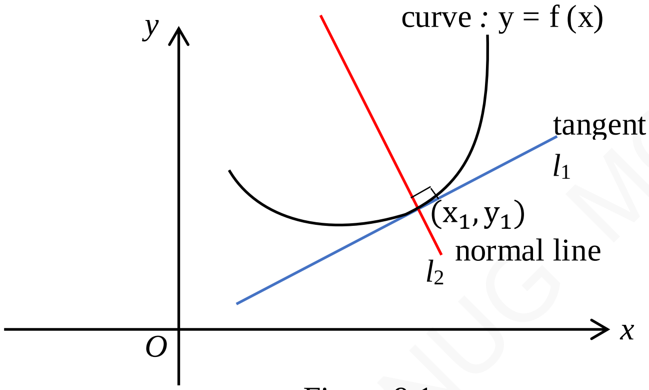


Figure 9.1

For the curve y = f (x), the gradient of the tangent l_1 at the point (x_1, y_1) is the value of $y'(x_1)$. The line l_2 which is perpendicular to the tangent l_1 at (x_1, y_1) is called the normal to the curve at (x_1, y_1). Hence its gradient is the value of $-\frac{1}{y'(x_1)}$ where $y'(x_1) \neq 0$.

The equation of the tangent at (x_1, y_1) is

$$y - y_1 = y'(x_1)(x - x_1)$$ and

the equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{1}{y'(x_1)}(x - x_1).$$

Example 1

Find the equations of tangent and normal line to the curve y = ln x at the point y = – 1.

Solution

$$y = \ln x,$$

$$y' = \frac{1}{x}$$

When y = –1,

$$-1 = \ln x$$

$$x = e^{-1}$$

$$x = \frac{1}{e}.$$

At the point $(\frac{1}{e}, -1)$ the gradient of the tangent is

$$y'(\frac{1}{e}) = \frac{1}{\frac{1}{e}} = e,$$

and the gradient of the normal is $-\frac{1}{e}$.

The equation of the tangent at $(\frac{1}{e}, -1)$ is

$$y + 1 = e\,(x - \frac{1}{e})$$

$$y + 1 = ex - 1$$

$$y = ex - 2.$$

The equation of the normal at $(\frac{1}{e}, -1)$ is

$$y + 1 = -\frac{1}{e}(x - \frac{1}{e})$$

$$y + 1 = -\frac{1}{e}x + \frac{1}{e^2}$$

$$y = -\frac{1}{e}x + \frac{1}{e^2} - 1.$$

Example 2

Find the equation of the tangent to y = sin x at the origin.

Solution

$$y = \sin x,$$

$$y' = \cos x.$$

At (0, 0), the gradient of the tangent is $y'(0) = \cos 0 = 1$.

The equation of the tangent at (0, 0) is

$$y - 0 = 1(x - 0)$$

$$y = x.$$

Example 3

Consider the curve $f(x) = x^2 + ax + b$ where a and b are constants. The tangent to this curve at the point $x = 2$ is $y = 2x + 1$. Find the values of a and b.

Solution

$$f(x) = x^2 + ax + b$$
$$f'(x) = 2x + a$$

At $x = 2$, $f'(2) = 4 + a$.

From the equation $y = 2x + 1$, the gradient of the tangent is 2.

$$4 + a = 2$$
$$a = -2$$
$$f(x) = x^2 - 2x + b$$

When $x = 2$, $f(2) = 4 - 4 + b$

$$f(2) = b.$$

The equation of the tangent at (2, b) is

$$y - b = 2(x - 2)$$
$$y - b = 2x - 4$$
$$y = 2x + b - 4.$$

By the comparing the given equation $y = 2x + 1$,

$$-4 + b = 1$$
$$b = 5.$$

$\therefore a = -2$ and $b = 5$.

<u>Alternative solution for example 3.</u>

$$f(x) = x^2 + ax + b$$
$$f'(x) = 2x + a$$

At $x = 2$, $f'(2) = 4 + a$.

From the equation $y = 2x + 1$, the gradient of tangent is 2.

$$4 + a = 2$$
$$a = -2$$
$$f(x) = x^2 - 2x + b \qquad \text{------- (1)}$$

From the equation $y = 2x + 1$,

when $x = 2$, $y = 2(2) + 1 = 5$

By using (1), $f(2) = 5$

$$(2)^2 - 2(2) + b = 5$$
$$4 - 4 + b = 5$$
$$b = 5$$

$a = -2$ and $b = 5$.

Exercise 9.1

1. Find the equations of tangent and normal line to the curve $y = 2 \ln x$ at (1, 0).
2. Find the equation of a line which passes through the origin and tangent to $y = e^x$.
3. Find the points of contact where horizontal tangent meet the curve

$$y = x^4 - 2x^2 + 2.$$

4. Consider the curve $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$ where a and b are constants.

   The normal to this curve at the point x = 4 is 4x + y = 22. Find the values of a and b.

9.2 Linear Approximation

Consider a function f that is differentiable at a point x = a. Recall that the tangent line to the graph of f at a is given by the equation

$$y = f(a) + f'(a)(x - a).$$

For example, consider the function $f(x) = \frac{1}{x}$ at a = 2. Since f is differentiable at x = 2 and $f'(x) = -\frac{1}{x^2}$, we see that $f'(2) = -\frac{1}{4}$. Therefore, the tangent line to the graph of f at a = 2 is

$$y = \frac{1}{2} - \frac{1}{4}(x - 2).$$

The following figure 9.2 shows a graph of $f(x) = \frac{1}{x}$ along with the tangent line to f at x = 2. Note that for x near 2, the graph of the tangent line is close to the graph of f. As a result, we can use the equation of tangent line to approximate f (x) for x near 2.

For example, if x = 2.1, the y value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(2.1 - 2) = 0.475.$$

The actual value of f (2.1) is given by

$$f(2.1) = \frac{1}{2.1} \approx 0.47619.$$

Therefore, the tangent line gives us a fairly good approximation of f (2.1). However, note that for values of x far from 2, the equation of the tangent line does not give us a good approximation. For example, if x = 10, the y-value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(10 - 2) = \frac{1}{2} - 2 = -1.5,$$

whereas the value of the function at x = 10 is f (10) = 0.1.

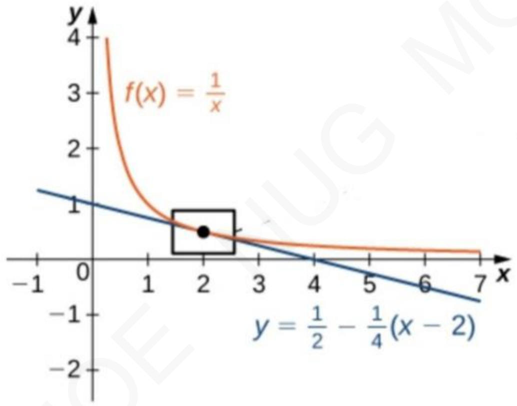


Figure 9.2

In general, for a differentiable function f, the equation of the tangent line to f at x = a can be used to approximate f (x) for x near a. Therefore, we can write

$$\boxed{f(x) \approx f(a) + f'(a)(x - a)} \text{ for x near a.} \qquad (1)$$

The linear function

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

is called the **linear approximation** or tangent line approximation of f at x = a. Now, (1) can be written in f (x) $\approx$ L(x) shortly. This function L is also known as the linearization of f at x = a. The difference of f(x) and L(x) is the error of the linearization.

Example 4

Find the linearization of $f(x) = x^2 + 2x$ at $x = 2$. Compare the approximate value and the true value at $x = 2.5$, $x = 2.05$ and $x = 2.005$.

Solution

$$f(x) = x^2 + 2x$$

$$f'(x) = 2x + 2.$$

$f(2) = 4 + 4 = 8$, $\quad f'(2) = 4 + 2 = 6.$

$$L(x) = f(2) + f'(2)(x - 2)$$

$$= 8 + 6(x - 2)$$

$$= 6x - 4.$$

$$f(x) \approx L(x) \text{ for x near } 2.$$

When $x = 2.5$, $\quad f(2.5) = (2.5)^2 + 2(2.5) = 11.25$

$$L(2.5) = 6(2.5) - 4 = 11.$$

Error $= 11.25 - 11 = 0.25$.

The approximate value differs from the true value by less than 0.25.

When $x = 2.05$, $\quad f(2.05) = (2.05)^2 + 2(2.05) = 8.3025$

$$L(2.05) = 6(2.05) - 4 = 8.3.$$

Error $= 8.3025 - 8.3 = 0.0025$.

The approximate value differs from the true value by less than 0.25×10^{-2}.

When $x = 2.005$, $\quad f(2.005) = (2.005)^2 + 2(2.005) = 8.030025$

$$L(2.005) = 6(2.005) - 4 = 8.03.$$

Error $= 8.030025 - 8.03 = 0.000025$.

The approximate value differs from the true value by less than 0.25×10^{-4}.

Example 5

Given that $y = x^{\frac{1}{2}}$, determine the approximate value of $\sqrt{101}$ by using linear approximation.

Solution

$$f(x) = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

When x = 100,

$$f(100) = \sqrt{100} = 10$$

$$f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$

$$L(x) = f(100) + f'(100)(x - 100)$$

$$= 10 + \frac{1}{20}(x - 100)$$

$$= 10 + 0.05x - 5$$

$$= 5 + 0.05x.$$

$$f(x) \approx L(x) \quad \text{for } x \text{ near } 100.$$

At x = 101,

$$f(101) \approx L(101)$$

$$\sqrt{101} \approx 5 + 0.05(101)$$

$$\sqrt{101} \approx 10.05.$$

Differentials

We have seen that linear approximations can be used to estimate function values. They can also be used to estimate the amount a function value changes as a result of a small change in the input. To discuss this more formally, we define a related concept: differentials. Differentials provide us with a way of estimating the amount a function changes as a result of a small change in input values.

When we first looked at derivatives, we used the Leibniz notation $\frac{dy}{dx}$ to represent the derivatives of y with respect to x. Although we used the expressions dy and dx in the notation, they did not have meaning on their own. Here we see a meaning to the expressions dy and dx. Suppose $y = f(x)$ is a differentiable function. Let dx be an independent variable that can be assigned any nonzero real number, and define the dependent variable dy by

$$dy = f'(x)\,dx \qquad (2)$$

It is important to notice that dy is a function of both x and dx. The expressions dy and dx are called **differentials**. We can divide both sides of Equation (2) by dx, which yields

$$\frac{dy}{dx} = f'(x) \qquad (3)$$

This is the familiar expression we have used to denote a derivative. Equation (2) is known as the differential form of Equation (3).

Example 6

For the function $y = x^2 + 2x$, find dy and evaluate when $x = 3$ and $dx = 0.1$.

Solution

Since $f(x) = x^2 + 2x$, we have $f'(x) = 2x + 2$ and

therefore $dy = (2x + 2)dx$.

When $x = 3$ and $dx = 0.1$,

$$dy = (2\cdot 3 + 2)(0.1)$$
$$= 0.8.$$

We now connect differentials to linear approximations. Differentials can be used to estimate the change in the value of a function resulting from a small change in input values. Consider a function f that is differentiable at point a. Suppose the input x changes by a small amount. We are interested in how much the output y changes.

If x changes from a to a + dx, then the change in x is dx (also denoted Δx), and the change in y is given by

$$\Delta y = f(a + dx) - f(a).$$

Instead of calculating the exact change in y, however, it is often easier to approximate the change in y by using a linear approximation. For x near a, f (x) can be approximated by the linear approximation.

$$L(x) = f(a) + f'(a)(x - a).$$

Therefore, if dx is small,

$$f(a + dx) \approx L(a + dx) = f(a) + f'(a)(a + dx - a).$$

That is, $$f(a + dx) - f(a) \approx L(a + dx) - f(a) = f'(a)\,dx.$$

In summary, $\Delta y \approx L(a + dx) - f(a) = dy$.

Therefore, we can use the differential $dy = f'(a)\,dx$ to approximate the change in y if x changes from $x = a$ to $x = a + dx$. We can see this in the following graph

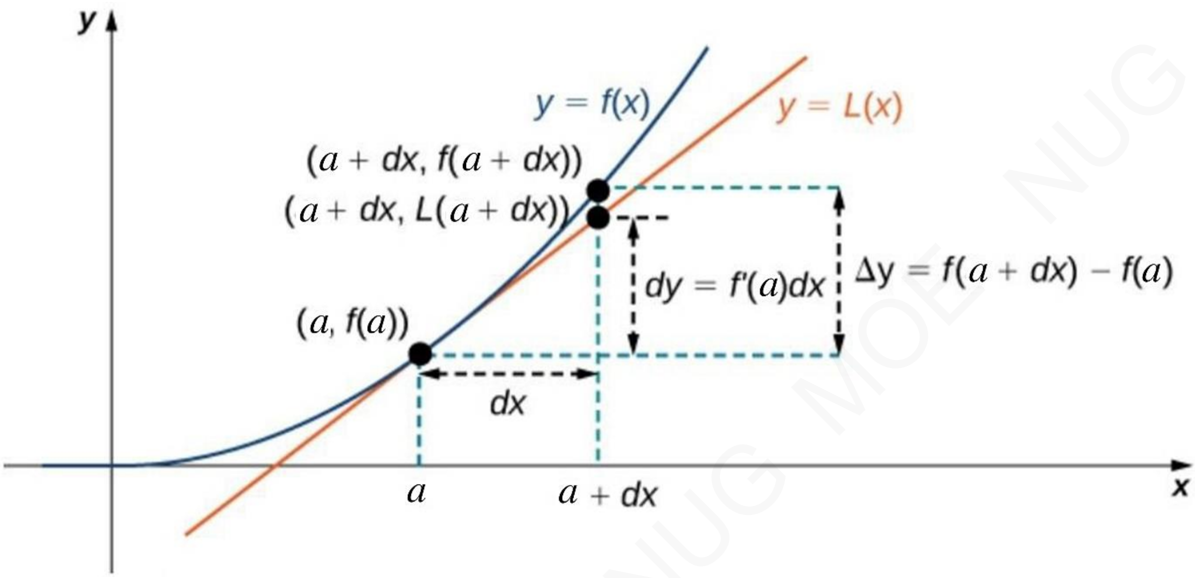


Figure 9.3

Example 7

If the radius of a circle increases from r = 5 cm to 5.01 cm, find the approximate increase in the area. Estimate the area of the enlarge circle and find the error.

Solution

Let A be the area of the circle of radius r.

$$A(r) = \pi r^2$$
$$A'(r) = 2\pi r$$
$$A(5) = 25\pi, \qquad A'(5) = 10\pi,$$
$$dr = 5.01 - 5 = 0.01$$
$$dA = A'(5)dr = 10\pi(0.01) = 0.1\pi.$$

The approximate increase in the area is 0.1π cm^2.

$$L(r) = A(5) + A'(5)(r - a)$$
$$L(r) = 25\pi + 10\pi(r - 5)$$
$$= 25\pi + 10\pi r - 50\pi$$
$$= 10\pi r - 25\pi$$

$$L(5.01) = 10\pi(5.01) - 25\pi$$
$$= 25.1\pi\ .$$

The approximate area of enlarge circle = $25.1\,\pi$ cm^2.

The true area of enlarge circle = A (5.01) = $(5.01)^2$ = $25.1001\,\pi$ cm^2

$\therefore$ Error = A (5.01) – L (5.01) = $25.1001\,\pi - 25.1\,\pi = 0.0001\,\pi$ cm^2. -

Exercise 9.2

1. Find the linearization of

   (a) $f(x) = \sqrt{1+x}$ at x = 3. Compare the approximate value and true value at x = 3.2, x = 3.02 and x = 3.002.

   (b) $f(x) = \frac{x}{x+1}$ at x = 1. Compare the approximate value and true value at x = 1, x = 1.01 and x = 1.001.

2. Show that the linearization of $f(x) = (1+x)^k$ at x = 0 is $L(x) = 1 + kx$.

3. Use the linearization to approximate the following values.

   (i) $\sqrt{80}$ (ii) $\sqrt[3]{65}$ (iii) $\frac{1}{\sqrt{1.22}}$

4. If $y = 4\sqrt{x} + 3x^2$, find the approximate change in y when x changes from 9 to 8.98.

5. If $y = 3\sqrt{72 + x^2}$, find the approximate change in y when

   (i) x increases from 3 to 3.01,

   (ii) x decreases from 3 to 2.98.

6. Find the approximate change in the volume of a sphere when its radius decreases from 5 cm to 4.97 cm. Estimate the volume of the reduced sphere and find the error.

9.3 Extreme values of functions

This section shows to locate and identify maximum or minimum values of a function from its derivative. Here, the domain of the functions we consider are intervals.

- **Definitions**

Lef f be a function with domain D. Then f has an **global maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all x in D and an **global minimum** value at c within its domain D if $f(x) \geq f(c)$ for all x in D.

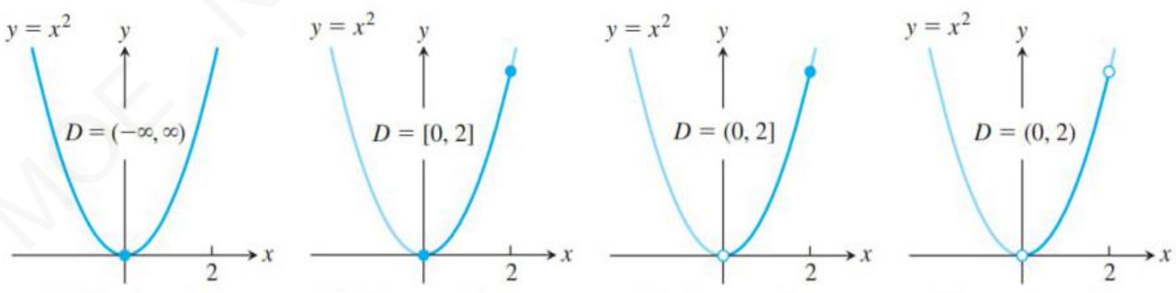
Maximum and minimum values are called extreme values of the function f.

Example 8

The global extrema of the following functions on their domains can be seen in Figure 9.4. Each function has the same defining equation, $y = x^2$, but the domains vary.

Solution

	Function	Domain D	Global extrema on D
(a)	$y = x^2$	$(-\infty, \infty)$	No global maximum
			Global minimum of 0 at x = 0
(b)	$y = x^2$	[0, 2]	Global maximum of 4 at x = 2
			Global minimum of 0 at x = 0.
(c)	$y = x^2$	(0, 2]	Global maximum of 4 at x = 2.
			No global minimum.
(d)	$y = x^2$	(0, 2)	No global extrema.



(a) gld min only (b) gld max and min (c) gld max only (d) no max or min

Figure 9.4

- **Definitions**

A function f has a **local maximum** value at a point c within its domain D if $f(c) \geq f(x)$ when x is near c.

A function f has a **local minimum** value at a point c within its domain D if $f(c) \leq f(x)$ when x is near c.

A global maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence a list of all local maxima will automatically include the global maximum if there is one. Similarly, a list of all local minima will include the global minimum if there is one.

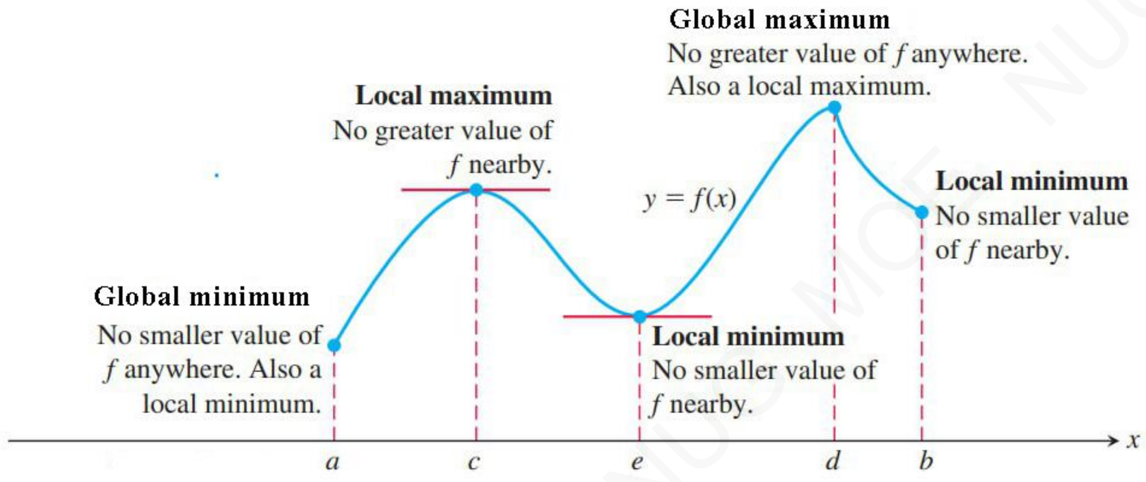


Figure 9.5

- **Definitions**

An interior point of the domain of a function f where f′ is zero or undefined is a **critical point** of f.

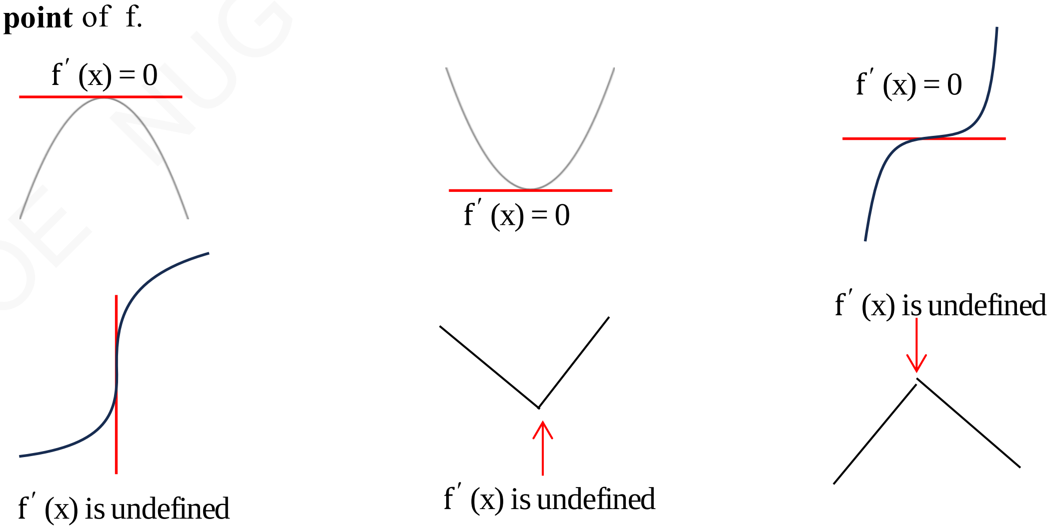


Figure 9.6

Example 9

Find the critical points of:

(a) $f(x) = x^3 - 3x + 1$ (b) $f(x) = (x-2)^{\frac{2}{3}}$ (c) $f(x) = \frac{1}{x}$

Solution

(a)

$$f(x) = x^3 - 3x + 1$$

$$f'(x) = 3x^2 - 3.$$

For critical points, $f'(x) = 0$

So, $3x^2 - 3 = 0$

$$x = \pm 1.$$

Therefore, the critical points are –1 and 1.

(b)

$$f(x) = (x-2)^{\frac{2}{3}}$$

$$f'(x) = \frac{2}{3}(x-2)^{-\frac{1}{3}} = \frac{2}{3(x-2)^{\frac{1}{3}}}.$$

f ' (x) is undefined when x = 2.

So, the critical point is 2.

(c)

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}.$$

When x = 0, f ' (x) is undefined.

But x = 0 is not in the domain of the given function.

Therefore, there are no critical point.

To find the global extrema of a continuous function f on a closed interval [a, b]:

1. Find all critical points of f on the interval.
2. Evalulate the values of f at all critical points and end points.
3. Take the largest and smallest of these values.

Example 10

Find the global minimum and global maximum values of each of the following functions:

(a) $f(x) = x^2$ on $[-2, 1]$, (b) $f(x) = x^{\frac{2}{3}}$ on $[-2, 3]$.

Solution

(a) To evaluate the values of the function at the critical points and endpoints, we will find the first derivatives,

$$f'(x) = 2x.$$

The only critical point occurs where $f'(x) = 2x = 0$, namely $x = 0$.

Critical point value : $f(0) = 0^2 = 0$.

Endpoint values : $f(-2) = (-2)^2 = 4$

$f(1) = 1^2 = 1$.

The function has the global maximum value of 4 at $x = -2$ and the global minimum value of 0 at $x = 0$.

(b) To evaluate the values of the function at critical points and endpoints, we will find the first derivative.

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$f'(x)$ has no zeros but is undefined at the interior point $x = 0$.

For the given function, we get the critical point $x = 0$.

Critical point values : $f(0) = 0$.

Endpoint values : $f(-2) = (-1)^{\frac{2}{3}} = \sqrt[3]{4}$

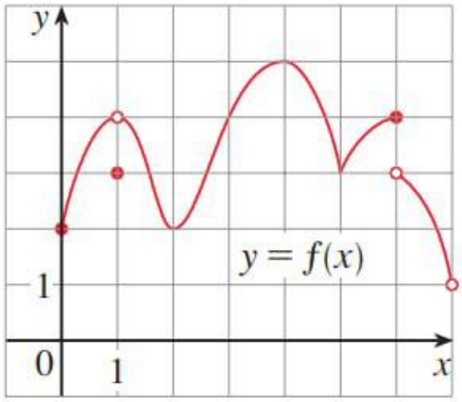
$f(3) = (3)^{\frac{2}{3}} = \sqrt[3]{9}$.

The function has the global maximum value of $\sqrt[3]{9}$ at $x = 3$ and the global minimum value of 0 at $x = 0$.

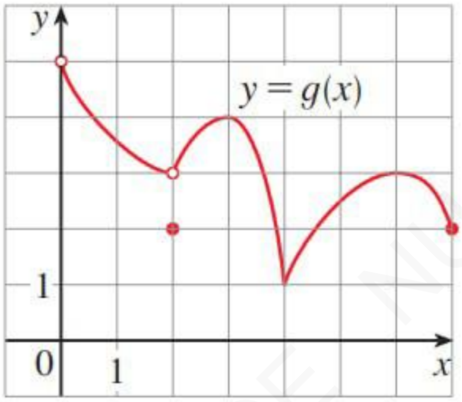
Exercise 9.3

1. Use the graph to state the global and local maximum and minimum values of the functions.

(a)



(b)



2. Find the critical points of:

(a) $f(x) = x^2 - 2x + 5$ (b) $f(x) = x^3 + 6x^2 + 3x + 10$

(c) $f(x) = \frac{1}{x-3}$ (d) $f(x) = (x^2 - 2x)^{\frac{2}{3}}$

(e) $f(x) = \dfrac{1}{x^2 - 2x + 1}$ (f) $f(x) = \frac{x^2}{x^2+2}$

3. Find the global minimum and maximum values of each of the following function:

(a) $f(x) = x^2 + 5x - 3$ on $[-4, 5]$ (b) $f(x) = \dfrac{x}{x^2+1}$ on $[-3, 3]$

4. Find the global extreme values of the following functions:

(a) $f(x) = 3x - x^3 + 2$ on $[-5, 5]$, (b) $f(x) = \sin 2x$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

9.4 First Derivative Test and Second Derivative Test for Local Extrema

Before we show how to test the critical points of a function to identify whether local extreme values are present let us discuss the behavior of increasing and decreasing functions.

- **Definitions**

A function f is **increasing** on an interval (a, b) if for any two numbers x_1 and x_2 in (a, b),

$$x_1 < x_2 \text{ implies } f(x_1) < f(x_2).$$

A function f is **decreasing** on an interval (a, b) if for any two numbers x_1 and x_2 in (a, b),

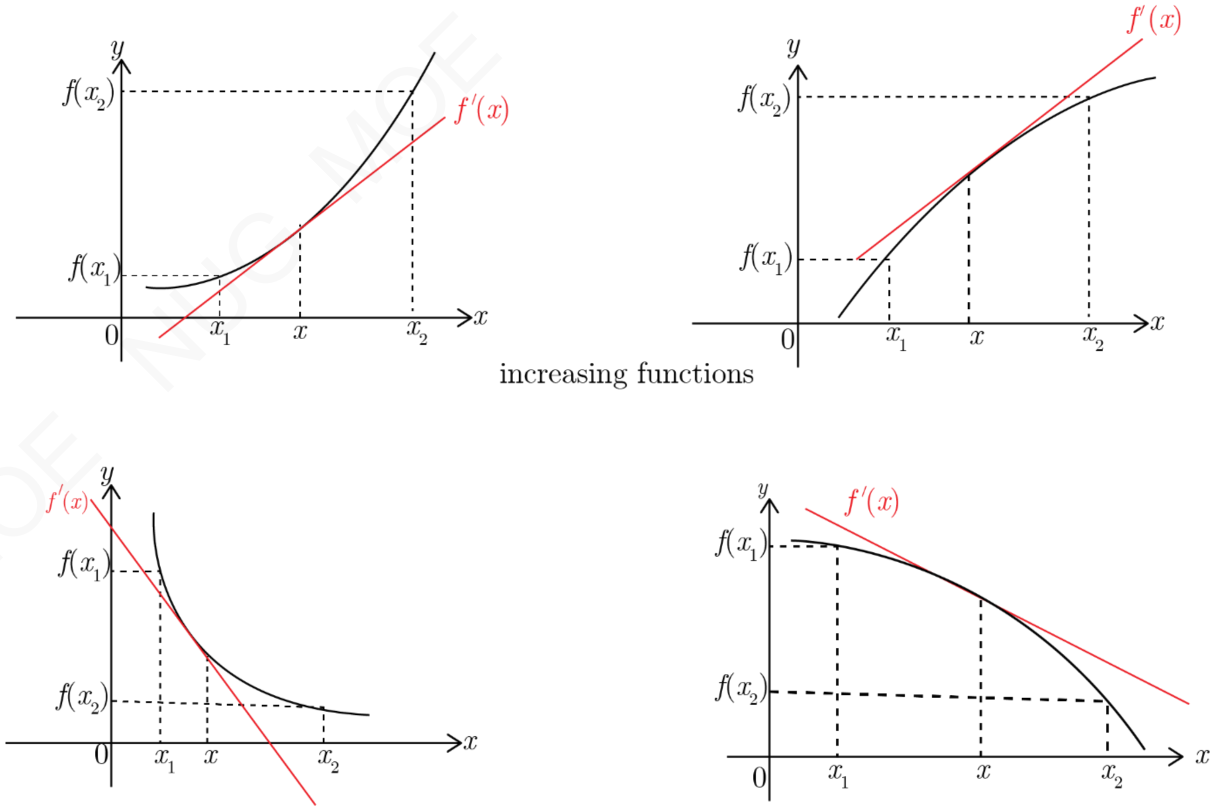
$$x_1 < x_2 \text{ implies } f(x_1) > f(x_2).$$

Increasing / Decreasing Test

Suppose that f is differentiable on (a, b).

(a) If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on (a, b).

(b) If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on (a, b).



increasing functions

decreasing functions

Figure 9.7

Example 11

Show that the function $f(x) = 3x^2 + 2$ is decreasing on the interval $x < 0$ and increasing on the interval $x > 0$.

Solution

$$f(x) = 3x^2 + 2.$$

$$f'(x) = 6x.$$

For $x < 0$, $f'(x) < 0$.

Thus f is decreasing on $x < 0$.

For $x > 0$, $f'(x) > 0$.

Thus f is increasing on $x > 0$.

Example 12

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and those on which f is decreasing.

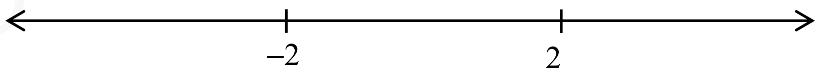
Solution

$$f(x) = x^3 - 12x - 5$$

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2).$$

$f'(x)$ is zero at $x = -2$ and $x = 2$.

So, the critical points are –2 and 2.



interval	**x < – 2**	**–2 < x < 2**	**x > 2**
Sign of f'	+	–	+
Behavior of f	increasing	decreasing	increasing

Thus, the given function is increasing on $x < -2$ and $x > 2$,

and the given function is decreasing on $-2 < x < 2$.

Example 13

Find the open intervals on which the following functions are increasing or decreasing.

(a) $f(x) = (x^2 - 4)^{\frac{2}{3}}$ (b) $f(x) = \frac{3x-7}{x-2}$

Solution

(a) $f(x) = (x^2 - 4)^{\frac{2}{3}}$

$f'(x) = \frac{2}{3}(x^2-4)^{-\frac{1}{3}} \cdot 2x$

$= \frac{4x}{3(x^2-4)^{\frac{1}{3}}}$

$f'(x)$ is zero at x = 0.

$f'(x)$ is undefined when $3(x^2-4)^{\frac{1}{3}} = 0$.

So, f ' (x) is undefined when x = –2 and x = 2.

interval	**x < – 2**	**–2 < x < 0**	**0 < x < 2**	**x > 2**
Sign of f '	–	+	–	+
Behavior of f	decreasing	increasing	decreasing	increasing

Therefore, f is decreasing on x < –2 and 0 < x < 2,

and f is increasing on –2 < x < 0 and x > 2.

(b) $f(x) = \frac{3x-7}{x-2}, \; x \neq 2,$

$f(x) = 3 - \frac{1}{x-2}$

$f'(x) = \frac{1}{(x-2)^2}.$

There is no critical point because x = 2 ∉ domain of f.

Interval	**x < 2**	**x > 2**
Sign of f'	+	+
Behaviour of f	increasing	increasing

So, f is increasing on x < 2 and x > 2.

First Derivative Test for local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides of c), then f has no local extremum at c.

The test for local extrema at endpoints is similar, but there is only one side to consider in determining whether f is increasing or decreasing, based on the sign of f'.

Example 14

Find the critical points of $f(x) = x^3 - 3x + 1$. Find the open intervals on which the function is increasing or decreasing. Identify the function's local maximum and minimum values.

Solution

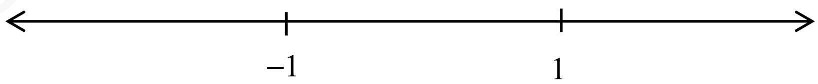
$$f(x) = x^3 - 3x + 1$$

$$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1).$$

For the critical points, $f'(x) = 0$.

It gives $x = -1$ and $x = 1$.

So, the critical points are -1 and 1.



interval	**x < – 1**	**–1 < x < 1**	**x > 1**
Sign of f'	+	–	+
Behaviour of f	increasing	decreasing	increasing

f has a local maximum value at $x = -1$.

At $x = -1$, $f(-1) = (-1)^3 - 3(-1) + 1 = 3$.

Therefore, the local maximum value of f is 3.

Next, f has a local minimum value at $x = 1$.

At $x = 1$, $f(1) = 1 - 3 + 1 = -1$.

Therefore, the local minimum value of f is -1.

Example 15

Show that each of the following functions is increasing on both sides of critical point.

(a) $f(x) = x^3$ (b) $f(x) = x^{\frac{1}{3}}$

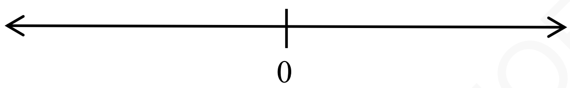
Solution

(a) $f(x) = x^3$

$f'(x) = 3x^2$

For the critical points, $f'(x) = 0$

It gives the critical point $x = 0$.



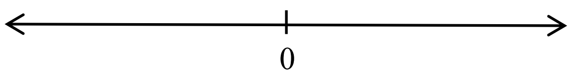
interval	x < 0	x > 0
Sign of f'	+	+
Behavior of f	increasing	increasing

Thus, the function $f(x) = x^3$ is increasing on both sides of the critical point $x = 0$.

(b) $f(x) = x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$.

Since $f'(x)$ is undefined, $x = 0$ is the critical point of f.

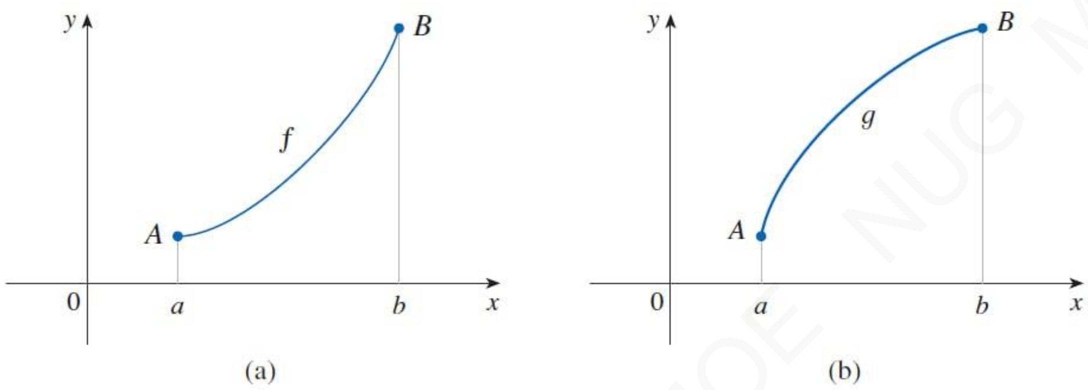


interval	x < 0	x > 0
Sign of f'	+	+
Behaviour of f	increasing	increasing

Therefore, the function $f(x) = x^{\frac{1}{3}}$ is increasing on both sides of the critical point $x = 0$.

- **Concavity**

The following figures 9.8 shows the graphs of two increasing functions on (a, b). Both graphs join point A to point B but they look different because they bend in different directions. How can we distinguish between these two types of behaviour?



Figures 9.8

As we approach the point b from the point a along the curve, of the function f, the slopes of the tangent lines are increasing on (a, b) and the curve lies above its tangent lines. As we move away from the point a to the point b along the curve of the function g, the slopes of the tangent lines are decreasing on (a, b) and the curve lies below its tangent lines. This behavior can be seen in the following figure 9.9. This bending behavior is called the **concavity** of the curve.

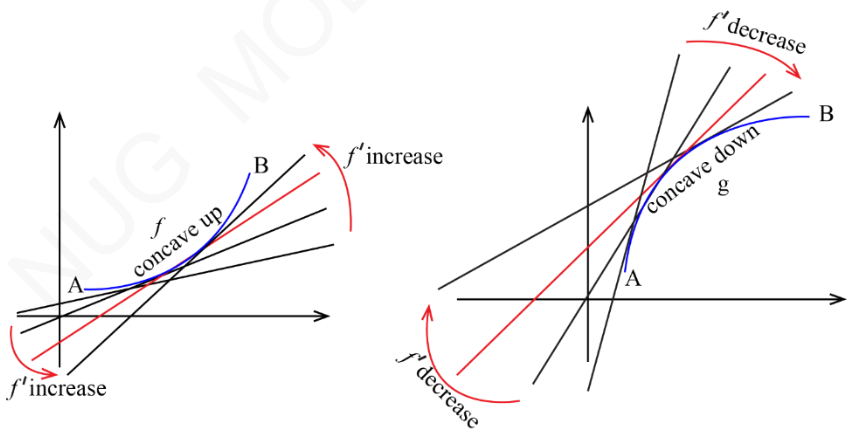


Figure 9.9

- **Definition**

The graph of a differentiable function f is

(a) **concave up** on an open interval (a, b) if f' is increasing on (a, b);

(b) **concave down** on an open interval (a, b) if f' is decreasing on (a, b).

A function whose graph is concave up is also often called convex.

If the function f has a second derivative, we conclude that f' increases if $f'' > 0$ on (a, b), and decrease if $f'' < 0$ on (a, b).

Concavity Test

Let f be twice-differentiable on an open interval (a, b).

1. If $f''(x) > 0$ at each point $x \in (a, b)$, then f is concave up on (a, b).
2. If $f''(x) < 0$ at each point $x \in (a, b)$, then f is concave down on (a, b).

Example 16

(a) From the figure 9.10, we can see that the curve $f(x) = x^3$ is concave down on $x < 0$, where $f''(x) = 6x < 0$, and concave up on $x > 0$, where $f''(x) = 6x > 0$.

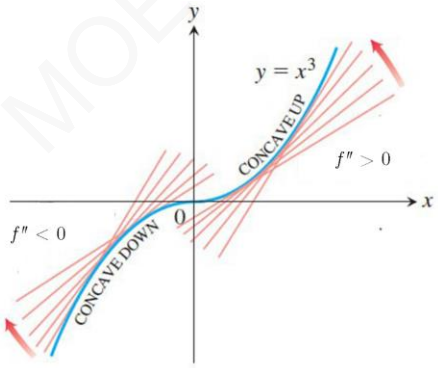


Figure 9.10

(b) The curve $f(x) = x^2$, from the figure 9.11, is concave up everywhere because its second derivative $f''(x) = 2$ is always positive.

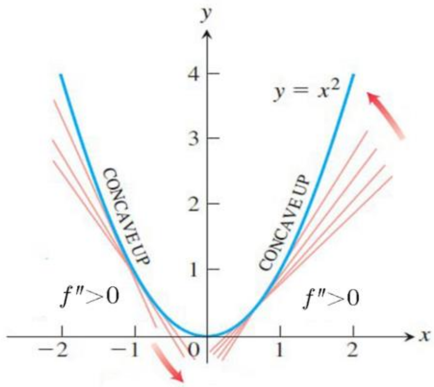


Figure 9.11

- **Definition**

A point (c, f (c)) where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

By the above definition, one of two things can happen at a point of inflection.

At a point of inflection (c, f (c)), either $f''(c) = 0$ or $f''(c)$ is undefined.

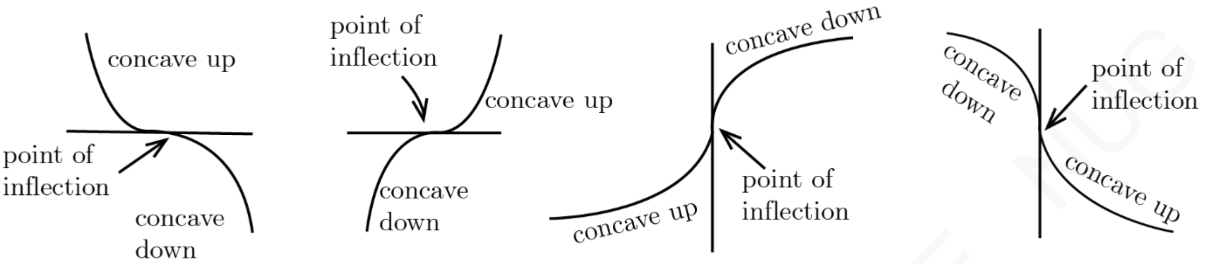


Figure 9.12

Example 17

Determine the concavity and find the inflection points of the following functions:

(a) $f(x) = x^{\frac{5}{3}}$ (b) $f(x) = x^{\frac{1}{3}}$ (c) $f(x) = x^3 - 3x^2 + 2$.

Solution

(a) $f(x) = x^{\frac{5}{3}}$

$f'(x) = \frac{5}{3}x^{\frac{2}{3}}$

$f''(x) = \frac{10}{9}x^{-\frac{1}{3}} = \frac{10}{9\sqrt[3]{x}}$.

Thus $f''(x)$ is undefined at x = 0.

interval	x < 0	x > 0
Sign of f''	–	+
Concavity of f	concave down	concave up

When x = 0, $f(0) = 0^{\frac{5}{3}} = 0$.

Therefore (0, 0) is a point of inflection.

(b) $f(x) = x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}} = -\frac{2}{9x\sqrt[3]{x^2}}$

So $f''(x)$ is undefined at $x = 0$.

interval	**x < 0**	**x > 0**
Sign of f''	+	–
Concavity of f	concave up	concave down

When $x = 0$, $f(0) = 0^{\frac{1}{3}} = 0$.

Therefore (0, 0) is a point of inflection.

(c) $f(x) = x^3 - 3x^2 + 2$

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6$

So, $f''(x) = 0$ at $x = 1$.

interval	**x < 1**	**x > 1**
Sign of f''	–	+
Concavity of f	concave down	concave up

When $x = 1$, $f(1) = 1^3 - 3(1)^2 + 2 = 0$.

Therefore (1, 0) is a point of inflection.

Example 18

Determine the concavity and find the inflection points (if any) of the following functions:

(a) $f(x) = x^4$ (b) $f(x) = x^{\frac{2}{3}}$ on (–2, 3)

Solution

(a) $f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

Thus $f''(x) = 0$ at $x = 0$.

interval	x < 0	x > 0
Sign of f''	+	+
concavity of f	concave up	concave up

The concavity of the curve does not change.

Although f '' (0) = 0, the function f has no inflection point at x = 0.

(b) $f(x) = x^{\frac{2}{3}}$ on (–2, 3)

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$$f''(x) = -\frac{2}{9}x^{-\frac{4}{3}} = -\frac{2}{9x\sqrt[3]{x}}$$

Thus $f''(x)$ is undefined at x = 0.

interval	x < 0	x > 0
Sign of f''	–	–
concavity of f	concave down	concave down

The concavity of the curve does not change.
Although $f''(0)$ is undefined, the given function has no inflection point at x = 0.

From the above examples 17 and 18, we see that when the second derivative is zero or does not exist at x = c, an inflection point may or may not occur there.

Another application of the second derivative is the following test for identifying local maximum and minimum values. It is a consequence of the Concavity Test, and it serves as an alternative to the First Derivative Test.

The Second Derivative Test for Local Extrema

suppose f'' is continuous on an open interval that contains x = c.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at x = c.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at x = c.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. In this case, the function f may have a local maximum, a local minimum, or neither.

Example 19

Discuss the function $f(x) = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima.

Solution

$$f(x) = x^4 - 4x^3$$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

For the critical points, $f'(x) = 0$

$$4x^2(x-3) = 0$$

$$x = 0 \text{ or } x = 3$$

When $x = 3$, $f''(3) > 0$.

So, f has a local minimum value at $x = 3$.

When $x = 0$, $f''(0) = 0$.

So, the Second Derivative Test fails

Interval	**x < 0**	**x > 0**
Sign of f'	–	–
Behaviour of f	decreasing	decreasing

Thus, the First Derivative Test tell us that f does not have a local maximum or minimum at 0.

To determine the concavity and the inflection points,

$$f''(x) = 0$$

$$12x(x-2) = 0$$

$$x = 0 \text{ or } x = 2.$$

interval	**x < 0**	**0 < x < 2**	**x > 2**
Sign of f''	+	–	+
Concavity of f	concave up	concave down	concave up

Since the concavity of f changes at $x = 0$ and $x = 2$, f has the inflection points at

$x = 0$ and $x = 2$.

When $x = 0$, $f(0) = 0$ and

when $x = 2$, $f(2) = -16$.

Therefore (0, 0) and (2, –16) are the inflection points.

Example 20

Find and classify the critical points of $f(x) = x^4 - 4x^3 + 5$.

Also find the points of inflection.

Solution

$f(x) = x^4 - 4x^3 + 5,$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

$f''(x) = 12x^2 - 24x = 12x(x - 2)$

For the critical points, $f'(x) = 0$

$4x^2(x - 3) = 0$

$x = 0$ or $x = 3$.

The critical points are $x = 0$ and $x = 3$.

When $x = 3$, $f''(3) > 0$.

Therefore, f has local minimum at $x = 3$.

When $x = 0$, $f''(0) = 0$.

The second derivative test fails.

Since $f'(x) < 0$ when $x < 0$ and $f'(x) < 0$ when $x > 0$, f has no local extremum at $x = 0$.

To find the points of inflection,

$f''(x) = 0$

$12x(x - 2) = 0$

$x = 0$ or $x = 2$.

interval	**$x < 0$**	**$0 < x < 2$**	**$x > 2$**
Sign of f''	+	–	+
Concavity of f	concave up	concave down	concave up

Since the concavity of f changes at $x = 0$ and $x = 2$, f has the inflection points at $x = 0$ and $x = 2$.

When $x = 0$, $f(0) = 5$.

When $x = 2$, $f(2) = 2^4 - 4(2) + 5$

$= 16 - 8 + 5$

$= 13.$

(0, 5) and (2, 13) are the points of inflection.

Example 21

Find the range of $f(x) = \ln x - x$.

Solution

$$f(x) = \ln x - x$$

$$f'(x) = \frac{1}{x} - 1$$

$$f''(x) = -\frac{1}{x^2}$$

For critical points, $f'(x) = 0$

$$\frac{1}{x} - 1 = 0.$$

Therefore, $x = 1.$

When $x = 1$, $f''(1) = -1 < 0.$

When $x = 1$, $f(1) = \ln 1 - 1 = 0 - 1 = -1.$

Since we do not need to evaluate the values of f at the end points, the global maximum value of f is -1 at $x = 1$.

i.e. $f(x) \leq -1$ for all x.

Therefore the range of f is $\{y : y \leq -1\}$.

Example 22

Find the least amount of material needed to build an open cylindrical vessel with a capacity of $400\ \pi cm^3$.

Solution

Let r be the radius and h be the height of the cylindrical vessel.

The volume of vessel $= 400\pi$

$$\pi r^2 h = 400\pi$$

$$h = \frac{400}{r^2}$$

Let A be the area of material for vessel.

$$A = \pi r^2 + 2\pi rh$$

$$A = \pi r^2 + 2\pi r\,\frac{400}{r^2}$$

$$= \pi r^2 + \frac{800\pi}{r}$$

$$\frac{dA}{dr} = 2\pi r - \frac{800\pi}{r^2}$$

$$\frac{d^2A}{dr^2} = 2\pi + \frac{1600\pi}{r^3}$$

For the critical points, $\frac{dA}{dr} = 0$

$$2\pi r - \frac{800\pi}{r^2} = 0$$

$$r^3 = 400$$

$$r = \sqrt[3]{400}$$

When $r = \sqrt[3]{400}$, $\frac{d^2A}{dr^2} = 6\pi > 0.$

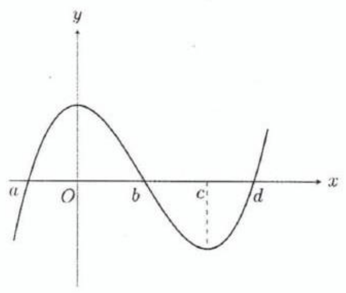
Thus the area of the material is the least when $r = \sqrt[3]{400}$.

Therefore the least amount of material $= \pi(\sqrt[3]{400})^2 + \frac{800\pi}{\sqrt[3]{400}}$

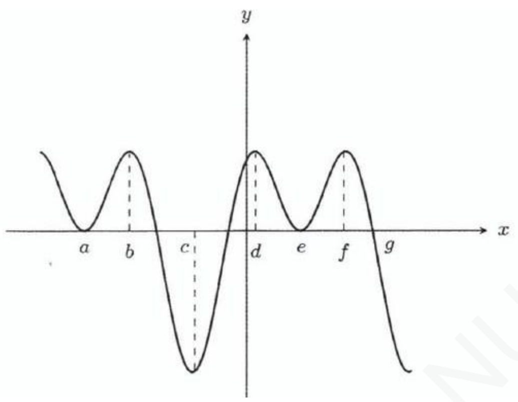
$$= \frac{1200\pi}{\sqrt[3]{400}}\text{cm}^2$$

Exercise 9.4

1. Consider the following graphs.



(i)



(ii)

(a) State all open intervals where the function is increasing.

(b) State all open intervals where the function is decreasing.

2. Find the open intervals on which the following functions are increasing or decreasing.

Find the function's local maximum and minimum values.

(a) $f(x) = x^3 - 3x^2 + 5$ (b) $f(x) = (x^2 - 2x)^{\frac{2}{5}}$ (c) $f(x) = \frac{x^2+5}{x}$

(d) $f(x) = -x^3 - 6x^2 + 12$ (e) $f(x) = \frac{x-3}{x-1}$ (f) $f(x) = x^{\frac{2}{3}}(x^2 - 16)$

3. The graph of a function $y = f(x)$ is shown. At which points are the following true?

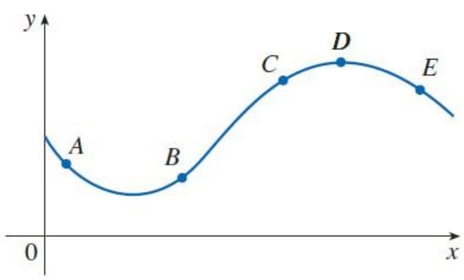
(a) $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are both positive.

(b) $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are both negative.

(c) $\frac{dy}{dx}$ is negative but $\frac{d^2y}{dx^2}$ is positive.

(d) $\frac{dy}{dx}$ is positive but $\frac{d^2y}{dx^2}$ is negative.

(e) $\frac{dy}{dx}$ is zero but $\frac{d^2y}{dx^2}$ is negative.



4. Determine the open intervals on which the graph of the following functions

(a) $f(x) = \frac{1}{x^2+1}$ (b) $f(x) = \frac{9}{88}x^{\frac{11}{3}}$

is concave up or concave down. State their inflection points.

5. Determine the concavity and find the inflection points of the following functions:

(a) $f(x) = (x - 1)^5$ (b) $g(x) = x^4 - 2x^3$ (c) $h(x) = \sin x$ on $[-\pi, \pi]$.

6. Find and classify the critical points of the following functions. Hence find their inflection points.

(a) $f(x) = x^5 - 5x + 5$ (b) $g(x) = xe^{-x}$.

7. Find the range of $f(x) = x - e^x$.

8. Find the minimum value of the sum of a positive number and its reciprocal.

9. A rectangular box has a square base of side x cm. If the sum of one side of the square and height is 15 cm, express the volume of the box in terms of x. Use this expression to determine the maximum volume of the box.

10. A rectangular field is surrounded by a fence on three of its sides and a straight hedge on the fourth side. If the length of the fence is 320 meters, find the maximum area of the field enclosed.

11. If a piece of string of fixed length is made to enclose a rectangle, show that the enclosed area is the greatest when the rectangle is a square.

12. Two sides of a triangle have lengths x and y, and the angle between them is θ. What value of θ will maximize the triangle's area?

Chapter 10

METHODS OF INTEGRATION

In this chapter, we will show you how to use integration as the reverse process of differentiation. Integrating the fundamental functions, trigonometric, exponential and logarithmic functions.

10.1 Antiderivatives

Consider $f'(x) = 3x^2$.

The natural question arises, what is the function f (x) ? Namely, what is f in terms of x?

We know that differentiation decreases the power by 1, so f must contain x^3.

If $f(x) = x^3$, then $f'(x) = 3x^2$, or

if $f(x) = x^3 + 2$, then $f'(x) = 3x^2$, or

if $f(x) = x^3 - \frac{1}{2}$, then $f'(x) = 3x^2$,etc.

It means that there are many such functions of the form

$$f(x) = x^3 + C$$

where C is an arbitrary constant.

We say that x^3 is the **antiderivative** of $3x^2$.

Next consider $f'(x) = x$.

We think the same way as above, the original function f must contain x^2. However,

$$\frac{d}{dx}x^2 = 2x,$$

we see the extra factor 2. If we multiply both sides by $\frac{1}{2}$, then

$$\frac{d}{dx}\left(\frac{1}{2}x^2\right) = x.$$

If $f(x) = \frac{1}{2}x^2 + C$, where C is an arbitrary constant, then $f'(x) = x$, so $\frac{1}{2}x^2$ is the **antiderivative** of x.

If F(x) is a function where $F'(x) = f(x)$, then the antiderivative of f (x) is F(x).

We call the antiderivative as **integral** and write

$$\int x \, dx = \frac{1}{2}x^2 + C,$$

where C is the constant of integration.

We read this as "the integral of x with respect to x."

In general,

$$\text{if } F'(x) = f(x) \text{ then } \int f(x)\ dx = F(x) + C$$

where dx means that the integration is taking place with respect to the variable x. Here f (x) is called the **integrand**. The variable of integration in an integral plays no essential role. It might be x, or t, or u, or anything else:

$$\int f(x)\ dx,\quad \int f(t)\ dt\ ,\quad \int f(u)\ du, \text{ etc.}$$

Example 1

Find the antiderivative of

(a) $\sqrt{x}$ (b) $3x^5$ (c) e^{3x} (d) $\frac{1}{x^3}$.

Solution

(a) Since $\frac{d}{dx}x^{\frac{3}{2}} = \frac{3}{2}x^{\frac{1}{2}}$ and $\frac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}}\right) = x^{\frac{1}{2}} = \sqrt{x}$,

the antiderivative of $\sqrt{x}$ is $\frac{2}{3}x^{\frac{3}{2}}$.

(b) Since $\frac{d}{dx}x^6 = 6x^5$ and $\frac{d}{dx}\left(\frac{1}{2}x^6\right) = 3x^5$,

the antiderivative of $3x^5$ is $\frac{1}{2}x^6$.

(c) Since $\frac{d}{dx}e^{3x} = 3e^{3x}$ and $\frac{d}{dx}\left(\frac{1}{3}e^{3x}\right) = e^{3x}$,

the antiderivative of e^{3x} is $\frac{1}{3}e^{3x}$.

(d) Since $\frac{d}{dx}x^{-2} = -2x^{-3}$ and $\frac{d}{dx}\left(-\frac{1}{2x^2}\right) = \frac{1}{x^3}$,

the antiderivative of $\frac{1}{x^3}$ is $-\frac{1}{2x^2}$

We now describe the integration of fundamental functions. We know that

$$\frac{d}{dx}x^{n+1} = (n+1)\ x^n.$$

The reverse of this process is

$$\int x^n\ dx = \frac{1}{n+1}\ x^{n+1} + C \quad \text{for } n \neq -1.$$

When n = 0, it is seen that

$$\int x^0\ dx = \int 1\ dx = \int\ dx = x + C$$

$$\int\ dx = x + C.$$

When n = −1, let us consider the differentiation of ln x.

Since $\frac{d}{dx}\ln\ x = \frac{1}{x}$, it follows that $\int \frac{1}{x}\ dx = \ln\ x + C$, for $x > 0$.

For $x < 0$, $\frac{d}{dx}\ln\ |x| = \frac{d}{dx}\ln\ (-x) = \frac{1}{-x}(-1) = \frac{1}{x}$, by Chain Rule.

$$\int \frac{1}{x}\ dx = \ln\ |x| + C,\ x \neq 0.$$

Rules of Integration

Suppose $f(x)$ and $g(x)$ are continuous functions and $k \in \mathbb{R}$.

1. $\int k\ dx = kx + C$
2. $\int k\ f(x)\ dx = k \int f(x)\ dx$
3. $\int [f(x) \pm g(x)]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$.

Example 2

Evaluate each of the following integrals.

(a) $\int (4x^5 + 1)\ dx$ (b) $\int \left(2x^6 - \frac{1}{3}x^3 + \frac{3}{x} \right) dx$ (c) $\int 5x\sqrt{x}\ dx$ (d) $\int \frac{(x-1)^2}{\sqrt{x}}\ dx$.

Solution

(a) $\int (4x^5 + 1)\ dx = 4 \int x^5\ dx + \int 1\ dx$

$= \left(4\frac{x^6}{6} + C_1\right) + (x + C_2)$; C_1 and C_2 are constants of integration

$= \left(\frac{2}{3}x^6 + x\right) + C_1 + C_2$

$= \left(\frac{2}{3}x^6 + x\right) + C$, $C = C_1+C_2$ is another constants of integration.

From now on, we shall write constant of integration only in the answer.

(b) $\int \left(2x^6 - \frac{1}{3}x^3 + \frac{3}{x} \right) dx = 2\int x^6 dx - \frac{1}{3}\int x^3\ dx + 3\int \frac{1}{x}\ dx$

$= \frac{2}{7}x^7 - \frac{1}{12}x^4 + 3\ \ln\ |x| + C.$

(c) $\int 5x\sqrt{x}\ dx = 5 \int x^{\frac{3}{2}}\ dx$

$= 5\left(\frac{2}{5}x^{\frac{5}{2}}\right) + C$

$= 2x^{\frac{5}{2}} + C.$

(d) $\int \frac{(x-1)^2}{\sqrt{x}}\ dx = \int \frac{x^2-2x+1}{x^{\frac{1}{2}}}\ dx$

$= \int \left(\frac{x^2}{x^{\frac{1}{2}}} - \frac{2x}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}}\right) dx$

$= \int x^{\frac{3}{2}}\ dx - 2\int x^{\frac{1}{2}}\ dx + \int x^{-\frac{1}{2}}\ dx$

$= \frac{2}{5}x^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C.$

Integrating Exponential Functions

$\frac{d}{dx}a^x = a^x \ln a, \quad \int a^x\,dx = \frac{1}{\ln a}a^x + C, \; a > 0, \; a \neq 1.$

When $a = e$, $\quad \frac{d}{dx}e^x = e^x, \quad \int e^x\,dx = e^x + C.$

Example 3

Find the following integrals.

(a) $\int \frac{e^{-x}+1}{e^{-x}}\,dx$ (b) $\int (e^x + 2x)\,dx$ (c) $\int \left(e^x \log 5 + \frac{1}{x^2}\right)dx$ (d) $\int 3^x \ln 3\,dx$.

Solution

(a) $\int \frac{e^{-x}+1}{e^{-x}}\,dx = \int (1 + e^x)\,dx = \int 1\,dx + \int e^x\,dx = x + e^x + C.$

(b) $\int (e^x + 2x)\,dx = \int e^x\,dx + 2\int x\,dx = e^x + x^2 + C.$

(c) $\int \left(e^x \log 5 + \frac{1}{x^2}\right)dx = \int e^x \log 5\,dx + \int x^{-2}\,dx = e^x \log 5 - \frac{1}{x} + C.$

(d) $\int 3^x \ln 3\,dx = \frac{1}{\ln 3}3^x \ln 3 + C = 3^x + C$.

Integrating Trigonometric Functions

1. $\frac{d}{dx}\sin x = \cos x, \; \int \cos x\,dx = \sin x + C.$
2. $\frac{d}{dx}(-\cos x) = -(-\sin x) = \sin x, \; \int \sin x\,dx = -\cos x + C$.
3. $\frac{d}{dx}\tan x = \sec^2 x, \; \int \sec^2 x\,dx = \tan x + C.$

Example 4

Evaluate.

(a) $\int (3\cos x - 5\sin x)\,dx$ (b) $\int (e^x + 2\sin x)\,dx$

(c) $\int \frac{e^x - \sqrt{x}}{2}\,dx$ (d) $\int (1 + \tan^2 x)\,dx$

Solution

(a) $\int (3\cos x - 5\sin x)\,dx = 3\int \cos x\,dx - 5\int \sin x\,dx = 3\sin x + 5\cos x + C.$

(b) $\int (e^x + 2\sin x)\,dx = \int e^x\,dx + 2\int \sin x\,dx = e^x - 2\cos x + C.$

(c) $\int \frac{e^x - \sqrt{x}}{2}\,dx = \frac{1}{2}\int e^x\,dx - \frac{1}{2}\int x^{\frac{1}{2}}\,dx = \frac{1}{2}e^x - \frac{1}{3}x^{\frac{3}{2}} + C.$

(d) $\int (1 + \tan^2 x)\,dx = \int \sec^2 x\,dx = \tan x + C.$

Integrating f (ax + b)

We have learned the reverse process of differentiation as an integration. Now we consider the integral of function which used the chain rule for differentiation. Consider the integral $\int (4x+1)^5\,dx$.

By the Chain rule, $\frac{d}{dx}(4x+1)^6 = 6(4x+1)^5\frac{d}{dx}(4x+1) = 24(4x+1)^5$.

Multiplying by $\frac{1}{24}$ on both side, we get

$$\frac{1}{24}\frac{d}{dx}(4x+1)^6 = (4x+1)^5$$

$$\frac{d}{dx}\left(\frac{1}{24}(4x+1)^6\right) = (4x+1)^5$$

Therefore $\int (4x+1)^5\,dx = \frac{1}{24}(4x+1)^6 + C.$

For $n \neq -1$, $\frac{d}{dx}\left(\frac{1}{a\,(n+1)}(a\,x+b)^{n+1}\right) = (a\,x+b)^n$

$$\int (a\,x+b)^n\,dx = \frac{1}{a}\frac{(a\,x+b)^{n+1}}{(n+1)} + C, \quad n \neq 1.$$

When $n = 1$, $\frac{d}{dx}\left(\frac{1}{a}\ln\,(a\,x+b)\right) = \frac{1}{a}\left(\frac{a}{a\,x+b}\right) = \frac{1}{a\,x+b}$, $a \neq 0$.

$$\int \frac{1}{a\,x+b}\,dx = \frac{1}{a}\ln|a\,x+b| + C.$$

In general, if f is a differentiable function of x, then

$$\frac{d}{dx}f(a\,x+b) = f'(a\,x+b)\frac{d}{dx}(a\,x+b) = a\,f'(a\,x+b)$$

and reversing this we get

$$\int f'(a\,x+b)\,dx = \frac{1}{a}f\,(a\,x+b) + C.$$

For the trigonometric functions of the form f (ax + b),

$$\int \cos\,(a\,x+b)\,dx = \frac{1}{a}\sin\,(a\,x+b) + C.$$

$$\int \sin\,(a\,x+b)\,dx = -\frac{1}{a}\cos\,(a\,x+b) + C.$$

$$\int \sec^2\,(a\,x+b)\,dx = \frac{1}{a}\tan\,(a\,x+b) + C.$$

For the exponential functions of the form f (px + q),

$$\int a^{px+q}\,dx = \frac{1}{p}\frac{1}{\ln a}a^{px+q} + C, \ a > 0, \ a \neq 1.$$

When $a = e$,

$$\int e^{px+q}\,dx = \frac{1}{p}e^{px+q} + C.$$

Example 5

Evaluate the following integrals.

(a) $\int \frac{1}{3-4x}\ dx$ (b) $\int \sqrt{8x-7}\ dx$ (c) $\int e^{2x+1}\ dx$

(d) $\int 2^{3x+1}\ dx$ (e) $\int \cos^2 x\ dx$ (f) $\int \sin 4x \cos 3x\ dx$

Solution

(a) $\int \frac{1}{3-4x}\ dx = \frac{1}{4} \ln |3-4x| + C.$

(b) $\int \sqrt{8x-7}\ dx = \frac{1}{8}\left[\frac{2}{3}(8x-7)^{\frac{3}{2}}\right] + C = \frac{1}{12}(8x-7)^{\frac{3}{2}} + C.$

(c) $\int e^{2x+1}\ dx = \frac{1}{2} e^{2x+1} + C.$

(d) $\int 2^{3x+1}\ dx = \frac{1}{3}\frac{1}{\ln 2} 2^{3x+1} + C.$

(e) We use $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, then

$$\int \cos^2 x\ dx = \frac{1}{2}\int (1+\cos 2x)\ dx$$
$$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$
$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

(f) We use $\sin 4x \cos 3x = \frac{1}{2}(\sin 7x + \sin x)$

$$\int \sin 4x \cos 3x\ dx = \frac{1}{2}\int (\sin 7x + \sin x)\ dx$$
$$= \frac{1}{2}\left(-\frac{1}{7}\cos 7x - \cos x\right) + C$$
$$= -\frac{1}{14}\cos 7x - \frac{1}{2}\cos x + C.$$

Exercise 10.1

1.Evaluate the following integrals:

(a) $\int 4x^8\ dx$ (b) $\int \frac{3}{2} x^2 \sqrt[3]{x}\ dx$ (c) $\int (5^x + 2)\ dx$

(d) $\int \sin^2 x\ dx$ (e) $\int \frac{x+3}{\sqrt{x}}\ dx$ (f) $\int \left(\frac{1}{2x} + 5\right) dx$

(g) $\int \left(e^x + \frac{2}{x}\right) dx$ (h) $\int \left(\frac{1}{x^5} + 4e^x\right) dx$ (i) $\int \left(\frac{3}{x} + e^x + 10\right) dx$

(j) $\int \sin^2 3x\ dx$ (k) $\int \sin 5x \sin 2x\ dx$ (l) $\int \cos 7x \cos 4x\ dx$

2. Evaluate the following integrals:

(a) $\int (1-2x)^3\ dx$ (b) $\int \sin\ (2\pi x+7)\ dx$ (c) $\int \cos(3x-7)\ dx$

(d) $\int 3^{5x-2}\ dx$ (e) $\int \frac{1}{7x-6}\ dx$ (f) $\int \frac{\sin\ 2x}{\sin\ x}\ dx$

(g) $\int \sec^2(2x+3)\ dx$ (h) $\int e^{7x-3}\ dx$ (i) $\int (1+\tan^2 2\pi x)\ dx$

10.2 Substitution Method

We use the substitution method when the given integral can be transformed to the simpler integral by a change of variable. We describe how to get the integrals $\int f(g(x))g'(x)\ dx$and $\int \frac{g'(x)}{g(x)}\ dx$ where g is differentiable and f is continuous. Then, we apply this method to the integral containing the trigonometric functions.

We recall the idea of differentials from article 9.2 in previous chapter.

Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable, the differential dy is given by

$$dy = f'(x)\ dx$$

Here, dy is a dependent variable that depends on x and dx.

1. $\int \mathbf{f(g(x))g'(x)\ dx}$

Suppose the function g is differentiable and f is continuous.

Let $u = g(x)$. Then $du = g'(x)\ dx$.

Substitute these into $\int f(g(x))g'(x)\ dx$, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

To compute the integral by substitution method, the basic steps are as follows:

Step 1 Select a substitution $u = g(x)$.

Step 2 Differentiate the substitution and arrange to write dx in terms of du .

Step 3 Substitute the expression from step 2, and transform the entire integral from x-variable to u-variable form.

Step 4 Integrate with respect to u.

Step 5 Rewrite the answer in terms of x.

Example 6

Evaluate each of the following integrals.

(a) $\int x\, e^{x^2}\, dx$ (b) $\int \sin^4 x \cos x\, dx$ (c) $\int x^2\left(\frac{x^3}{3}+1\right)^3 dx$

(d) $\int \sqrt{1-5x}\, dx$ (e) $\int 3x\sqrt{x^2+5}\, dx$ (f) $\int \cos^3 x\, dx$

Solution

(a) $\int x\, e^{x^2}\, dx$

Let $u = x^2$. Then $du = 2x\, dx$, $\frac{1}{2}du = x\, dx$.

$\int x\, e^{x^2}\, dx = \int e^{x^2}\, x\, dx = \int e^u \frac{1}{2}\, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$

(b) $\int \sin^4 x \cos x\, dx$

Let $u = \sin x$. Then, $du = \cos x\, dx$.

$\int \sin^4 x \cos x\, dx = \int u^4\, du = \frac{u^5}{5} + C = \frac{\sin^5 x}{5} + C.$

(c) $\int x^2\left(\frac{x^3}{3}+1\right)^3 dx$

Let $u = \frac{x^3}{3} + 1$. Then $du = x^2 dx$.

$\int x^2\left(\frac{x^3}{3}+1\right)^3 dx = \int u^3 du = \frac{u^4}{4} + C = \frac{1}{4}\left(\frac{x^3}{3}+1\right)^4 + C.$

(d) $\int \sqrt{1-5x}\, dx$

Let $u = 1 - 5x$. Then $du = -5\, dx$, $-\frac{1}{5}\, du = dx$.

$\int \sqrt{1-5x}\, dx = \int u^{\frac{1}{2}} \left(-\frac{1}{5}\right) du = -\frac{1}{5}\ \frac{2}{3}\ u^{\frac{3}{2}} + C = -\frac{2}{15}\ (1-5x)^{\frac{3}{2}} + C.$

(e) $\int 3x\sqrt{x^2+5}\, dx$

Let $u = x^2 + 5$. Then $\frac{1}{2}du = x\, dx$.

$\int 3x\sqrt{x^2+5}\, dx = 3\int u^{\frac{1}{2}}\, \frac{1}{2}du = \frac{3}{2}\ \frac{2}{3}\ u^{\frac{3}{2}} + C = (x^2+5)^{\frac{3}{2}} + C.$

(f) $\int \cos^3 x\, dx$

We use $\cos^2 x = 1 - \sin^2 x$,then

$\int \cos^3 x\, dx = \int (1-\sin^2 x) \cos x\, dx.$

Let $u = \sin x$, then $du = \text{cox}\ x\, dx$

$\int \cos^3 x\, dx = \int (1-u^2)\, du = u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C.$

2. $\int \frac{g'(x)}{g(x)}\,dx$

Let u = g(x). Then $du = g'(x)\,dx$.

Thus, $$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C.$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + C.$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C.$$

Example 7

Evaluate the following integrals.

(a) $\int \frac{2x-1}{x^2-x-6}\,dx$ (b) $\int \frac{x^2+2x-1}{x^2-1}\,dx$ (c) $\int \frac{1}{1+e^x}\,dx$

Solution

(a) $\int \frac{2x-1}{x^2-x-6}\,dx$

Let $u = x^2 - x - 6$. Then $du = (2x-1)\,dx$.

$$\int \frac{2x-1}{x^2-x-6}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|x^2 - x - 6| + C.$$

(b) $\int \frac{x^2+2x-1}{x^2-1}\,dx$

We can write as $\int \frac{x^2+2x-1}{x^2-1}\,dx = \int \left(1 + \frac{2x}{x^2-1}\right)\,dx$

Let $u = x^2 - 1$. Then $du = 2x\,dx$.

$\int \frac{x^2+2x-1}{x^2-1}\,dx = \int \left(1 + \frac{2x}{x^2-1}\right)\,dx = \int dx + \int \frac{1}{u}\,du = x + \ln|u| + C = x + \ln|x^2 - 1| + C.$

(c) $\int \frac{1}{1+e^x}\,dx$

We can rewrite by multiplying and dividing by e^{-x}.

$$\int \frac{1}{1+e^x}\,dx = \int \frac{1}{1+e^x}\,\frac{e^{-x}}{e^{-x}}\,dx = \int \frac{e^{-x}}{e^{-x}+1}\,dx.$$

Let $u = e^{-x} + 1$. Then $du = -e^{-x}dx$.

$$\int \frac{1}{1+e^x}\,dx = \int \frac{e^{-x}}{e^{-x}+1}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln(e^{-x} + 1) + C.$$

We now explain some integrals involving the trigonometric functions using method of substitution.

(a) $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

Proof.

We have $\int \sec x \, dx = \int \sec x \frac{\sec x+\tan x}{\sec x+\tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x+\tan x} dx.$

Let $u = \sec x + \tan x$. Then $du = (\sec x \tan x + \sec^2 x) \, dx.$

So, $\int \sec x \, dx = \int \frac{1}{u} du = \ln|u| + C = \ln|\sec x + \tan x| + C.$

(b) $\int \csc x \, dx = -\ln|\csc x + \cot x| + C$

Proof.

We have $\int \csc x \, dx = \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x} dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx$

Let $u = \csc x + \cot x$. Then $du = -(\csc x \cot x + \csc^2 x) \, dx.$

So, $\int \csc x \, dx = -\int \frac{1}{u} du = -\ln|u| + C = -\ln|\csc x + \cot x| + C.$

Exercise 10.2

1.Integrate the following functions using the given substitutions.

(a) $4x^3\sqrt{x^4 - 1}$; $u = x^4 - 1$

(b) $\cos^3 x \sin x$; $u = \cos x$

(c) $\frac{1}{x \ln|x|}$; $u = \ln|x|$

(d) $\sin^5 x \cos x$; $u = \sin x$

(e) $\frac{\ln x}{x}$, $x > 0$; $u = \ln x$

(f) $x^3 e^{x^4}$; $u = x^4$

2.Using the substitution method to evaluate the following integrals.

(a) $\int x\sqrt{1-x} \, dx$

(b) $\int (2x + 1)(x^2 + x)^7 \, dx$

(c) $\int \sin^3 x \, dx$

(d) $\int x^2 \sqrt{x^3 - 2} \, dx$

(e) $\int \frac{\sec^2 x}{\tan x} dx$

(f) $\int \frac{x}{\sqrt{x+1}} dx$

3.Evaluate the integral $\int \frac{x}{(x^2+1) \ln (x^2+1)} dx.$

10.3 Integration by Parts

We use the method of integration by parts to integrate the product of two functions. In Section 10.1, we explain that the integral of the sum of functions is the sum of respective integrals. But the integral of the product function is not the product of respective integrals. Therefore, we use another technique and it is called the integration by parts.

It is based on product rule of differentiation,

$$(u\,v)' = u'\,v + u\,v'$$

where u and v are functions of x. Then integrating both sides, we get

$$\int (u\,v)'\,dx = \int u'\,v\,dx + \int u\,v'\,dx$$

Applying the antiderivative in left hand side, we get

$$u\,v = \int u'\,v\,dx + \int u\,v'\,dx$$

or

$$\int u\,v'\,dx = u\,v - \int u'\,v\,dx$$

Since $u'\,dx = du$ and $v'\,dx = dv$, we get

$$\int u\,dv = u\,v - \int v\,du.$$

To compute the integral of product of two functions using integration by parts, the basic steps are as follows:

Step 1 Choose u and dv.

Step 2 Differentiate u and integrate dv.

Step 3 Substitute the expression from step 2 in

$$\int u\,dv = u\,v - \int v\,du.$$

Step 4 Simplify.

Example 8

Evaluate the following integrals.

(a) $\int x\,e^x\,dx$ (b) $\int \ln x\,dx$ (c) $\int x\cos x\,dx$

(d) $\int x\sin x\,dx$ (e) $\int e^x\sin x\,dx$ (f) $\int x^2\ln x\,dx$

Solution

(a) $\int x\,e^x\,dx$

Let $u = x$, $dv = e^x\,dx$

Then $du = dx$, $v = \int e^x\,dx = e^x$.

$\int x\,e^x\,dx = x\,e^x - \int e^x\,dx = x\,e^x - e^x + C.$

(b) $\int \ln x \, dx$

Let $u = \ln x, \quad dv = dx$

Then $du = \frac{1}{x} dx, \quad v = \int dx = x.$

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} dx + C = x \ln x - x + C.$$

(c) $\int x \cos x \, dx$

Let $u = x, \quad dv = \cos x \, dx$

Then $du = dx, \quad v = \int \cos x \, dx = \sin x.$

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

(d) $\int x \sin x \, dx$

Let $u = x, \quad dv = \sin x \, dx$

Then $du = dx, \quad v = \int \sin x \, dx = -\cos x.$

$$\int x \sin x \, dx = -x \cos x - \int (-\cos x) \, dx = -x \cos x + \sin x + C.$$

(e) $\int e^x \sin x \, dx$

Let $u = e^x, \quad dv = \sin x \, dx$

Then $du = e^x dx, \quad v = \int \sin x \, dx = -\cos x$

$$\int e^x \sin x \, dx = -e^x \cos x - \int (-\cos x) e^x \, dx$$

$$= -e^x \cos x + \int e^x \cos x \, dx.$$

Let $I = \int e^x \sin x \, dx$. Then we have

$$I = -e^x \cos x + \int e^x \cos x \, dx. \quad (1)$$

Let $u = e^x, \quad dv = \cos x \, dx$

Then $du = e^x dx, \quad v = \int \cos x \, dx = \sin x.$

Thus

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx. \quad (2)$$

Substituting (2) into (1), we get

$$I = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$= e^x (\sin x - \cos x) - I$$

$$I = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

Therefore

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

(f) $\int x^2 \ln x\, dx$

Let $u = \ln x$, $dv = x^2 dx$

Then $du = \frac{1}{x} dx$, $v = \int x^2 dx = \frac{x^3}{3}$.

$\int x^2 \ln x\, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \frac{1}{x} dx = \frac{x^3}{3} \ln x - \frac{1}{3} \frac{x^3}{3} + C = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C.$

Exercise 10.3

1. Use the integration by parts to evaluate the following integrals.
   (a) $\int s\, e^{-2s} ds$ (b) $\int \ln (x+1)\, dx$ (c) $\int t \sin 2t\, dt$
   (d) $\int x\, 2^x\, dx$ (e) $\int x \cos 5x\, dx$ (f) $\int e^x \cos x\, dx$

10.4 Partial Fraction Method

We use the partial fraction method to integrate the rational functions.

Example 9

Evaluate the integral $\int \frac{1}{x^2+2x-3} dx$.

Solution

$$\int \frac{1}{x^2+2x-3} dx$$

We rewrite the rational fraction as

$$\frac{1}{x^2+2x-3} = \frac{1}{(x+3)(x-1)}.$$

However, we want this fraction as the sum of linear fractions as below.

$$\frac{1}{x^2+2x-3} = \frac{A}{x+3} + \frac{B}{x-1}$$

where A and B are constants that will be determined later. Then, it can be rewritten as

$$\frac{1}{x^2+2x-3} = \frac{A(x-1)+B(x+3)}{(x+3)(x-1)}.$$

Therefore,

$$1 = A(x-1) + B(x+3)$$

$$1 = (A+B)\, x + (-A+3B).$$

Equating the coefficients of corresponding powers of x, we get

$$A + B = 0$$

$$-A + 3b = 1$$

which give $A = -\frac{1}{4}$ and $B = \frac{1}{4}$.

Therefore

$$\frac{1}{x^2+2x-3} = \frac{-\frac{1}{4}}{x+3} + \frac{\frac{1}{4}}{x-1}$$

Thus

$$\int \frac{1}{x^2+2x-3}\, dx = -\frac{1}{4}\int \frac{1}{x+3}\, dx + \frac{1}{4}\int \frac{1}{x-1}\, dx$$

$$= -\frac{1}{4}\ln|x+3| + \frac{1}{4}\ln|x-1| + C$$

$$= \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C.$$

Example 10

Evaluate the integral $\int \frac{1}{2x^2+3x+1}\, dx$.

Solution

$$\int \frac{1}{2x^2+3x+1}\, dx$$

First we write $\frac{1}{2x^2+3x+1} = \frac{1}{(2x+1)\,(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}$

and so $\frac{1}{2x^2+3x+1} = \frac{A(x+1)+B(2x+1)}{(2x+1)\,(x+1)}$.

Therefore, $1 = A(x+1) + B(2x+1)$.

When $x = -1$, $1 = B(-1)$, so $B = -1$.

When $x = -\frac{1}{2}$, $1 = A(\frac{1}{2})$, so $A = 2$.

$$\int \frac{1}{2x^2+3x+1}\, dx = \int \frac{2}{2x+1}\, dx + \int \frac{-1}{x+1}\, dx$$

$$= \ln|2x+1| - \ln|x+1| + C$$

$$= \ln\left|\frac{2x+1}{x+1}\right| + C$$

Example 11

Evaluate the integral $\int \frac{x^3+x^2-4x}{x^2-4}\, dx$.

Solution

$$\int \frac{x^3+x^2-4x}{x^2-4}\, dx$$

We can write as

$$\int \frac{x^3+x^2-4x}{x^2-4}\, dx = \int \left(x+1+\frac{4}{x^2-4}\right) dx.$$

First we write $\frac{4}{x^2-4} = \frac{4}{(x+2)\ (x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$

and so $\frac{4}{x^2-4} = \frac{A(x-2)+B(x+2)}{(x+2)\ (x-2)}$

Therefore, $4 = A(x-2) + B(x+2)$.

When $x = 2$, $4 = A(4)$, so $B = 1$.

When $x = -2$, $4 = A(-4)$, so $A = -1$.

$$\int \frac{x^3+x^2-4x}{x^2-4}\, dx = \int (x+1)\, dx + \int \frac{-1}{x+2}\, dx + \int \frac{1}{x-2}\, dx$$

$$= \frac{x^2}{2} + x - \ln\ |x+2| + \ln|x-2| + C$$

$$= \frac{x^2}{2} + x + \ln\ \left|\frac{x-2}{x+2}\right| + C.$$

Example 12

Evaluate the integral $\int \frac{2x+1}{(x-1)^2}\, dx$.

Solution

$$\int \frac{2x+1}{(x-1)^2}\, dx$$

We can rewrite as $\frac{2x+1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$

and so $\frac{2x+1}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2}$.

Therefore, $2x+1 = A(x-1) + B = Ax + (-A+B)$.

Equating the coefficients of corresponding powers of x, we get

$A = 2$ and $-A + B = 1$

which give $B = 3$.

Therefore $\frac{2x+1}{(x-1)^2} = \frac{2}{x-1} + \frac{3}{(x-1)^2}$.

Thus

$$\int \frac{2x+1}{(x-1)^2}\, dx = \int \frac{2}{x-1}\, dx + \int \frac{3}{(x-1)^2}\, dx$$

$$= 2\ln|x-1| - \frac{3}{x-1} + C.$$

Example 13

Find the function f (x) satisfying the equation

$$f'(x) = \frac{x-1}{\sqrt{x}} \qquad \text{with} \quad f(1) = 0.$$

Solution

$$f'(x) = \frac{x-1}{\sqrt{x}}$$

By integrating we get

$$f(x) = \int \frac{x-1}{\sqrt{x}}\, dx = \int \left(x^{\frac{1}{2}} - x^{\frac{-1}{2}}\right) dx$$

$$= \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + C.$$

Since f (1) = 0, it follows that

$$f(1) = \frac{2}{3}(1)^{\frac{3}{2}} - 2(1)^{\frac{1}{2}} + C$$

$$0 = \frac{2}{3} - 2 + C$$

$$C = \frac{4}{3}$$

Therefore $f(x) = \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + \frac{4}{3}$.

Example 14

Find the function f (x) satisfying the equation

$$f''(x) = \frac{e^x - e^{-x}}{2}, \text{with } f'(0) = 1 \text{ and } f(0) = 1.$$

Solution

$$f''(x) = \frac{e^x - e^{-x}}{2}$$

By integrating we get

$$f'(x) = \int \frac{e^x - e^{-x}}{2}\, dx = \frac{1}{2}(e^x + e^{-x}) + C.$$

Since $f'(0) = 1$, it follows that

$$f'(0) = \frac{1}{2}(e^0 + e^0) + C$$

$$1 = 1 + C$$

$$C = 0.$$

Therefore, $f'(x) = \frac{1}{2}(e^x + e^{-x})$.

Again, by integrating we get $f(x) = \int \frac{e^x + e^{-x}}{2}\, dx = \frac{1}{2}(e^x - e^{-x}) + C.$

Since $f(0) = 1$, it follows that

$$f(0) = \frac{1}{2}(e^0 - e^0) + C$$

$$1 = 0 + C$$

$$C = 1.$$

Therefore, $f(x) = \frac{1}{2}(e^x - e^{-x}) + 1$.

Exercise 10.4

1. Use the partial fractions method to evaluate the following integrals.

   (a) $\int \frac{1}{2x^2+5x+3}\,dx$ (b) $\int \frac{2x-1}{(x-3)^2}\,dx$

   (c) $\int \frac{x+1}{(2x+5)(x+4)}\,dx$ (d) $\int \frac{2x^2-1}{x^2-1}\,dx$

2. Find the function f (x) that satisfying the equation

   $$f'(x) = \sin 4x \cdot \cos 2x \text{ with } f(\tfrac{\pi}{2}) = 0.$$

3. Find the function g(x) that satisfying the equation

   $$g'(x) = x^2e^{x^3} \text{ with } g(0) = -\frac{2}{3}.$$

4. Find the function h(x) that satisfying the equation

   $$h'(x) = \frac{x}{x^2-1} \text{ with } h(2) = \frac{1}{2}.$$

5. Find the function f(x) that satisfying the equation

   $$f''(x) = 2x - 1 \text{ with } f(0) = -1 \text{ and } f'(1) = 2.$$

6. Find the function g(x) that satisfying the equation

   $$g''(x) = x\sin x \text{ with } g(\tfrac{\pi}{2}) = 0 \text{ and } g'(0) = 0.$$

Chapter 11

APPLICATION OF INTEGRALS

In this chapter, we discuss application of integrals into two parts such that application of indefinite integral and application of definite integral.

11.1 Indefinite Integral

- A function F is an antiderivative of a function f if $F'(x) = f(x)$.
- If F and G are both antiderivatives of f, they differ by a constant: $F(x) = G(x) + K$ for some constant K.
- We use the symbol $\int f(x)dx$, called an indefinite integral, to represent the family of all antiderivatives of f, and we write

$$\int f(x)dx = F(x) + C \text{ where C is a constant.}$$

Theorems of Linearity

1. If f and g are continuous functions then

$$\int \big(f(x) \pm g(x)\big)dx = \int f(x)dx \pm \int g(x)dx.$$

2. If f is a continuous function and K is a constant then

$$\int Kf(x)dx = K\int f(x)dx.$$

We consider the indefinite integral to find the equation of a curve, displacement and velocity.

11.2 Application of Indefinite Integral

(i) Finding the equation of a curve given the gradient of the tangent to the curve

Example 1

Find the equation of the curve that passes through the point $(-2, 1)$ given that the gradient of the tangent to the curve is $-11x^2$.

Solution

We are told that gradient of the tangent to a curve is equal to $-11x^2$, but we also know that the gradient of a tangent line at a value of x must be equal to $\frac{dy}{dx}$ (the rate of change of y with respect to x).

Therefore, $\frac{dy}{dx} = -11x^2$.

We want to find an expression for our curve y. Since the derivative of y is $-11x^2$,we must also have that the antiderivative of $-11x^2$ is equal to y.

We can try to find this antiderivative by using indefinite integration:

$y = \int -11x^2\,dx$

$= \frac{-11x^3}{3} + C$ ---------------- (1)

So, we have found a general solution for our curve, which includes the constant of integration C. To find the value of C, we can use the fact that the curve passes through the point (−2, 1).

Therefore, putting $= -2$ and $y = 1$ in equation (1), $1 = \frac{-11(-2)^3}{3} + C$.

$C = -\frac{85}{3}$.

The equation of the curve, $y = -\frac{11x^3}{3} - \frac{85}{3}$.

Example 2

Find the equation of a curve passing through the point (−2, 3), given that the slope of the tangent to the curve at any point (x, y) is $\frac{2x}{y^2}$.

Solution

Slope of tangent $= \frac{dy}{dx}$

Thus $\frac{dy}{dx} = \frac{2x}{y^2}$

$y^2 dy = 2x\,dx$

Integrating both sides

$\int y^2 dy = \int 2x\,dx$

$\frac{y^3}{3} = x^2 + C$

$y^3 = 3x^2 + 3C$ ------------- (1)

Given that curve passes through (−2, 3).

Putting $x = -2, y = 3$ in equation (1),

$3^3 = 3(-2)^2 + 3C$

$27 = 12 + 3C$

$C = 5$.

Putting C in equation (1),

$y^3 = 3x^2 + 15$.

The equation of curve, $y = (3x^2 + 15)^{\frac{1}{3}}$.

(ii) Displacement from Velocity and Velocity from Acceleration

We can find an expression for velocity by differentiating the expression for displacement:

$$v = \frac{ds}{dt}.$$

Similarly, we can find the expression for the acceleration by differentiating the expression for velocity, and this is equivalent to finding the second derivative of the displacement:

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

It follows (since integration is the opposite process to differentiation) that to obtain the displacement, s of an object at time t (given the expression for velocity, v) we would use:

$$s = \int v\,dt .$$

Similarly, the velocity of an object at time t with acceleration a, is given by:

$$v = \int a\,dt.$$

Example 3

A car starts from rest at $s = 3$ m from the origin and has acceleration at time t given by $a = 2t - 5\ ms^{-2}$. Find the velocity and displacement of the car at $t = 4$ s.

Solution

We find the velocity using:

$$v = \int a\,dt.$$

So in this example we have:

$$v = \int (2t - 5)dt$$
$$= t^2 - 5t + C \quad \text{................(1)}$$

Putting $t = 0$, $v = 0$ in equation (1), so $C = 0$.

So the expression for velocity as a function of time is:

$$v = t^2 - 5t\ ms^{-1}.$$

When $t = 4$ second,

$$v = 4^2 - 5(4) = -4\ ms^{-1}.$$

Now to find the displacement.

$$s = \int v\,dt.$$

Then

$$s = \int (t^2 - 5t)\,dt$$

$$= \frac{t^3}{3} - \frac{5t^2}{2} + K.$$

Now when $t = 0$, $s = 3$, so we substitute to obtain:

$$3 = \frac{0^3}{3} - \frac{5(0)t^2}{2} + K .$$

So, $K = 3$ and therefore the general expression for s is:

$$s = \frac{t^3}{3} - \frac{5t^2}{2} + 3 .$$

When $t = 4$ seconds, $s = \frac{4^3}{3} - \frac{5(4)^2}{2} + 3 = -15.67$ m.

Example 4

A proton moves in an electric field such that its acceleration (in cms^{-2}) is $a = -20\,(1 + 2t)^{-2}$, where t is in seconds. Find the velocity as a function of time if $v = 30\ cms^{-1}$ when $t = 0$.

Solution

$$v = \int a\, dt.$$

So

$$v = \int \frac{-20}{(1+2t)^2}\, dt.$$

Put $u = 1 + 2t$ then $du = 2\, dt$, so $dt = \frac{du}{2}$

$$v = \int \frac{-10}{u^2}\, du$$

$$= \int -10\, u^{-2}\, du$$

$$= \frac{10}{u} + K$$

$$= \frac{10}{1+2t} + K .$$

When $t = 0$, $v = 30\ cms^{-1}$, so we substitute to obtain:

$$K = 20.$$

So the expression for velocity as a function of time is:

$$v = \left(\frac{10}{1+2t} + 20\right) cms^{-1}.$$

Example 5

A flare is ejected vertically upwards from the ground at 15 ms^{-1}. Find the height of the flare after 2.5 second. The object has acting on it the force due to gravity, so its acceleration is -9.8 ms^{-2}.

Solution

$v = \int a\, dt$

$= \int (-9.8)\, dt$

$= -9.8t + C.$

Now at $t = 0$, the velocity = 15 ms^{-1}. So, $C = 15$.

So, the expression for velocity becomes:

$v = -9.8t + 15.$

Now, we need to find the displacement, so we integrate our expression for velocity:

$s = \int v\, dt$

$= \int (-9.8t + 15)\, dt$

$= -4.9t^2 + 15t + K.$

Now, we know from the question that when $t = 0$, $s = 0$.

This gives $K = 0$.

So $s = -4.9t^2 + 15t$.

At time $t = 2.5$ second, $s = -4.9(2.5)^2 + 15(2.5) = 6.875$ m.

11.3 Definite Integral

If $F'(x) = f(x)$ where $f(x)$ is continuous on the interval $a \leq x \leq b$, then the definite integral of $f(x)$ on $[a, b]$ is defined by

$$\int_a^b f(x)dx = F(b) - F(a)$$

where the numbers a and b are called the limits of integration. Here a is the lower limit and b is the upper limit.

$\int_a^b f(x)dx$ reads "the integral from $x = a$ to $x = b$ of $f(x)$ with respect to x".

We write $F(b) - F(a) = F(x)]_a^b$ or $[F(x)]_a^b$

depending on whether F has one or more terms.

As an example, consider $\int_a^b x^3 dx$.

We can compute that

$$\int_a^b x^3\, dx = \left[\frac{x^4}{4}\right]_a^b = \frac{b^4}{4} - \frac{a^4}{4}.$$

The Fundamental Theorem of Calculus

1. Let f be continuous on [a, b]. Then for each $x \in [a, b]$,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

2. Let F' be continuous on [a, b]. Then for each $x \in [a, b]$,

$$\int_a^x F'(t)dt = F(x) - F(a).$$

Properties of the Definite Integral

Suppose f and g are continuous on [a, b] and $K \in \mathbb{R}$.

1. $\int_a^b Kf(x)dx = K\int_a^b f(x)dx.$

2. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$

3. $\int_a^a f(x)dx = 0.$

4. $\int_a^b f(x)dx = -\int_b^a f(x)dx.$

5. $\int_d^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $c \in (a, b)$.

Example 6

Evaluate the following definite integrals.

(a) $\int_{-1}^{2} x^3 dx$

(b) $\int_0^{\frac{\pi}{2}} \sin x dx$

(c) $\int_{-1}^{1} \sqrt{x+2}\, dx$

(d) $\int_{\frac{\pi}{4}}^{\pi} |\cos x| dx$

Solution

(a) $\int_{-1}^{2} x^3 dx = \frac{x^4}{4}\Big]_{-1}^{2} = \frac{2^4}{4} - \frac{(-1)^4}{4} = 3\frac{3}{4}.$

(b) $\int_0^{\frac{\pi}{2}} \sin x\, dx = -\cos x\Big]_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + \cos 0 = -0 + 1 = 1.$

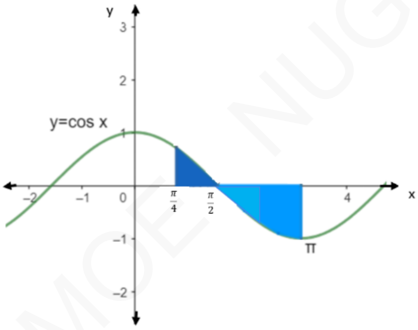
(c) $\int_{-1}^{1} \sqrt{x+2}\,dx \quad = \int_{-1}^{1} (x+2)^{\frac{1}{2}}dx$

$$= \frac{2}{3}(x+2)^{\frac{3}{2}}\Big]_{-1}^{1}$$

$$= \frac{2}{3}\left((3)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right)$$

$$= \frac{2}{3}(3\sqrt{3} - 1).$$

(d) $\int_{\frac{\pi}{4}}^{\pi} |\cos x| d$



When $\frac{\pi}{4} \le x \le \frac{\pi}{2}$, $|\cos x| = \cos x$.

When $\frac{\pi}{2} \le x \le \pi$, $|\cos x| = -\cos x$.

$$\int_{\frac{\pi}{4}}^{\pi} |\cos x|dx \quad = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)dx$$

$$= \sin x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \sin x\Big]_{\frac{\pi}{2}}^{\pi}$$

$$= \left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right) - \left(\sin\pi - \sin\frac{\pi}{2}\right)$$

$$= 2 - \frac{\sqrt{2}}{2}.$$

Example 7

Evaluate the integral $\int_{1}^{2} x\sqrt{2-x}\,dx$.

Solution

Let $u = 2 - x$. Then $du = -dx$.

When $x = 1$ then $u = 1$ and when $x = 2$ then $u = 0$

Changing the variable from x to u, we get

$$\int_{1}^{2} x\sqrt{2-x}\,dx = -\int_{1}^{0}(2u^{\frac{1}{2}} - u^{\frac{3}{2}})\,du$$

$$= \int_{0}^{1}(2u^{\frac{1}{2}} - u^{\frac{3}{2}})\,du$$

$$= \left[\frac{4}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_{0}^{1}$$

$$= \frac{14}{15}.$$

Exercise 11.1

1. Find the equation of the curve that passes through the point (4, 3) given that the gradient of the tangent to the curve is $3x - 5$.
2. The slop at the point (x, y) on the graph of a function is $\frac{5x-2}{x}$. Find the equation of the curve if it passes through the point (e, 5e + 3).
3. The velocity of a particle is given by $v(t) = 2t + 3\ ms^{-1}$.
   (i) Find the equation for the displacement of the particle if it was at the origin initially.
   (ii) Find the displacement after 1 second.
4. A particle is traveling with velocity $v(t) = 6t^2 ms^{-1}$, and has displacement 100 m after 5 seconds. Find the displacement of the particles.
5. The acceleration of a particle is given by $a(t) = 6t\ ms^{-2}$ and the position of the particle is 10 m at t = 0 and 14 m at t= 2. Calculate the displacement function of the particle.
6. A particle is traveling with acceleration $a(t) = \cos t\ ms^{-2}$ and is stationary at t = 0. Find the velocity function for this particle.
7. Evaluate the following definite integrals.
   (a) $\int_0^1 \frac{e^x}{e^x+1} dx$
   (b) $\int_{-1}^1 x\sqrt{x+2}dx$
   (c) $\int_0^6 \left(\frac{1}{3}x - 1\right)^3 dx$
   (d) $\int_2^4 \frac{x+5}{x^2-6x+5} dx$
   (e) $\int_0^2 e^{-x}\cos 2x dx$
   (f) $\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \tan 2x dx$
8. Find the integral of $\int_0^{2\pi} \cos mx \sin nx\ dx$ for (a) m = n and (b) m ≠ n.
9. Evaluate the following integrals.
   (a) $\int_0^4 \left|\sqrt{2x} - 2\right| dx$ (b) $\int_{-1}^1 |e^x - 1|\ dx$ (c) $\int_0^{2\pi} |\sin x|\ dx$

11.4 Application of definite integral

(i) Area between the Curve and x-axis

In this section, we discuss the area between the curve and x-axis using the definite integral. If f(x) is positive and continuous on the closed interval $a \leq x \leq b$, then the area bounded by $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$ is given by $\int_a^b f(x)dx$.

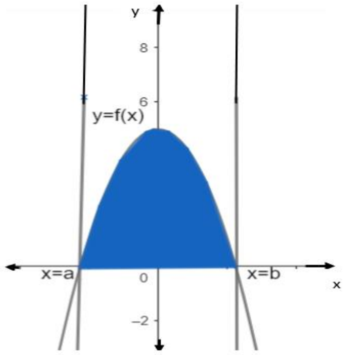


Figure 11.1

If f(x) is negative and continuous on the closed interval $a \leq x \leq b$, then the area bounded by $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$ is given by $\left|\int_a^b f(x)dx\right|$.

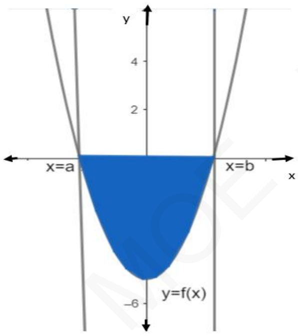


Figure 11.2

Consider the given function

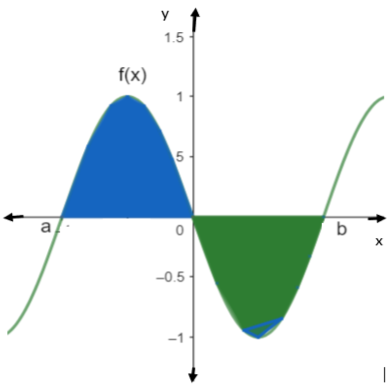


Figure 11.3

On [a, 0], f(x) is positive and continuous. Thus, the area between the curve, f(x) and x-axis is $\int_a^b f(x)dx$.

On [0, b], f(x) is negative and continuous. Thus, the area between the curve, f(x) and x-axis is $\left|\int_a^b f(x)dx\right|$.

Total area of the graph of f(x) and the x-axis on [a, b] = $\int_a^b f(x)dx + \left|\int_a^b f(x)dx\right|$.

Example 8

Find the area enclosed between the graph of

(a) $f(x) = 3$ and the x-axis over $[1,4]$.

(b) $f(x) = 3x + 2$ and the x-axis over $\left[-\frac{2}{3}, 0\right]$.

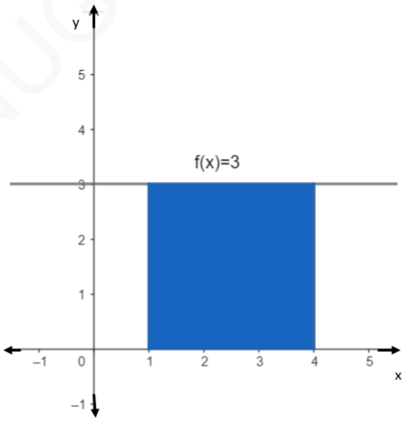
(c) $f(x) = -x^2 + 4$ and the x-axis over $[-2.2]$.

Solutions

(a) Area $= \int_1^4 3\,dx = [3x]_1^4$

$= 12 - 3$

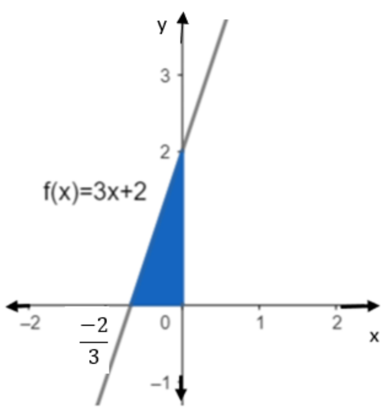
$= 9$ unit^2.



(b) Area $= \int_{-\frac{2}{3}}^{0} (3x+2)\,dx = \left[3\frac{x^2}{2} + 2x\right]_{-\frac{2}{3}}^{0}$

$= 0 - \frac{3}{2}\left(-\frac{2}{3}\right)^2 - 2\left(-\frac{2}{3}\right)$

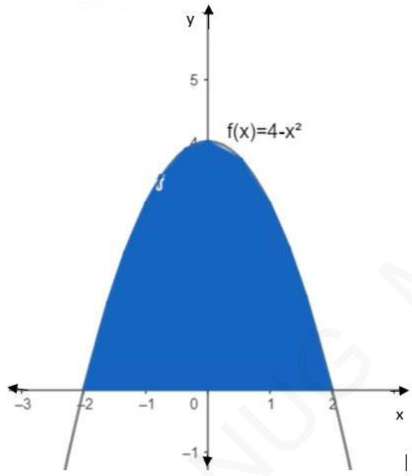
$= \frac{2}{3}$ unit^2.



(c) Area $= \int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$

$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$

$= 10\frac{2}{3}$ unit^2.



Example 9

(a) Compute the definite integral $\int_0^{\frac{5}{2}} (x^2 - 2x)dx$.

(b) Find the total area enclosed between the graph of $f(x) = x^2 - 2x$ and the x-axis over $\left[0, \frac{5}{2}\right]$.

Solution

(a) $\int_0^{\frac{5}{2}} (x^2 - 2x)\,dx = \left[\frac{x^3}{3} - x^2\right]_0^{\frac{5}{2}}$

$= \frac{1}{3}\left(\frac{5}{2}\right)^3 - \left(\frac{5}{2}\right)^2$

$= -1\frac{1}{24}$.

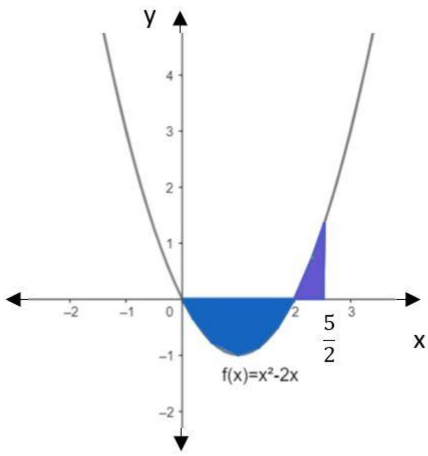
(b) For the intersection point of x-axis and the curve, f(x) = 0

$x^2 - 2x = 0$

$x(x - 2) = 0$

$x = 0, 2.$

Thus, the graph of f(x) cuts x-axis at 0 and 2.



We divide the domain in two intervals: the interval $[0,2]$ and the interval $\left[2, \frac{5}{2}\right]$.

On $[0, 2]$, $\int_0^2 (x^2 - 2x)dx = \left[\frac{x^3}{3} - x^2\right]_0^2 = \left(\frac{8}{3} - 4\right) - 0 = -\frac{4}{3}$.

Area of the graph of f(x) and the x-axis on $[0, 2] = \left|-\frac{4}{3}\right| = \frac{4}{3}$ unit^2.

On $[2, \frac{5}{2}]$,

$$\int_2^{\frac{5}{2}} (x^2 - 2x)dx = \left[\frac{x^3}{3} - x^2\right]_2^{\frac{5}{2}}$$
$$= \left(\frac{125}{24} - \frac{25}{4}\right) - \left(\frac{8}{3} - 4\right)$$
$$= \frac{7}{24}.$$

Area of the graph of f(x) and the x-axis on $[2, \frac{5}{2}] = \frac{7}{24}$ unit^2.

Total Area of the graph of f(x) and the x-axis on $\left[0, \frac{5}{2}\right] = \frac{4}{3} + \frac{7}{24} = 1\frac{5}{8}$ unit^2.

Example 10

Figure shows the graph of the function $f(x) = \cos x$.

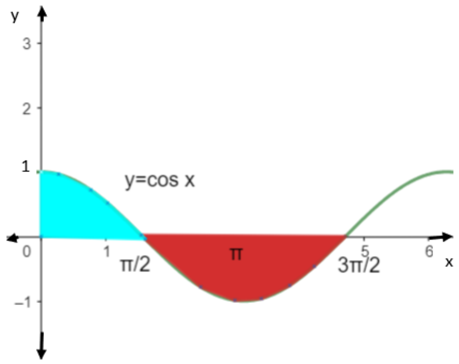
Find the area enclosed between the graph of $f(x) = \cos x$ and the lines

(a) $x = 0$ and $x = \frac{\pi}{2}$.

(b) $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

(c) $x = 0$ and $x = \frac{3\pi}{2}$.

(d) Find the definite integral $\int_0^{\frac{3\pi}{2}} \cos x dx$.



Solution

(a) $\int_0^{\frac{\pi}{2}} \cos x\, dx = \sin x\Big]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1.$

Area $= 1$ unit^2.

(b) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\, dx = \sin x\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \sin\frac{3\pi}{2} - \sin\frac{\pi}{2} = -2.$

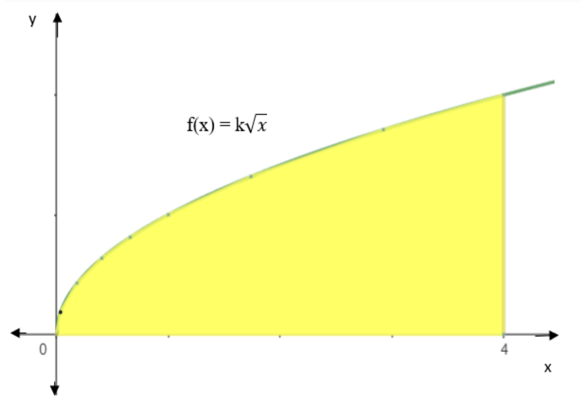
Area $= |-2| = 2$ unit^2.

(c) Area between the graph $f(x) = \cos x$ and the lines $x = 0$ and $x = \frac{3\pi}{2}$ $= 1\ \text{unit}^2 + 2\ \text{unit}^2 = 3\ \text{unit}^2$.

(d) $\int_0^{\frac{3\pi}{2}} \cos x\, dx = \sin x\Big]_0^{\frac{3\pi}{2}} = \sin\frac{3\pi}{2} - \sin 0 = -1$.

Example 11

The shaded area is 16 unit ^{2}. Find the value of k.



Solution

Area $= \int_0^4 k\sqrt{x}dx = \frac{2k}{3}x^{\frac{3}{2}}\Big]_0^4 = \frac{2k}{3}\cdot 4^{\frac{3}{2}} = \frac{16k}{3}\ \text{unit}^2$.

By the problem, $\frac{16k}{3} = 16$, so $k = 3$.

(ii) Area between Two Curves

In this section, we discuss the area between the two curves using the definite integral. Let f(x) and g(x) be continuous functions over an interval [a, b] such that $f(x) \geq g(x)$ on [a, b].

The area between $y = f(x)$ and $y = g(x)$ is given by $\int_a^b [f(x) - g(x)]\, dx$.

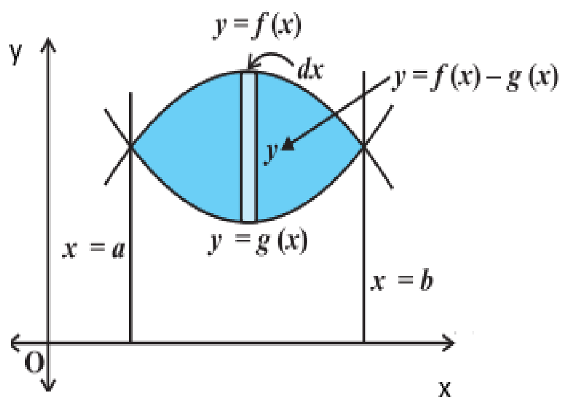


Figure 11.4

If f (x) ≥ g (x) in [a, c] and f (x) ≤ g (x) in [c, b], where a < c < b as shown in the Fig, then the area of the regions bounded by curves can be written as:

Total Area = $\int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx$.

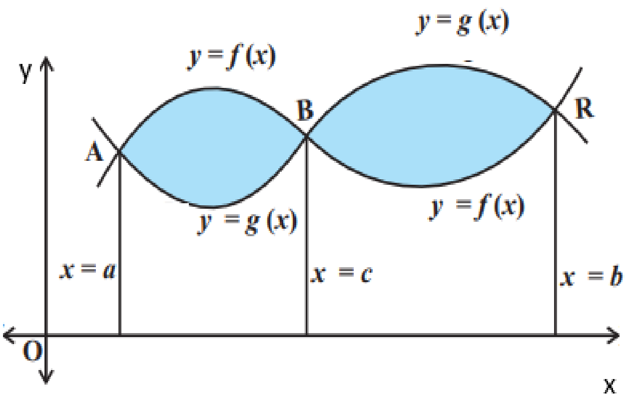


Figure 11.5

Example 12

Find the area between $y = x^2 - 2x + 2$ and $y = 2x - 1$.

Solution

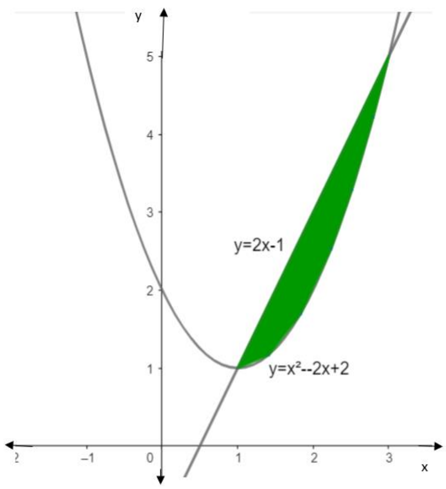
To find the x-coordinate of intersection points,

$$x^2 - 2x + 2 = 2x - 1$$

$$x^2 - 4x + 3 = 0$$

$$(x - 1)(x - 3) = 0$$

$$x = 1, 3.$$



$$\text{Area} = \int_1^3 [(2x-1)-(x^2-2x+2)]\,dx$$

$$= \int_1^3 (-x^2+4x-3)\,dx$$

$$= \left[-\frac{x^3}{3}+\frac{4x^2}{2}-3x\right]_1^3$$

$$= 0 - \left(-\frac{4}{3}\right)$$

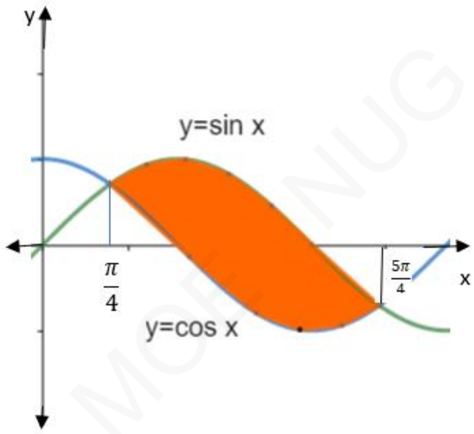
$$= 1\frac{1}{3}\ \text{unit}^2.$$

Example 13

Find the area between $y = \sin x$ and $y = \cos x$ for $\frac{\pi}{4} \leq x \leq \frac{5}{4}\pi$.

Solution

From the figure, we see that $\sin x \geq \cos x$ when $\frac{\pi}{4} \leq x \leq \frac{5}{4}\pi$.



$$\text{Area} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\, dx$$

$$= [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \left(-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4}\right) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)$$

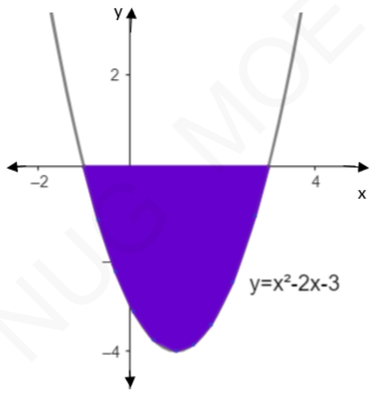
$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= 2\sqrt{2} \text{ unit}^2.$$

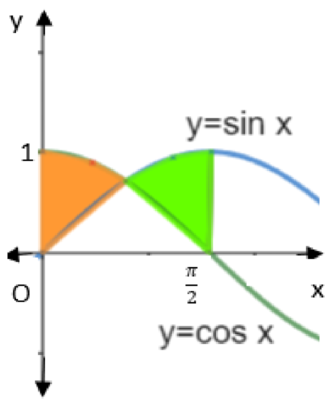
Exercise 11.2

1. Find the shaded areas.

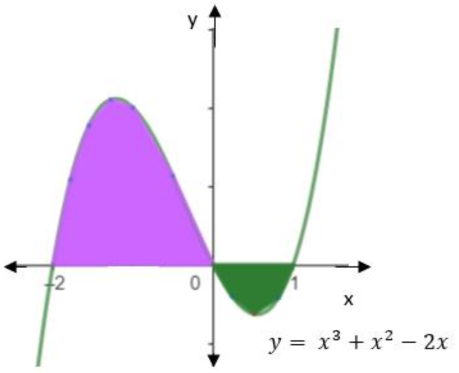
(a)



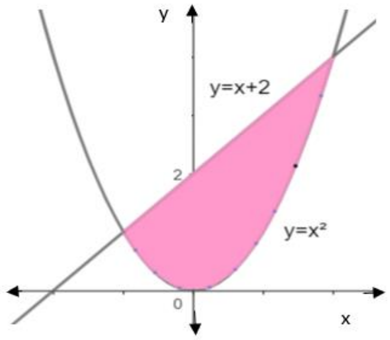
(b)



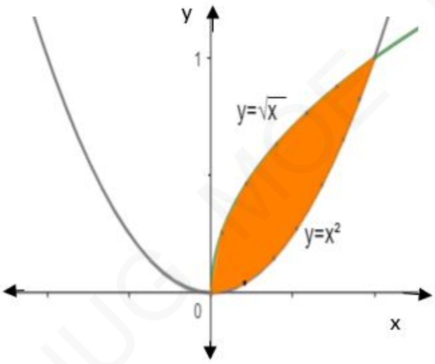
(c)



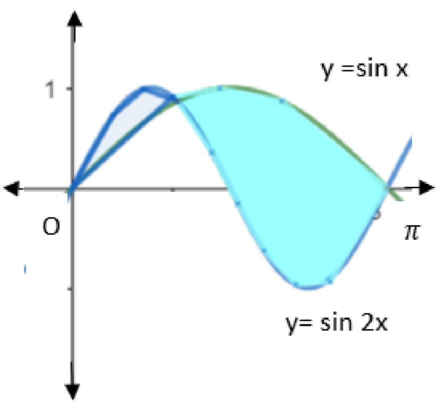
(d)



(e)



(f)



2. Find the area under each curve between the given x-values:

(a) $f(x) = \ln x$ from $x = 1$ to $x = e$.

(b) $f(x) = 12 - 3x^2$ from $x = -1$ to $x = 2$.

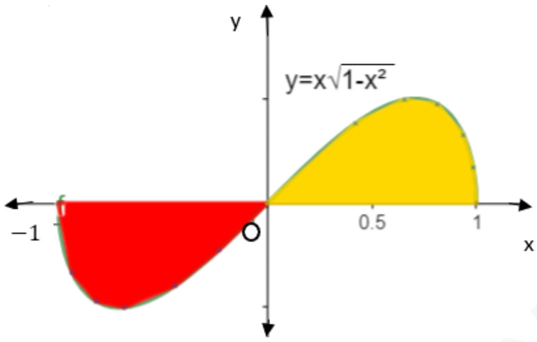
(c) $f(x) = xe^{x^2}$ from $x = 0$ to $x = 1$.

(d) $f(x) = \sin x$ from $x = 0$ to $x = \frac{3\pi}{2}$.

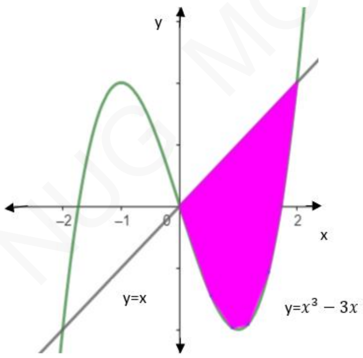
(e) $f(x) = (1 + \cos x)\sin x$ from $x = 0$ to $x = \pi$.

3. (a) Compute $\int_{-1}^{1} x\sqrt{1 - x^2}dx$.

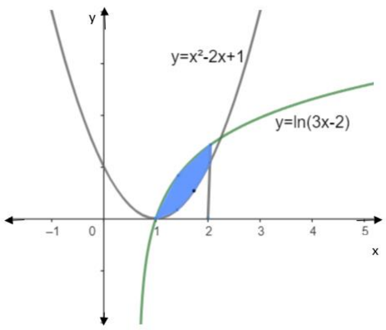
(b) Find the area enclosed between the graph of $f(x) = x\sqrt{1 - x^2}$ and the x-axis over $[-1,1]$.



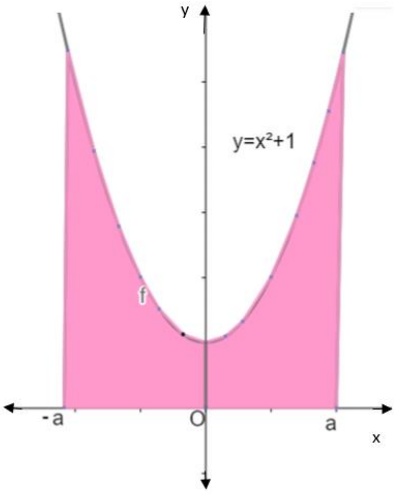
10. Find the area enclosed between the graphs of $y = x^3 - 3x$ and $y = x$ for $0 \le x \le 2$.



11. Find the area enclosed of the graphs of $y = \ln(3x-2)$ and $y = x^2 - 2x + 1$ for $1 \le x \le 2$.



6. The shaded area is 8a $unit^2$. Find the value of a.



(iii) Volume Using Cross-Sections

We have seen how to compute certain areas by using integration; some volumes may also be computed by evaluating an integral. Generally, the volumes that we can compute this way have cross-sections that are easy to describe.

A cross-section of a solid is the plane region generated by the intersection of the solid with a plan. The cross-section is mainly two types, namely **Horizontal cross-section** and **Vertical cross-section.**

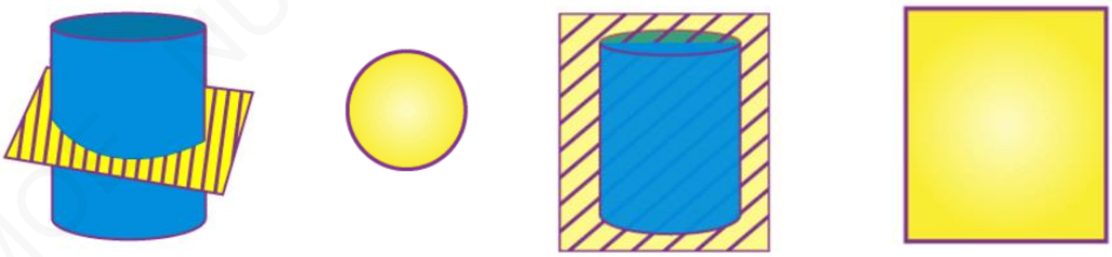


Figure 11.6

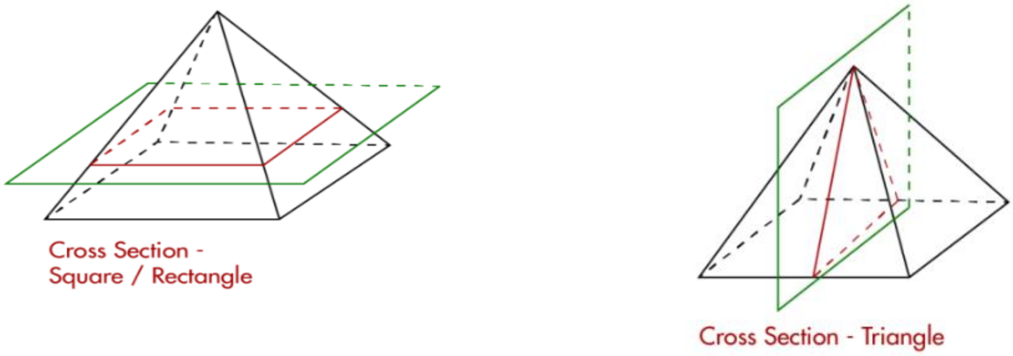


Figure 11.7

In this section, we use definite integrals to find volumes of three-dimensional solids. We consider three approaches—the slicing method, the disc method and the washer method —for finding these volumes, depending on the characteristics of the solid.

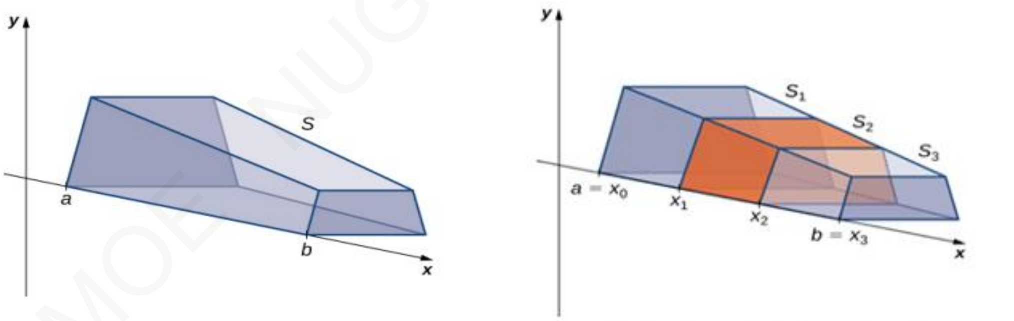
(iv) Volume by The Slicing Method

In this section, we define volumes of solids using the areas of their cross-section. The slices have thickness dx.

The area A of a cross-section = A(x).

The volume of the slice, dv = A(x) dx.

The volume of the whole solid from $x = a$ to $x = b = \int_a^b A(x)dx$.



The solid S has been divided into three slices perpendicular to the x-axis.

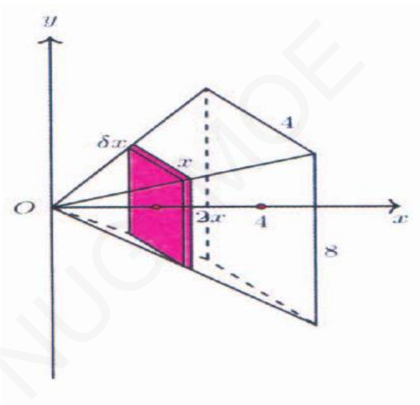
Figure 11.8

Note: we assume the slices are perpendicular to the x-axis. Therefore, the area formula is in terms of x and the limits of integration lie on the x-axis.

Shape	**Area**
Square	$A = s^2$
Rectangle	$A = b\times h$
Circle	$A = \pi r^2$
Equilateral Triangle	$A = \frac{\sqrt{3}}{4} s^2$
Triangle	$A = \frac{1}{2}(b \times h)$
Trapezoid	$A = h(\frac{b_1+b_2}{2})$
Parallelogram	$A = b\times h$

Example 14

A pyramid with the height of 4 m has a rectangle base shown in figure. Find the volume of the pyramid.



Solution

The cross-section area at x is $A(x) = 2x^2$.

$$V = \int_0^4 A(x)dx \quad = \int_0^4 2x^2 dx$$
$$= \frac{2x^3}{3}\Big]_0^4$$
$$= \frac{128}{3}\ m^3.$$

(v) Volume of Revolution by The Disc Method

Suppose that $y = f(x)$ is a continuous, non-negative function on the interval $[a, b]$.

The disc method is used when we rotate a single curve $y = f(x)$ about $x = a$ and $x = b$ about the x-axis through 360°.

We can have a function, like this one:

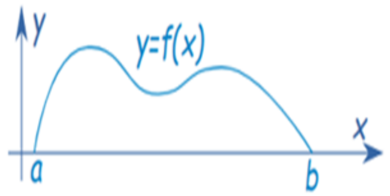


Figure 11.9

And revolve it around the x-axis like this:

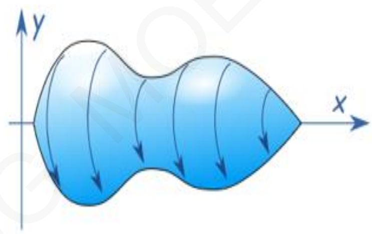


Figure 11.10

This solid is called a solid of revolution.

To find its volume we can add up a series of disks:

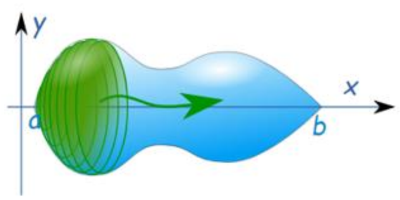


Figure 11.11

Each disk's face is a circle:

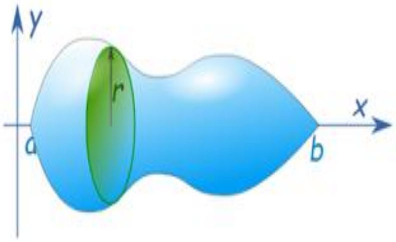


Figure 11.12

Solid of Revolution **Cross Sections of Solid**

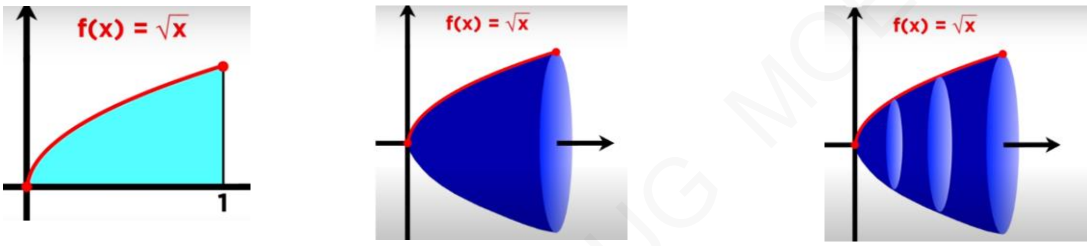


Figure 11.13

The area of a circle = πr^2.

The radius r is the value of the function at that point, $r = f(x)$.

The volume is found by summing all those discs using definite integration:

$$\text{Volume} = \int_a^b \pi\,[f(x)]^2\,dx .$$

Example 15

Find the volume of revolution formed when the curve

(a) $y = \sqrt{x}$ for $0 \le x \le 3$

(b) $y = x^2$ for $0 \le x \le 1$

are rotated through 360° about the x-axis.

Solution

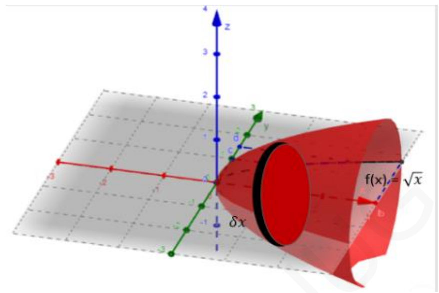
(a) Volume of revolution $= \pi \int_{0}^{3} (\sqrt{x})^2 \ dx$

Volume $= \int_{a}^{b} \pi\,[f(x)]^2 \ dx$

$$= \pi \int_{0}^{3} x \ dx$$

$$= \pi \left[\frac{x^2}{2}\right]_0^3$$

$$= \frac{9\pi}{2} \ \text{unit}^3.$$

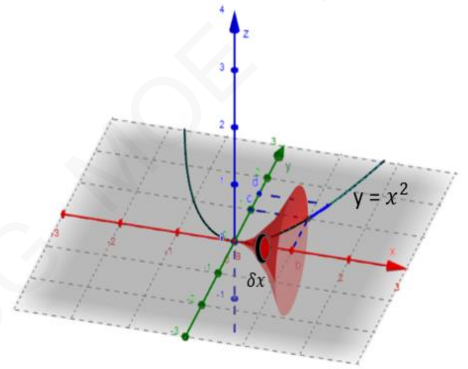


(b) Volume of revolution $= \pi \int_{0}^{1} (x^2)^2 \ dx$

$$= \pi \int_{0}^{1} x^4 \ dx$$

$$= \pi \left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{\pi}{5} \ \text{unit}^3.$$



Example 16

Find the volume of revolution formed when the curve y = sin x for $\frac{\pi}{4} \le x \le \pi$, is rotated through 360° about the x-axis.

Solution

Volume of revolution $= \pi \int_{\frac{\pi}{4}}^{\pi} \sin^2 x \ dx$

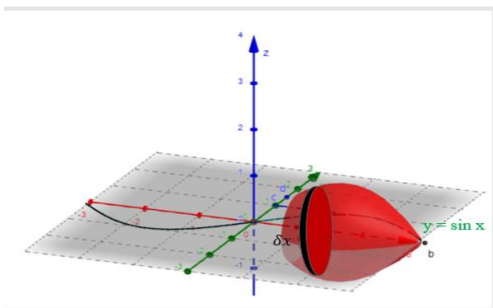
$$= \pi \int_{\frac{\pi}{4}}^{\pi} \frac{1 - \cos 2x}{2} \ dx$$

$$= \pi \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_{\frac{\pi}{4}}^{\pi}$$

$$= \pi\left(\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right) - \pi\left(\frac{\pi}{8} - \frac{\sin \frac{\pi}{2}}{4}\right)$$

$$= \pi\left(\frac{\pi}{2} - 0\right) - \pi\left(\frac{\pi}{8} - \frac{1}{4}\right)$$

$$= \left(\frac{3\pi^2}{8} + \frac{\pi}{4}\right) \ \text{unit}^3.$$



(vi) Volume of Revolution by The Washer Method

We can extend the disk method to find the volume of a hollow solid of revolution. We use the Washer Method if we want to find the volume between two functions. Assuming that the functions f(x) and g(x) are continuous and **non-negative** on the interval [a, b] and $g(x) \le f(x)$.

If the region bounded by $y = f(x)$ (on top) and $y = g(x)$, between $x = a$ and $x = b$ is revolved about the x-axis, then its volume of revolution is given by

$$\text{Volume of revolution} = \pi \int_a^b [(f(x))^2 - (g(x))^2]\ dx .$$

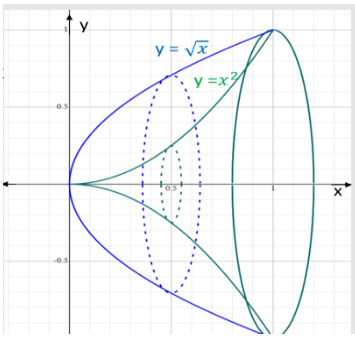
Example 17

Find the volume of revolution generated by revolving the regions between $y = x^2$ and $y = \sqrt{x}$ about the x-axis.

Solution

To find the intersection points of $y = x^2$ and $= \sqrt{x}$.

$$x^2 = \sqrt{x}$$
$$x^4 = x$$
$$x^4 - x = 0$$
$$x(x^3 - 1) = 0$$
$$x = 0 \text{ and } x = 1$$



The two graphs cut x-axis at 0 and 1. The curve $y = \sqrt{x}$ is bounded above.

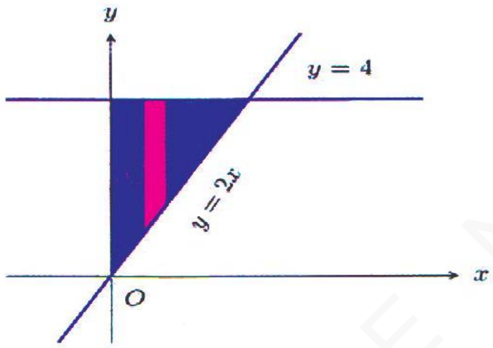
$$\text{Volume of revolution} = \pi \int_0^1 [(\sqrt{x})^2 - (x^2)^2]\ dx$$
$$= \pi \int_0^1 [x - x^4]\ dx$$
$$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$
$$= \frac{3\pi}{10}\ \text{unit}^3.$$

Exercise 11.3

1. Find the volume of the solid by using slicing method. The base of the solid is the region bounded by the graphs of $y = 2x$, $y = 4$ and $x = 0$. The cross-section perpendicular to the x-axis are

   (a) rectangle of height 8.

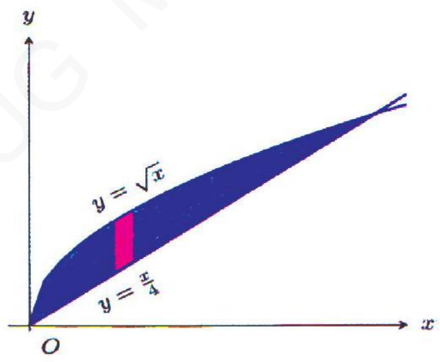
   (b) rectangle of perimeter 10.



2. Find the volume of the solid by using slicing method. The base of the solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{4}$. The cross-section perpendicular to the x-axis are

   (a) isosceles triangles of height 4.

   (b) semicircles with diameters running across the base of solid.



3. Find the volume of the solid by using slicing method. The solid lies between the planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-section perpendicular to the x-axis are circular disks whose diameters run from $y = -x^2 + 4$ and $y = x^2 + 2$.

4. By using the disc method, find the volume of revolution formed when the curve $y = f(x)$ for $a \le x \le b$, is rotated through $360°$ about the x-axis.

(a) $f(x) = 3x + 2;\ a = \frac{1}{2},\ b = 4.$

(b) $f(x) = \sqrt{\cos x};\ a = 0,\ b = \frac{\pi}{2}.$

(c) $f(x) = \frac{1}{x-1};\ a = 3,\ b = 5.$

(d) $f(x) = \sin x \cos x;\ a = 0,\ b = \frac{\pi}{2}.$

(e) $f(x) = \sqrt{x}\ \sqrt[4]{1+x^2};\ a = 2,\ b = 3.$

5. By using the washer method, find the volume of revolution generated by revolving the regions between $y = f(x)$ and $y = g(x)$ for $a \le x \le b$ about the x-axis.

(a) $f(x) = x^2,\ g(x) = 2x;\ a = 0,\ b = 2.$

(b) $f(x) = \tan x,\ g(x) = \frac{x}{2};\ a = 0,\ b = \frac{\pi}{4}.$

(c) $f(x) = 2\sqrt{x},\ g(x) = 2;\ a = 0,\ b = 1.$