Rewriting (2) in the equvalent form

$$y = \frac{3x-2}{2} \qquad (3)$$

Substituting $y = \frac{3x-2}{2}$ in (1),

$$\begin{aligned}
x^2 - x\frac{3x-2}{2} + 2\left(\frac{3x-2}{2}\right)^2 &= 8 \\
x^2 - \frac{x(3x-2)}{2} + \frac{2(3x-2)^2}{4} &= 8 \\
2x^2 - x(3x-2) + (3x-2)^2 &= 16 \\
2x^2 - 3x^2 + 2x + 9x^2 - 12x + 4 &= 16 \\
8x^2 - 10x - 12 &= 0 \\
4x^2 - 5x - 6 &= 0 \\
(4x+3)(x-2) &= 0 \\
x = -\frac{3}{4} \quad &\text{or} \quad x = 2
\end{aligned}$$

Substituting $x = -\frac{3}{4}$ in (3),

$$y = \frac{3(-\frac{3}{4}) - 2}{2} = -\frac{17}{8}$$

Substituting $x = 2$ in (3),

$$y = \frac{3(2)-2}{2} = 2$$

Hence the solution set is $\{(-\frac{3}{4}\,, -\frac{17}{8}), (2,2)\}$.

Example 10.

A rectangular garden has an area of 720 m^2. If the length is reduced by 6 m and the breadth is increased by 6 m, then the resulting area is the same as the original area. Find the length and breadth of the garden.

Solution

Suppose that the length of the garden is x m and the breadth of the garden is y m. Then the new length is $(x-6)$ m and the new breadth is $(y+6)$ m.

$$\begin{aligned} xy &= 720 \qquad (1) \\ (x-6)(y+6) &= 720 \qquad (2) \end{aligned}$$

Rewriting (2) in the equivalent form

$$\begin{aligned} xy + 6x - 6y - 36 &= 720 \\ xy + 6x - 6y &= 756 \qquad (3) \end{aligned}$$

Substituting $xy = 720$ in (3),

$$\begin{aligned} 720 + 6x - 6y &= 756 \\ 6x - 6y &= 36 \\ x - y &= 6 \\ x &= 6 + y \qquad (4) \end{aligned}$$

Substituting $x = 6 + y$ in (1),

$$\begin{aligned} (6+y)y &= 720 \\ y^2 + 6y &= 720 \\ y^2 + 6y - 720 &= 0 \\ (y+30)(y-24) &= 0 \\ y + 30 = 0 \quad &\text{or} \quad y - 24 = 0 \\ y = -30 \quad &\text{or} \quad y = 24 \end{aligned}$$

Since y must be positive, $y = -30$ is impossible. So $y = 24$.

Substituting $y = 24$ in (4), $x = 24 + 6 = 30$.

Hence the length is 30 m and the breadth is 24 m.

Exercise 5.5

1. Find the solution set of each of the systems of equations:

   (a) $x^2 - y^2 = 9$, $x + y = 1$ (b) $y = \frac{8}{x}$, $y = 7 + x$ (c) $x^2 + 5x + y = 4$, $x + y = 8$

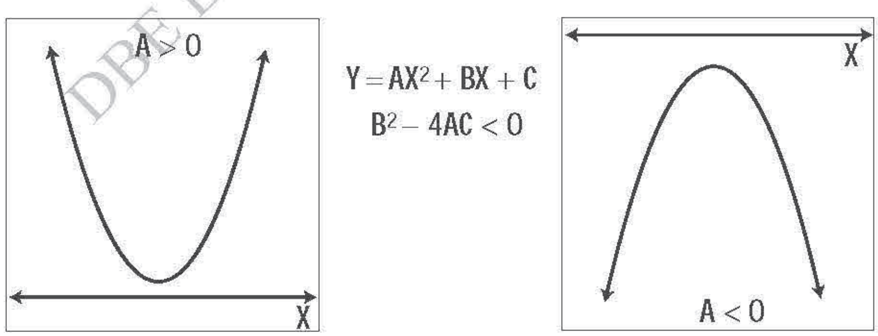
2. The sum of squares of two numbers is 58. If the first number and twice the second add up to 13, find the numbers.

3. The sum of the reciprocals of two positive numbers is $\frac{7}{36}$ and the product of the numbers is 108. Find the numbers.

5.8 Quadratic Inequality

From the discriminant $b^2 - 4ac$ of quadratic function $y = ax^2 + bx + c$ and sign of a, the coefficient of x^2, we can determine the solution set of x when the value of y is less than zero, equals zero, and is greater than zero.

Case 1. $\boxed{b^2 - 4ac < 0}$

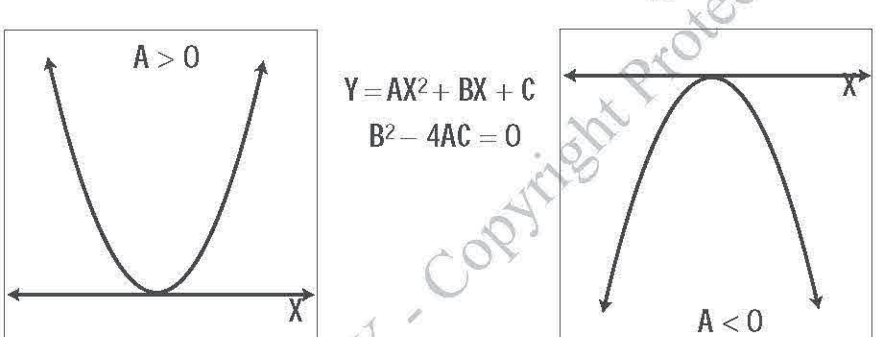


- If $b^2 - 4ac < 0$ and $a > 0$, the parabola opens up and does not cut the x-axis. Therefore all of the values of y are positive for all values of x.
- If $b^2 - 4ac < 0$ and $a < 0$, the parabola opens down and does not cut the x-axis. Therefore all of the values of y are negative for all values of x.

$a > 0$		$a < 0$	
	solution set		solution set
$y < 0$	$\varnothing$	$y < 0$	$\mathbb{R}$
$y = 0$	$\varnothing$	$y = 0$	$\varnothing$
$y > 0$	$\mathbb{R}$	$y > 0$	$\varnothing$

Case 2. $\boxed{b^2 - 4ac = 0}$

If $b^2 - 4ac = 0$, then $y = ax^2 + bx + c = a(x + \frac{b}{2a})^2$.



- If $b^2 - 4ac = 0$ and $a > 0$, the parabola opens up and meets the x-axis only at the vertex $(-\dfrac{b}{2a}, 0)$. Therefore all values of y are positive for all values of x except $-\dfrac{b}{2a}$. The value of y is zero, when $x = -\dfrac{b}{2a}$.
- If $b^2 - 4ac = 0$ and $a < 0$, the parabola opens down and meets the x-axis only at the vertex $(-\dfrac{b}{2a}, 0)$. Therefore all values of y are negative for all values of x except $-\dfrac{b}{2a}$. The value of y is zero, when $x = -\dfrac{b}{2a}$.

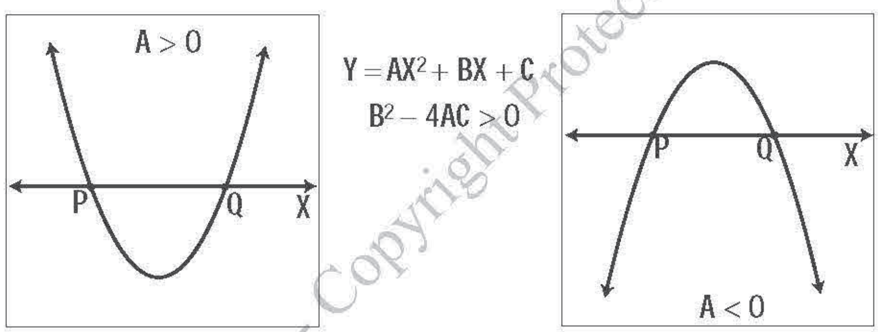
$a > 0$		$a < 0$	
	solution set		solution set
$y < 0$	$\varnothing$	$y < 0$	$\mathbb{R} \setminus \{-\frac{b}{2a}\}$
$y = 0$	$\{-\frac{b}{2a}\}$	$y = 0$	$\{-\frac{b}{2a}\}$
$y > 0$	$\mathbb{R} \setminus \{-\frac{b}{2a}\}$	$y > 0$	$\varnothing$

Case 3. $\boxed{b^2 - 4ac > 0}$

If $b^2 - 4ac > 0$, then

$$y = ax^2 + bx + c = a(x-p)(x-q),$$

where p and q are x-intercepts $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ of the function. Let us assume that $p < q$.



- If $a > 0$, the parabola opens up. The values of y are positive for $x < p$ or $x > q$. The values of y are equal to zero when $x = p$ or $x = q$. The values of y are negative for $p < x < q$.
- If $a < 0$, the parabola opens down. The values of y are positive for $p < x < q$. The values of y are equal to zero when $x = p$ or $x = q$. The values of y are negative for $x < p$ or $x < q$.

$a > 0$		$a < 0$	
	solution set		solution set
$y < 0$	$\{x \mid p < x < q\}$	$y < 0$	$\{x \mid x < p \text{ or } x > q\}$
$y = 0$	$\{p, q\}$	$y = 0$	$\{p, q\}$
$y > 0$	$\{x \mid x < p \text{ or } x > q\}$	$y > 0$	$\{x \mid p < x < q\}$

Example 11.

Find the solution set of each of the inequalities $2x^2+x+3<0$, $2x^2+x+3\leq 0$, $2x^2+x+3>0$, and $2x^2+x+3\geq 0$.

Solution
Let $y=2x^2+x+3$. Comparing the function $y=2x^2+x+3$ to the standard form $y=ax^2+bx+c$, we get $a=2, b=1, c=3$. Then

$$b^2-4ac=1-4(2)(3)=-23<0$$

Since $a>0$ and $b^2-4ac<0$, the parabola of the function opens up and does not cut the x-axis. Therefore all of the values of y are positive for all values of x.

The solution set of $2x^2+x+3<0$ is $\varnothing$.
The solution set of $2x^2+x+3\leq 0$ is $\varnothing$.
The solution set of $2x^2+x+3>0$ is $\mathbb{R}$.
The solution set of $2x^2+x+3\geq 0$ is $\mathbb{R}$.

Example 12.

Find the solution set of each of the inequalities $-2x^2+x-3<0$, $-2x^2+x-3\leq 0$, $-2x^2+x-3>0$, and $-2x^2+x-3\geq 0$.

Solution
Let $y=-2x^2+x-3$. Comparing the function $y=-2x^2+x-3$ to the standard form $y=ax^2+bx+c$, we get $a=-2, b=1, c=-3$. Then

$$b^2-4ac=1-4(-2)(-3)=-23<0$$

Since $a<0$ and $b^2-4ac<0$, the parabola of the function opens down and does not cut the x-axis. Therefore all of the values of y are negative for all values of x.

The solution set of $-2x^2+x-3<0$ is $\mathbb{R}$.
The solution set of $-2x^2+x-3\leq 0$ is $\mathbb{R}$.
The solution set of $-2x^2+x-3>0$ is $\varnothing$.
The solution set of $-2x^2+x-3\geq 0$ is $\varnothing$.

Example 13.

Find the solution set of each of the inequalities $2x^2+8x+8<0$, $2x^2+8x+8\leq 0$, $2x^2+8x+8>0$, and $2x^2+8x+8\geq 0$.

Solution

Let $y=2x^2+8x+8$. Comparing the function $y=2x^2+8x+8$ to the standard form $y=ax^2+bx+c$, we get $a=2, b=8, c=8$. Then

$$b^2-4ac=64-4(2)(8)=0;$$

$$y=2x^2+8x+8=2(x+2)^2$$

Since $a>0$ and $b^2-4ac=0$, the parabola of the function opens up and meets the x-axis only at the vertex $(-2,0)$. Therefore all values of y are positive for all values of x except -2. The value of y is zero, when $x=-2$.

The solution set of $2x^2+8x+8<0$ is $\varnothing$.
The solution set of $2x^2+8x+8\leq 0$ is $\{-2\}$.
The solution set of $2x^2+8x+8>0$ is $\mathbb{R}\setminus\{-2\}$.
The solution set of $2x^2+8x+8\geq 0$ is $\mathbb{R}$.

Example 14.

Find the solution set of each of the inequalities $-2x^2-6x+8<0$, $-2x^2-6x+8\leq 0$, $-2x^2-6x+8>0$, and $-2x^2-6x+8\geq 0$.

Solution

Let $y=-2x^2-6x+8$. Comparing the function $y=-2x^2-6x+8$ to the standard form $y=ax^2+bx+c$, we get $a=-2, b=-6, c=8$. Then

$$b^2-4ac=36-4(-2)(8)=100>0;$$

$$y=-2x^2-6x+8=-2(x+4)(x-1)$$

Since $a<0$ and $b^2-4ac>0$, the parabola of the function opens down and cuts the x-axis at the two points $(-4,0)$ and $(1,0)$. Therefore when $x<-4$ or $x>1$, the values of y are negative; when $x=-4$ and $x=1$, the values of y are equal to zero; when $-4<x<0$, the values of y are positive.

The solution set of $-2x^2-6x+8<0$ is $\{x|\, x<-4 \text{ or } x>1\}$.
The solution set of $-2x^2-6x+8\leq 0$ is $\{x|\, x\leq -4 \text{ or } x\geq 1\}$.
The solution set of $-2x^2-6x+8>0$ is $\{x|\, -4<x<1\}$.
The solution set of $-2x^2-6x+8\geq 0$ is $\{x|\, -4\leq x\leq 1\}$.

Exercise 5.6

Find the solution set of each of the quadratic inequality:

1. $2x^2 - 3x + 2 > 0$	2. $5x^2 + 2 < 0$	3. $12x^2 + 7x - 10 \leq 0$
4. $2x^2 + 7x - 4 > 0$	5. $2x^2 - 8x + 8 \geq 0$	6. $5x^2 + 3x - 2 > 0$

Chapter 6

Absolute Value Functions

For a real number x, the **absolute value** or **modulus of** x, which is written as $|x|$, is defined as follow:

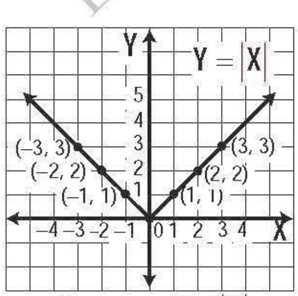
$$|x| = \begin{cases} x, & \text{when } x \geq 0, \\ -x, & \text{when } x < 0. \end{cases}$$

For example, $|3| = 3$ because $3 > 0$ and $|-3| = -(-3) = 3$ because $-3 < 0$.

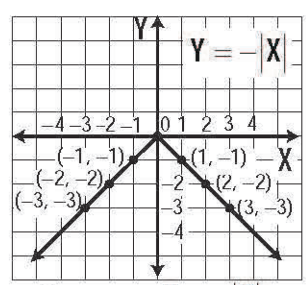
Absolute value of x can be seen as $|x| = \sqrt{x^2}$. For example,

$$|3| = \sqrt{3^2} = \sqrt{9} = 3 \text{ and } |-3| = \sqrt{(-3)^2} = \sqrt{9} = 3.$$

Functions like $y = |x|$ and $y = -|x|$ are called **absolute value functions**.



Graph of Y = |X|



Graph of Y = −|X|

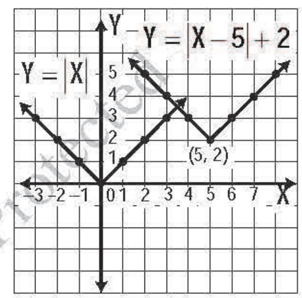
Note that the graph of $y = -|x|$ is the **reflection on the x-axis** of the graph of $y = |x|$.

6.1 Graph of the Function $y = |x - h| + k$

Graph of the absolute value function $y = |x - h| + k$ can be seen as the **translation** of h-units horizontally and k-units vertically of the graph $y = |x|$ as shown in the example below.

Example 1.

Consider the function $y = |x - 5| + 2$. The graph of $y = |x - 5| + 2$ is the translation of **positive** 5 units **horizontally** and **positive** 2 units **vertically** of the graph $y = |x|$.

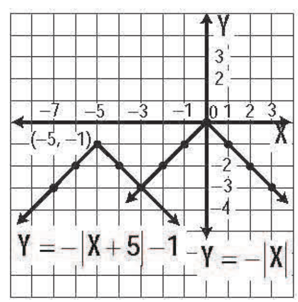


6.2 Graph of the Function $y = -|x - h| + k$

Graph of the absolute value function $y = -|x - h| + k$ can be seen as the **translation** of h-units horizontally and k-units vertically of the graph $y = -|x|$ as shown in the example below.

Example 2.

Consider the function $y = -|x + 5| - 1$. The graph of $y = -|x + 5| - 1$ is the translation of **negative** 5 units **horizontally** and **negative** 1 unit **vertically** of the graph $y = -|x|$.

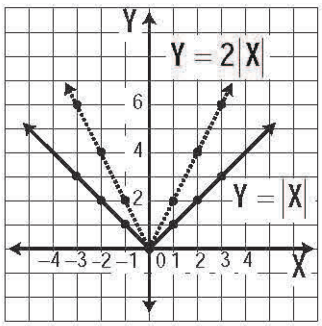


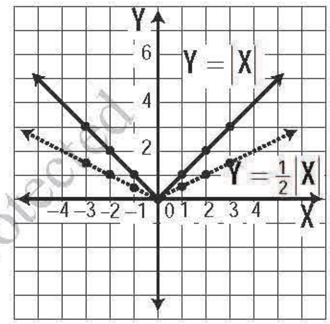
Exercise 6.1

1. Compare the graphs of the following functions to the graph of $y = |x|$.
   (a) $y = |x - 3| - 2$ (b) $y = |x + 1| + 3$ (c) $y = |x - 2| + 3$
2. Compare the graphs of the following functions to the graph of $y = -|x|$.
   (a) $y = -|x + 3| + 2$ (b) $y = -|x - 4| + 1$ (c) $y = -|x + 4| - 1$

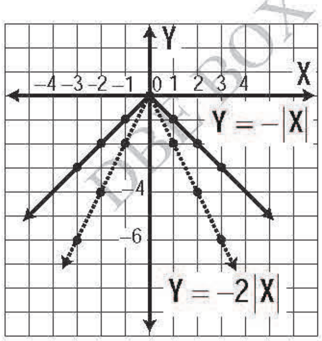
6.3 Graph of the Function $y = a|x|$

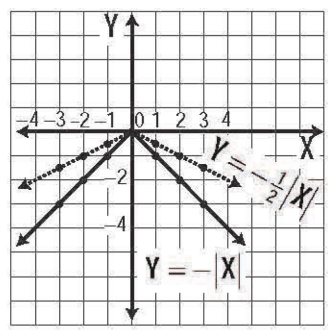
When a is **positive**, the graph of the function $y = a|x|$ is the vertical stretch of the scale factor a of the function $y = |x|$ as in the following figures.





When a is **negative**, the graph of the function $y = a|x|$ is the vertical stretch of the scale factor $|a|$ of the function $y = -|x|$ as in the following figures.





One can see that if the point (p, q) is on the graph $y = |x|$, then the point (p, aq) is on the graph $y = a|x|$. For example,

the point $(-2, 2)$ is on the graph $y = |x|$, then

- the point $(-2, 4)$ is on the graph $y = 2|x|$
- the point $(-2, 1)$ is on the graph $y = \frac{1}{2}|x|$

- the point $(-2, -4)$ is on the graph $y = -2|x|$
- the point $(-2, -1)$ is on the graph $y = -\frac{1}{2}|x|$

Since

$$\boxed{|pq| = \sqrt{(pq)^2} = \sqrt{p^2q^2} = \sqrt{p^2}\sqrt{q^2} = |p||q|},$$

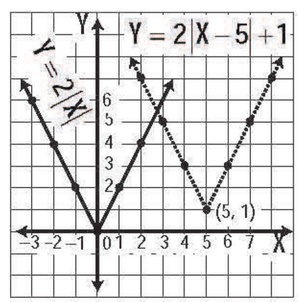
the function $y = |ax|$ can be seen as $y = |a||x|$.

6.4 Graph of the Function $y = a|x - h| + k$

The graph of the $y = a|x - h| + k$ can be seen as the **translation** of h-units horizontally and k-units vertically of the graph of $y = a|x|$ as shown in the examples below.

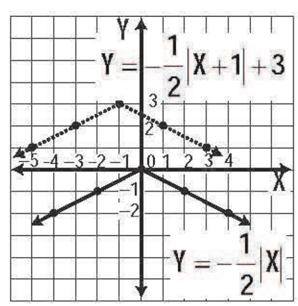
Example 3.

Consider the function $y = 2|x - 5| + 1$. The graph of $y = 2|x-5|+1$ is the translation of **positive** 5 units **horizontally** and **positive** 1 unit **vertically** of the graph $y = 2|x|$.



Example 4.

Consider the function $y = -\frac{1}{2}|x + 1| + 3$. The graph of $y = -\frac{1}{2}|x + 1| + 3$ is the translation of **negative** 1 unit **horizontally** and **positive** 3 units **vertically** of the graph $y = -\frac{1}{2}|x|$.



Features of the graph $y = a|x-h|+k$

- vertex: (h, k)
- axis of symmetry: $x = h$
- domain = the set $\mathbb{R}$ of all real numbers
- range $= \begin{cases} \{y \mid y \geq k\} & \text{when } a > 0 \\ \{y \mid y \leq k\} & \text{when } a < 0 \end{cases}$
- $\begin{cases} \text{when } a > 0, & k \text{ is the minimum value of } y \\ \text{when } a < 0, & k \text{ is the maximum value of } y \end{cases}$

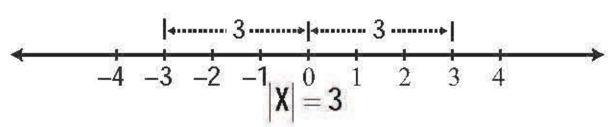
Exercise 6.2

Draw the graph of each of the following functions. Find also vertex, axis of symmetry, and range of each of the functions.

1. $y = |x| + 3$
2. $y = -|x+3| - 2$
3. $y = 3|x-4| + 1$
4. $y = -\frac{1}{2}|x| + 3$
5. $y = \frac{1}{2}|x-4| + 1$
6. $y = -\frac{1}{2}|x-3| - 1$

6.5 Equation $|x - p| = q$

By the definition of $|x|$, solutions of the equation $|x| = 3$ are $x = 3$ and $x = -3$. On the number line, the distance between 0 and x is $|x - 0| = |x|$. So solving equation like $|x| = 3$ is finding the points on number line which are distant 3 from 0.



Now we will consider the equation $|x - p| = q$.

- When $q < 0$, $|x - p| = q$ has no solution.
- When $q = 0$, $|x - p| = 0$ has only one solution p.

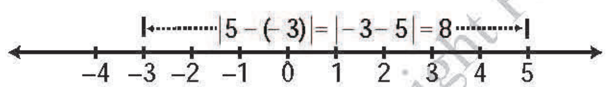
- When $q > 0$, the equation $|x - p| = q$ can be seen as

$$x - p = q \text{ or } x - p = -q.$$

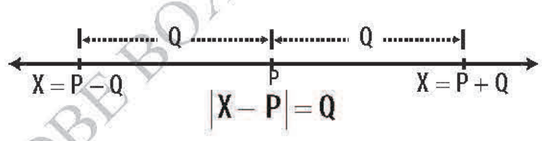
So the solutions are $x = p + q$ and $x = p - q$.

> On the number line, the distance between p and q is $|p - q|$ or $|q - p|$.

For example, the distance between -3 and 5 can be calculated as $|-3-5| = 8$ or $|5-(-3)| = 8$.



> On the number line, solving $|x - p| = q$ means finding the points x which are distant q from the point p.



Exercise 6.3

1. Find the solutions of the following equations. Illustrate each of the equations on the number line.

   (a) $|x - 5| = 3$ (b) $|x + 3| = 2$ (c) $|x - 4| = 1$

2. Find the solutions of the following equations.

   (a) $|2x - 5| = 4$ (b) $|-2x - 4| = 3$ (c) $|5x + 10| = 2$

6.6 Inequalities Involving $|x-p|$

In this section, inequalities of the forms

$$|x-p|<q, |x-p|\leq q, |x-p|>q \text{ and } |x-p|\geq q$$

will be considered. First note that

$$|x-p| = \begin{cases} x-p, & \text{when } x-p\geq 0, \\ -(x-p), & \text{when } x-p<0. \end{cases}$$

Case 1. $|x-p|<q$

When $q\leq 0$, the inequality $|x-p|<q$ has no solution.

When $q>0$, the inequality $|x-p|<q$ can be solved as follows:

- If $x-p\geq 0$, then $|x-p|=x-p<q$. So $0\leq x-p<q$.
- If $x-p<0$, then $|x-p|=-(x-p)$. So $-(x-p)<q$, thus $x-p>-q$. We have $-q<x-p<0$.

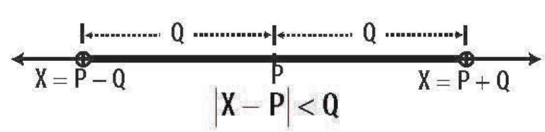
Therefore

$$-q<x-p<q$$

and then $p-q<x<p+q$.

The solution set of $|x-p|<q$ when $q>0$ is $\{x|\, p-q<x<p+q\}$

Since $|x-p|$ is the distance between x and p, solving $|x-p|<q$ means finding the points x which are distant less than q from the point p.



Case 2. $|x-p| \leq q$

When $q < 0$, the inequlaity $|x-p| \leq q$ has no solution.

When $q \geq 0$, the solution set of $|x-p| \leq q$ is $\{x|p-q \leq x \leq p+q\}$. It can be found as in the case of $|x-p| < q$.

The solution set of $|x-p| \leq q$ when $q \geq 0$ is $\{x|\, p-q \leq x \leq p+q\}$.

Case 3. $|x-p| > q$

When $q < 0$, the solution set of $|x-p| > q$ is $\mathbb{R}$.

When $q \geq 0$, the inequality $|x-p| > q$ can be solved as follows:

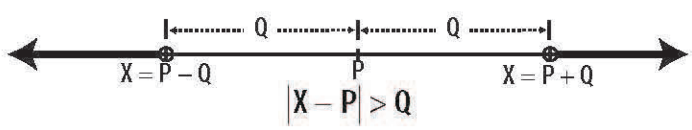
- If $x-p \geq 0$, then $|x-p| = x-p$. So $x-p > q$, and hence $x > p+q$.
- If $x-p < 0$, then $|x-p| = -(x-p)$. So $-(x-p) > q$, thus $x-p < -q$, and hence $x < p-q$.

Therefore

$$x < p-q \quad \text{or} \quad x > p+q.$$

The solution set of $|x-p| > q$, when $q \geq 0$ is $\{x|\, x < p-q \text{ or } x > p+q\}$.

Since $|x-p|$ is the distance between x and p, solving $|x-p| < q$ means finding the points x which are distant greater than q from the point p.



Case 4. $|x-p| \geq q$

When $q < 0$, the solution set of $|x-p| \geq q$ is $\mathbb{R}$.

When $q \geq 0$, the solution set of $|x-p| \geq q$ is $\{x|\, x \leq p-q \text{ or } x \geq p+q\}$. It can be found as in the case of $|x-p| > q$.

The solution set of $|x-p| \geq q$, when $q \geq 0$ is $\{x|\, x \leq p-q \text{ or } x \geq p+q\}$.

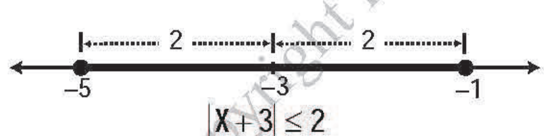
Example 5.

Find the solution set of the inequality $|x+3| \leq 2$. Illustrate the inequality on the number line.

Solution

$$\begin{aligned} |x+3| &\leq 2 \\ -2 \leq x+3 &\leq 2 \\ -5 \leq x &\leq -1 \end{aligned}$$

The solution set is $\{x|\, -5 \leq x \leq -1\}$.



Example 6.

Find the solution set of the inequality $|2x-3| > 5$.

Solution

$$\begin{array}{ccc} |2x-3| & > & 5 \\ 2x-3 < -5 & \text{or} & 2x-3 > 5 \\ x < -1 & \text{or} & x > 4 \end{array}$$

The solution set is $\{x|\, x < -1 \text{ or } x > 4\}$.

Exercise 6.4

1. Find the solution sets of the following inequalities. Illustrate each of the inequalities on the number line.
   (a) $|x-1| < 3$ (b) $|x+5| > 2$ (c) $|x-2| \leq 4$
   (d) $|x+1| \geq 2$ (e) $|x-3| > 0$ (f) $|x+3| \leq 0$

2. Find the solution sets of the following inequalities.
   (a) $|2x-1| < 4$ (b) $|3x+5| > 6$ (c) $|4x-2| \leq 4$
   (d) $|2x+1| \geq 2$ (e) $|2x-3| > 0$ (f) $|5x+3| \leq 0$

Chapter 7

Probability

Probability as a general concept can be defined as the chance of an event occurring. Many people are familiar with probability from observation or playing games of chance, such as card games, slot machines or lotteries. In addition to being used in games of chance, probability theory is used in the fields of insurance, investments, weather forecasting and in various other areas. Blaise Pascal (1623-1662) and Pierre de Fermat (1601-1665) laid the foundation of probability theory.

7.1 Calculating Probability

In this chapter, we are concerned with experiments such as tossing a coin, rolling a die, drawing a ball from a bag and so on. These experiments are said to be *random experiments* or chance experiments (for brevity, we will simply call them *experiments* instead), since exact results of these experiments cannot be predicted. A single specific result of an experiment is called an *outcome*. The set of all possible outcomes of an experiment is called the *sample space*. An *event* is a subset of a sample space.

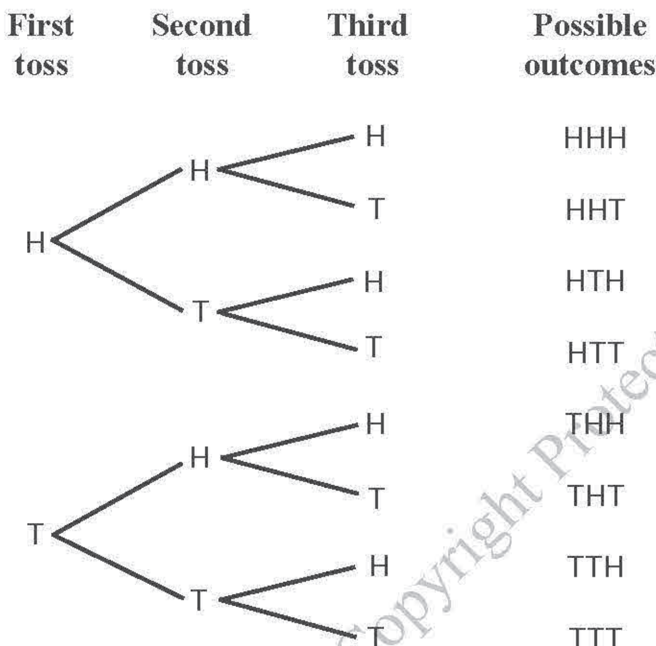
For the experiment "*tossing a coin*", there are two possible outcomes: head and tail (which will be denoted by H and T respectively). The sample space is {H, T}, and the events are ∅, {H}, {T} and {H, T}. But for the experiment "*tossing two coins*", the sample space is {(H, H), (H, T), (T, H), (T, T)} , which can also be expressed as {HH, HT, TH, TT}. The event {(H, T)} means that "*the first toss shows head and the second toss shows tail.*"

For the experiment "*rolling a die*", there are six possible outcomes: 1, 2, 3, 4, 5, 6. The sample space is {1, 2, 3, 4, 5, 6}. Some of the events are {3}, {4, 5, 6} and {2, 3, 5}, which can respectively be described as "*the result is* 3", "*the result is at least* 4" and "*the result is a prime number*".

The following **table** represents the sample space for rolling two dice. The first part of an ordered pair in the table represents the number appears on the first die and the second part represents the corresponding number on the second die. Sample spaces of similar experiments can also be obtained by constructing such tables.

First Die \ Second Die	**1**	**2**	**3**	**4**	**5**	**6**
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Sometimes, it is convenient to use a **tree diagram** to list all possible outcomes in a sample space. The following tree diagram displays the sample space for tossing a coin three times.



Let S be a finite sample space for an experiment such that all outcomes are *equally likely*, which means that they are random and have an equal likelihood of occurrence. Then the **probability of an event** A, denoted by $\mathrm{P}(A)$, in the sample space S, is defined by

$$\mathrm{P}(A) = \frac{\text{number of outcomes in the event } A}{\text{number of outcomes in the sample space } S}$$

In symbols,

$$\mathrm{P}(A) = \frac{\mathrm{n}(A)}{\mathrm{n}(S)}$$

where $\mathrm{n}(A)$ denotes the number of elements in the event A and $\mathrm{n}(S)$ denotes the number of outcomes in the sample space S.

Notice that:

- For any event A, P(A) is a real number such that $0 \leq P(A) \leq 1$.
- $P(\varnothing)=0$ and $P(S)=1$. (This means that the probability of an *impossible event* is 0 and that of a *sure event* is 1.)
- For any event A,
  $P(A) + P(\text{not}A) = 1$. **(Rule for complementary events)**

Example 1.

Find the probability of randomly selecting a red pen from a box that contains 2 red pens, 4 blue pens and 3 yellow pens.

Solution

Let A be the event of selecting a red pen.

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in the sample space}} = \frac{2}{9}.$$

Example 2.

If a whole number from 1 to 20 both inclusive is randomly selected, and if each number has an equal chance of being selected, what is the probability that the number will be

(a) even? (b) greater than 1? (c) prime?

Solution

The sample space is $S = \{1, 2, 3, ..., 20\}$. $n(S) = 20$.

(a) The event is $E_1 = \{2, 4, 6, ..., 20\}$. $n(E_1) = 10$.

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{10}{20} = \frac{1}{2}$$

(b) The event is $E_2 = \{2, 3, 4, ..., 20\}$. $n(E_2) = 19$.

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{19}{20}$$

(c) The event is $E_3 = \{2, 3, 5, 7, 11, 13, 17, 19\}$. $n(E_3) = 8$.

$$P(E_3) = \frac{n(E_3)}{n(S)} = \frac{8}{20} = \frac{2}{5}$$

Example 3.

In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood and, 2 had type AB blood. Find the following probabilities.

(a) A person has type O blood.

(b) A person has type A or type B blood.

(c) A person has neither type A nor type O blood.

(d) A person does not have type AB blood.

Solution

Type of blood	Frequency
A	22
B	5
AB	2
O	21
Total	50

(a) $P(O) = \frac{21}{50}$

(b) $P(A \text{ or } B) = \frac{27}{50}$

(c) $P(\text{neither A nor O}) = \frac{7}{50}$

(d) $P(\text{not AB}) = \frac{48}{50} = \frac{24}{25}$

Example 4.

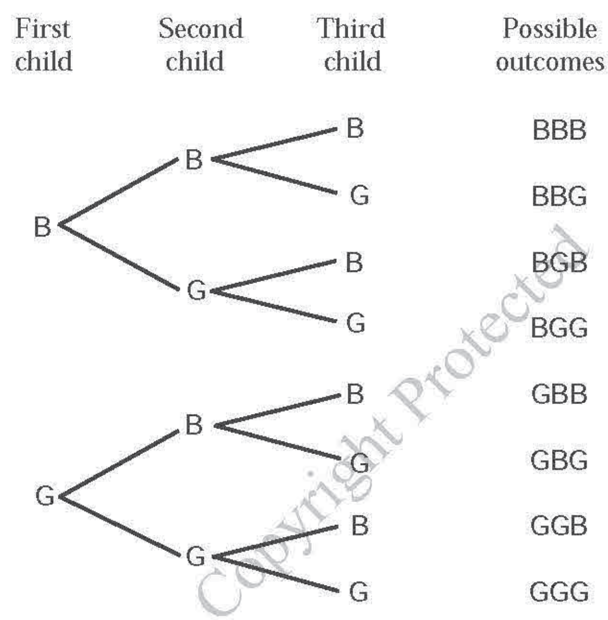
A family has three children. Find the probabilities of

(a) all boys, (b) exactly two boys,

(c) at most two boys, (d) at least one girl,

(e) at least one boy and at least one girl.

Solution



The sample space consists of 8 outcomes.

(a) The event is {BBB}.

P(all boys) $= \frac{1}{8}$

(b) The event is {BBG, BGB, GBB}.

P(exactly two boys) $= \frac{3}{8}$

(c) The event is { BBG, BGB, BGG, GBB, GBG, GGB, GGG}.

P(at most two boys) $= \frac{7}{8}$

(d) The event is {BBG, BGB, BGG, GBB, GBG, GGB, GGG}.

P(at least one girl) $= \frac{7}{8}$

(or) P(at least one girl) $= 1 - P(\text{all boys}) = 1 - \frac{1}{8} = \frac{7}{8}$

(by using rule for complementary events)

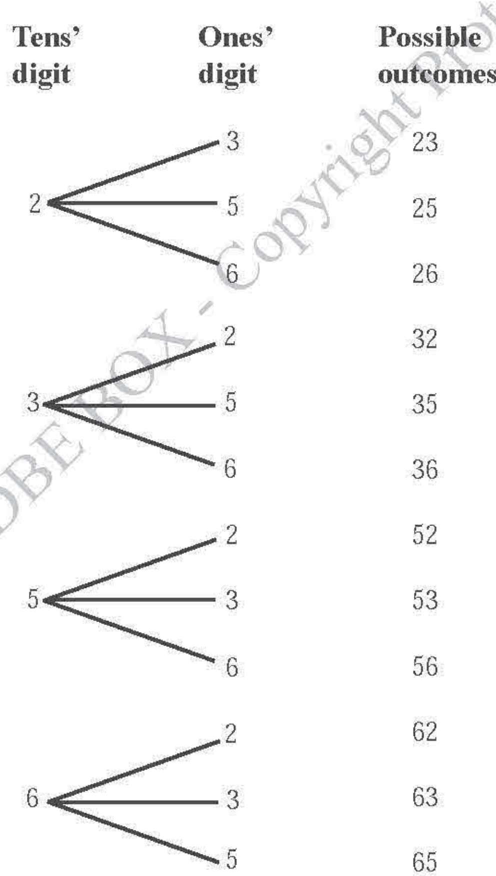
(e) The event is {BBG, BGB, BGG, GBB, GBG, GGB}.

P(at least one boy and at least one girl) $= \frac{6}{8} = \frac{3}{4}$

Example 5.

Draw a tree diagram to list all possible two-digit numerals which can be formed by using the digits 2, 3, 5 and 6 without repeating any digit. If one of these numerals is chosen at random, find the probability that it is divisible by 13. Find also the probability that it is either a prime or a perfect square. Find the probability that none of its digits is 6.

Solution



The sample space consists of 12 outcomes.

The event containing numerals divisible by 13 is $\{26, 52, 65\}$.

P(a numeral which is divisible by 13) $= \frac{3}{12} = \frac{1}{4}$

The event containing numerals which are primes or perfect squares is $\{23, 25, 36, 53\}$.

P(a numeral which is prime or a perfect square) $= \frac{4}{12} = \frac{1}{3}$

The event containing numerals which do not have digit 6 is $\{23, 25, 32, 35, 52, 53\}$.

P(a numeral none of its digits is 6) $= \frac{6}{12} = \frac{1}{2}$.

Example 6.

Two fair dice are thrown and the numbers appeared on top faces are recorded. Find the probability of each event:

(a) The first die shows 5.

(b) The sum of the numbers on the dice is 7.

(c) The product of the numbers on the two dice is greater than 24.

Solution

		Second Die					
		1	**2**	**3**	**4**	**5**	**6**
First Die	**1**	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

The sample space contains 36 outcomes.

(a) The event is $\{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$.

P(5 on the first die) $= \frac{6}{36} = \frac{1}{6}$.

(b) The event is $\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$.

P(the sum is 7) $= \frac{6}{36} = \frac{1}{6}$.

(c) The event is $\{(5, 5), (5, 6), (6, 5), (6, 6)\}$.
P(product is greater than 24) $= \frac{4}{36} = \frac{1}{9}$.

Exercise 7.1

1. A letter is chosen at random from the letters of the word ORANGE. What is the probability that it is a vowel?

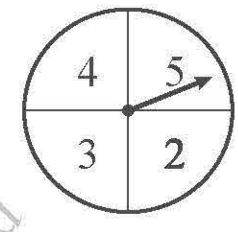
2. A bag contains 10 red balls and 30 black balls.
   (a) If a ball is drawn at random, what is the probability of getting a red ball?
   (b) Suppose the first ball drawn at random is red and is not replaced. If another ball is drawn at random, what is the probability that it will again be red?

3. How many three-digit numerals can be formed from 1, 5 and 7 without repeating any digit? Find the probability of a numeral which begins with 1.

4. A box contains five cards numbered as 2, 3, 4, 5 and 9. A card is chosen, the number is recorded, and the card is not replaced. Then another card is chosen and the number is recorded. Draw a tree diagram to get the possible outcomes. Find the probabilities of
   (a) getting two prime numbers,
   (b) getting two odd numbers and
   (c) getting a pair of numbers whose sum is a prime number.

5. A box contains four marbles of two blue, one red and one yellow. A marble is chosen, the colour is recorded, and the marble is not replaced. Then another marble is chosen and the colour is recorded. Draw a tree diagram to determine possible outcomes. Hence, find the probabilities of
   (a) getting two blue marbles and
   (b) getting two different colours.

6. A spinner is equally likely to point to any one of the numbers 2, 3, 4 and 5. Make a table of ordered pairs (first spin, second spin).
   Find the probability of
   (a) two odd numbers,
   (b) an even number followed by an odd number.



7. A coin is tossed and then a die is thrown. Head or tail and the number turns up are recorded each time. Draw a tree diagram and list the possible outcomes. Hence, find the probability that head and 6 turns up.

7.2 Probabilities of Combined Events

Two events are **independent** if the occurrence of any one of them does not affect the probability of the other. For example, when two fair coins are tossed, the event of getting head on the first coin and the event of getting tail on the second coin are independent.

Suppose an experiment consists of choosing a marble from a bag containing 3 red, 5 green, 2 blue and 6 yellow marbles, and rolling a die. Let us consider the events:

1. getting a blue marble from the bag

2. getting a prime number on the die

3. getting *both* a blue marble from the bag *and* a prime number on the die

Then the probability of the first event is $\frac{1}{8}$, the probability of the second event is $\frac{1}{2}$ and the probability of the third event is $\frac{1}{16}$. The first two events are independent. In fact, the product of the probabilities of the first two events *is the same as* the probability of the third (combined) event. The general rule for independent events is as follow:

Multiplication Rule :

A and B are *independent* events in a sample space *if and only if* $\mathrm{P}(A \text{ and } B) = \mathrm{P}(A) \times \mathrm{P}(B)$.

Example 7.

A bag contains 3 red balls, 2 blue balls and 5 white balls. A ball is selected at random and its colour noted. Then it is replaced. A second ball is selected and its colour noted. Find the probabilities of:
(a) selecting 2 blue balls,
(b) selecting 1 blue ball and then 1 white ball,
(c) selecting 1 red ball and then 1 blue ball.

Solution

The bag contains 10 balls (3 red, 2 blue and 5 white).

(a)
$$\begin{aligned}\text{P(2 blue balls)} &= \text{P(1st ball is blue and 2nd ball is blue)}\\ &= \text{P(1st ball is blue)} \times \text{P(2nd ball is blue)}\\ &= \frac{2}{10} \times \frac{2}{10}\\ &= \frac{1}{25}\end{aligned}$$

(b)
$$\begin{aligned}\text{P(1 blue ball and then 1 white ball)} &= \text{P(1st ball is blue)} \times \text{P(2nd ball is white)}\\ &= \frac{2}{10} \times \frac{5}{10}\\ &= \frac{1}{10}\end{aligned}$$

(c)
$$\begin{aligned}\text{P(1 red ball then 1 blue ball)} &= \text{P(1st ball is red and 2nd ball is blue)}\\ &= \text{P(1st ball is red)} \times \text{P(2nd ball is blue)}\\ &= \frac{3}{10} \times \frac{2}{10}\\ &= \frac{3}{50}\end{aligned}$$

The events in above example are independent. But in some cases, the events considered *may not* be independent, that is, *the occurrence of the first event changes the probability of the occurrence of the second event.* These conditions can be seen in the following example.

Example 8.
A bag contains 9 red marbles and 3 green marbles. For each case below, find the probability of randomly selecting a red marble on the first draw and a green marble on the second draw.

(a) The first marble is replaced.
(b) The first marble is not replaced.

Solution

There are 12 marbles (9 red and 3 green).

(a) P(1st marble is red and 2nd marble is green)

$$= P(\text{1st marble is red}) \times P(\text{2nd marble is green})$$
$$= \frac{9}{12} \times \frac{3}{12} = \frac{3}{16}$$

(b) P(1st marble is red and 2nd marble is green)

$$= P(\text{1st marble is red}) \times P(\text{2nd marble is green})$$
$$= \frac{9}{12} \times \frac{3}{11} = \frac{9}{44}$$

Example 9.

At a teachers' conference, there are 4 English teachers, 3 Mathematics teachers, and 5 Science teachers. If 4 teachers are selected for a committee, find the probability that at least one is a science teacher.
Solution

There are 12 teachers (4 English, 3 Mathematics and 5 Science).

P(at least one is a Science teacher)

$= 1-$ P(none of the 4 teachers is a science teacher)

=1− P(1st teacher is not a science teacher,
2nd teacher is not a science teacher,
3rd teacher is not a science teacher and
4th teacher is not a science teacher)

$$= 1 - \frac{7}{12} \times \frac{6}{11} \times \frac{5}{10} \times \frac{4}{9}$$

$$= 1 - \frac{7}{99} \quad = \frac{92}{99}$$

> **Addition Rule 1 :**
> For any two events A and B in a sample space,
> $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

Example 10.

In a hospital unit, there are 8 nurses and 5 doctors. Among them, there are 7 nurses and 3 doctors are females. If a staff person is selected, find the probability that the staff is a nurse or a male.

Solution

Staff	Female	Male	Total
Nurses	7	1	8
Doctors	3	2	5
Total	10	3	13

$$\begin{aligned} P(\text{nurse or male}) &= P(\text{nurse}) + P(\text{male}) - P(\text{male nurse}) \\ &= \frac{8}{13} + \frac{3}{13} - \frac{1}{13} \\ &= \frac{10}{13} \end{aligned}$$

Two events A and B are **mutually exclusive** if they cannot occur jointly, that is, they do not have common outcomes. For example, the occurrence of head and the occurrence of tail, in a single flip of a coin, are mutually exclusive.

If A and B are mutually exclusive events, then $P(A \text{ and } B) = 0$.

> **Addition Rule 2** :
>
> If A and B are *mutually exclusive* events in a sample space, then $P(A \text{ or } B) = P(A) + P(B)$.

Example 11.

A box contains 3 strawberry doughnuts, 4 jelly doughnuts, and 5 chocolate doughnuts. If a doughnut is selected at random, find the probability that it is either a strawberry doughnut or a chocolate doughnut.

Solution

The box contains 12 doughnuts (3 strawberry, 4 jelly and 5 chocolate).

P(strawberry doughnut or chocolate doughnut)

$$= P(\text{strawberry doughnut}) + P(\text{chocolate doughnut})$$

$$= \frac{3}{12} + \frac{5}{12}$$

$$= \frac{8}{12} = \frac{2}{3}$$

Exercise 7.2

1. At a conference, there are 7 mathematics instructors, 5 computer science instructors, 3 statistics instructors and 4 science instructors. If an instructor is selected, find the probability of getting a science instructor or a math instructor.

2. Two dice are rolled. Find the probability of getting
   (a) a sum greater than 8 or a sum less than 3.
   (b) a product greater than 9 or a product less than 16.

3. A bag contains 15 discs of which 3 are white, 5 are red and 7 are blue. Two discs are to be drawn at random, in succession, each being replaced after its colour has been noted. Calculate the probability that the two discs will be of the same colour.

4. In a survey about a change in public policy, 100 people were asked if they favor the change, oppose the change, or have no opinion about the change. The responses are indicated as below:

	The Youth	Senior Citizens	Total
Favor	18	9	27
Oppose	12	25	37
No opinion	20	16	36
Total	50	50	100

Find the probability that a randomly selected respondent to this survey *oppose* or *has no opinion* about the change policy.

5. The probabilities that the student A and B pass an examination are $\frac{2}{3}$ and $\frac{3}{4}$ respectively. Find the probabilities that:
(a) both A and B pass the examination.
(b) exactly one of A and B passes the examination.

6. Three groups of children consist of 3 boys and 1 girl, 2 boys and 2 girls, and 1 boy and 3 girls respectively. If a child is chosen from each group, find the probability that 1 boy and 2 girls are chosen.

7.3 Calculation of Expected Frequency

The **expected frequency** of an event is the number of times that we predict the event will occur, in a given number of trials, based on a calculation using probabilities. It can be calculated by the formula:

$$\begin{matrix} \textit{Expected frequency} \\ \textit{of an event} \end{matrix} = \begin{matrix} \textit{Probability} \\ \textit{of the event} \end{matrix} \times \begin{matrix} \textit{Number} \\ \textit{of trials} \end{matrix}$$

Consider tossing a coin 1000 times. Since each toss is a trial, the number of trials is 1000. Since the probability of getting head in each trial is 0.5, the expected frequency of getting head is 500 (= 0.5 ×1000). This means that we can expect 500 heads occurring in 1000 trials. The expected frequency is based on probability. On the other hand, you can actually flip a coin 1000 times and observe the frequency of head. Do you think that these two frequencies are the same?

Example 12.

Traffic analysis found that the probability that a motorist will turn right at the intersection is $\frac{1}{3}$. Out of 300 motorists, how many would you expect to turn right at that intersection?

Solution

$\text{P(turning right)} = \frac{1}{3}$

Number of trials or motorists = 300

$$\text{Expected number of motorists turning right at the intersection} = \frac{1}{3} \times 300 = 100$$

Example 13.

A spinner is equally to point to any one of the numbers 1, 2, 3, 4, 5, 6, 7. What is the probability of scoring a number divisible by 3? If the arrow is spun 700 times, how many would you expect a number not divisible by 3?

Solution

Among the given numbers, the numbers 3 and 6 are divisible by 3.

$\text{P(a number divisible by 3)} = \frac{2}{7}$

$\text{P(a number not divisible by 3)} = 1 - \frac{2}{7} = \frac{5}{7}$

The number of trials = 700

$$\text{Expected frequency of a number which is not divisible by 3} = \frac{5}{7} \times 700 = 500.$$

Exercise 7.3

1. After a large number of tossing a pin, the probability of 'pin up' was estimated to be 0.3. In 400 more trials, how many times would 'pin up' be expected?
2. If a die is rolled 60 times, what is the expected frequency of
   (a) 1 turns up?
   (b) a number divisible by 3 turns up?
   (c) a factor of 6 turns up?
3. Two honest coins are tossed. How many times would you expect to obtain two heads in 200 trials?
4. The probability of scoring 12 when throwing two dice at once is $\frac{1}{36}$. If such an experiment is repeated 720 times, what is the expected frequency of the score not being 12?
5. A spinner is equally likely to point to any one of the numbers: 1, 2, 3, ... , 10.
   (a) What is the probability of an odd number?
   (b) What is the probability of an even number?
   (c) If the arrow is spun 1000 times, what final score would you expect if all the individual scores are added together?

Chapter 8

Similarity

This chapter will be concerned with the study of the similarity of polygons especially triangles. We characterize the triangles which are similar by comparing their angles and their sides. We observe that the theorems on similar triangles depend very much on parallel lines that divides transverse line proportionally. Therefore ratio and proportion concepts should be reviewed before similarity.

The following properties of proportions are especially useful in geometry.

1. **The Means-extremes Product Property:** The product of the means equals the product of the extremes. If $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$, where a, d are extremes and b, c are means. If $\frac{a}{b} = \frac{b}{c}$, then b is called the **geometric mean** of a and c.

2. **Invertendo Property:** In a proportion, the ratios may be inverted. If $\frac{a}{b} = \frac{c}{d}$ then $\frac{b}{a} = \frac{d}{c}$.

3. **Alternando Property:** In a proportion, the means (or extremes) may be interchanged. If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{c} = \frac{b}{d}$ (or $\frac{d}{b} = \frac{c}{a}$).

4. **Componendo Property:** If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{b} = \frac{c+d}{d}$.

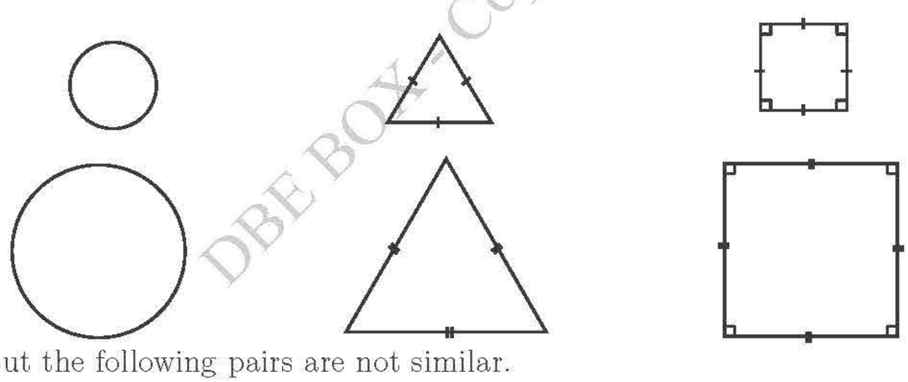
5. **Dividendo Property:** If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a-b}{b} = \frac{c-d}{d}$.

6. **Componendo and Dividendo Property:** If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

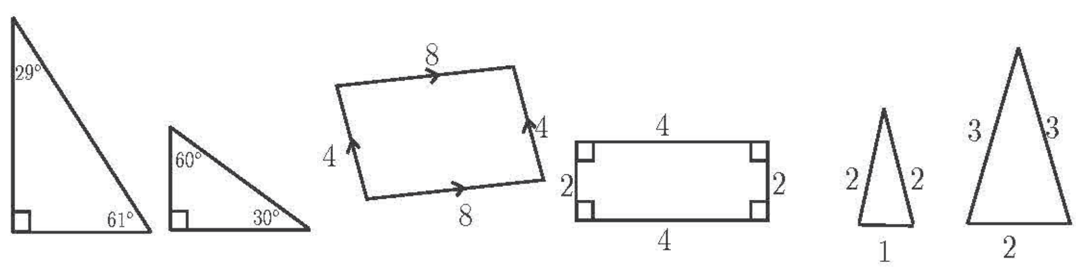
8.1 Ideas of Similarities and Similar Triangles

We now learn some properties of geometric figures that are of the same shape but not necessarily of the same size. Such figures are said to be similar.

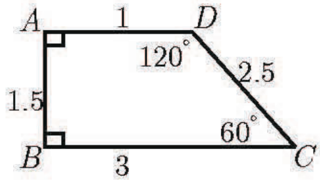
The following pairs of figures are similar.

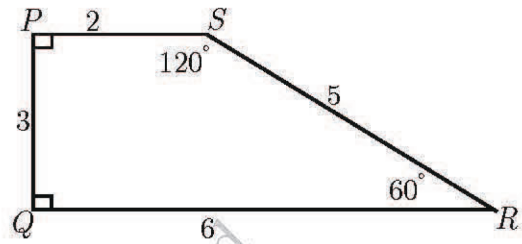


But the following pairs are not similar.



To consider the similarity of two figures we have to find the relation between their corresponding sides and corresponding angles.





Consider the correspondence $ABCD \leftrightarrow PQRS$, which matches A with P, B with Q, C with R and D with S.
We notice that the corresponding angles are equal, that is

$$\boxed{\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R, \angle D = \angle S.}$$

Although corresponding sides are not equal by forming the ratios of their lengths, we have

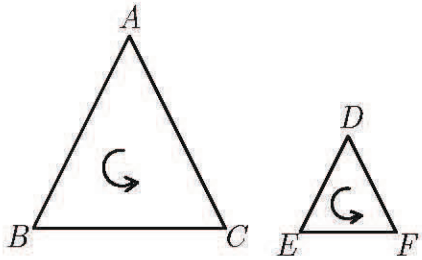
$$\frac{AB}{PQ} = \frac{1.5}{3} = \frac{1}{2}, \frac{BC}{QR} = \frac{3}{6} = \frac{1}{2}, \frac{CD}{RS} = \frac{2.5}{5} = \frac{1}{2}, \frac{AD}{PS} = \frac{1}{2}$$

$$\boxed{\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{AD}{PS}.}$$

In general, if all the ratios of the lengths of the corresponding sides are equal, we say that the **corresponding sides are proportional.**

Thus the conditions for two polygons to be similar are
(1) corresponding angles must be equal, and
(2) corresponding sides must be proportional.

Definition. Two triangles whose corresponding angles are equal and whose corresponding sides are proportional are said to be **similar.**



That is, in $\triangle ABC$ and $\triangle DEF$,
if $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, and $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$
then $\triangle ABC$ and $\triangle DEF$ are similar.

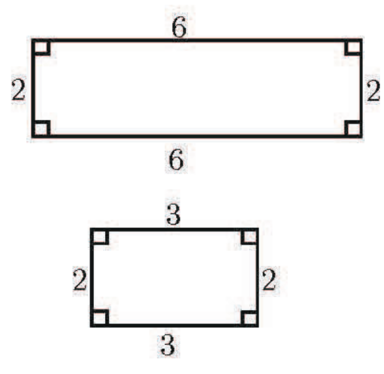
The symbol " $\sim$ " will be used to denote similarity.
That is, if $\triangle ABC$ and $\triangle DEF$ are similar, we can write $\triangle ABC \sim \triangle DEF$.

As a convention, similar triangles and similar polygons will be named so that the order of letters indicates the correspondence between the two figures. **Thus the statement $\triangle ABC \sim \triangle DEF$ will always indicate that which angles are corresponding angles and which sides are corresponding sides.**

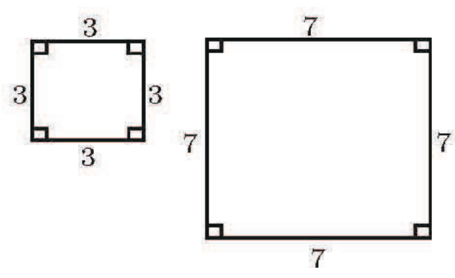
Exercise 8.1

1. State why the two polygons are, or are not, similar.

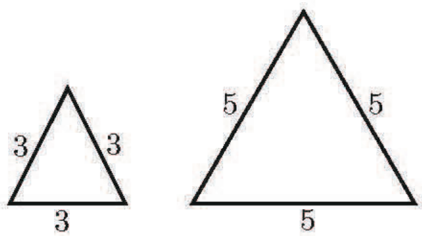
(a)



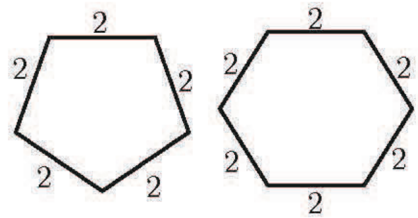
(b)



(c)



(d)



2. Complete the proportions.
   (a) If $\triangle ABC \sim \triangle DEF$ then $\frac{AB}{?} = \frac{BC}{?} = \frac{?}{DF}$.

   (b) If $\triangle GHI \sim \triangle KLM$ then $\frac{?}{HI} = \frac{?}{GH} = \frac{?}{GI}$.

3. State whether the proportions are correct for the indicated similar triangles.

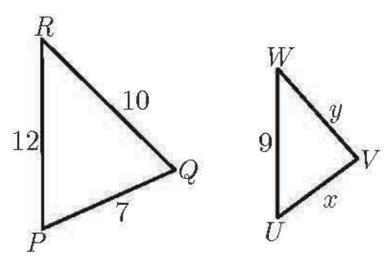
   (a) $\triangle ABC \sim \triangle XYZ$ $\quad \frac{AB}{XY} = \frac{BC}{YZ}$

   (b) $\triangle DEF \sim \triangle HIJ$ $\quad \frac{DE}{HI} = \frac{EF}{IJ}$

   (c) $\triangle RST \sim \triangle LMK$ $\quad \frac{RT}{LM} = \frac{ST}{MK}$

   (d) $\triangle XYZ \sim \triangle UVW$ $\quad \frac{XY}{UV} = \frac{XZ}{VW}$

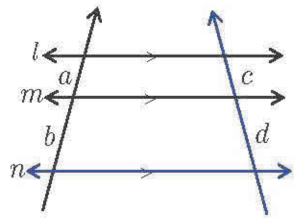
4. Given : $\triangle PQR \sim \triangle UVW$ and lengths of sides are as marked.
   Find : The values of x and y.



5. The measures of two angles of $\triangle XYZ$ are 82° and 16°. Find the measures of the angles of a triangle similar to $\triangle XYZ$.

8.2 The Basic Proportionality Theorem

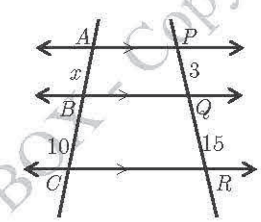
Postulate 1. If three parallel lines intersect two transversals, then the lines divide the transversals proportionally.
According to the above postulate, if $l \parallel m \parallel n$, then $\frac{a}{b} = \frac{c}{d}$.



Example 1.

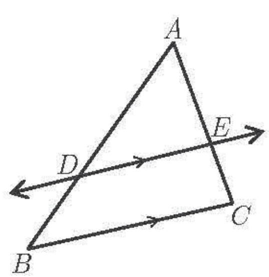
In the following figure if $AP \parallel BQ \parallel CR$, find the value of x.
Solution

$$\frac{x}{10} = \frac{3}{15}$$
$$\therefore x = 2.$$



Theorem 1 (The Basic Proportionality Theorem-BPT). If a line intersecting the interior of a triangle is parallel to one side, then the line divides the other two sides proportionally.

$$\text{In}\triangle ABC, \text{if } DE \parallel BC \text{ then } \frac{AD}{DB} = \frac{AE}{EC}.$$



Corollary 1.1. Using properties of proportions, it can be shown that the following three proportions are equivalent that is they have same value.

(1) $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

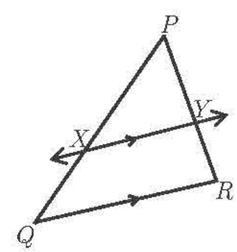
(2) $\dfrac{AD}{AB} = \dfrac{AE}{AC}$

(3) $\dfrac{AB}{DB} = \dfrac{AC}{EC}$

The following corollary is the converse of the BPT.

Corollary 1.2 (CBPT). If a line divides two sides of a triangle proportionally, then the line is parallel to the third side.

In$\triangle PQR$, if $\dfrac{PX}{XQ} = \dfrac{PY}{YR}$ then $XY \parallel QR$.



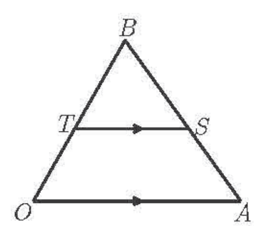
Exercise 8.2

1. Use the Basic Proportionality Theorem (BPT) and its corollary to complete the proportions for the adjoining figures.

In $\triangle AOB, TS \parallel OA$.

(a) $\dfrac{OT}{TB}$ =?

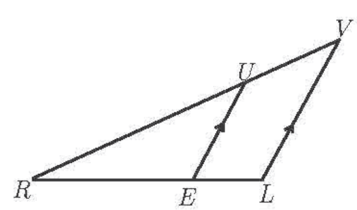
(b) $\dfrac{SA}{BA}$ =?



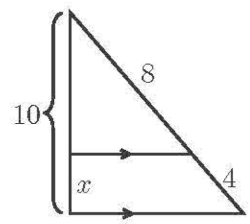
In $\triangle RVL, EU \parallel LV$.

(c) $\dfrac{EL}{RE}$ =?

(d) $\dfrac{RU}{RV}$ =?



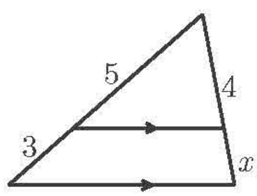
2. To find the value of x in the figure, a student wrote the proportion $\frac{x}{10} = \frac{4}{8}$.
   (a) Is this correct?
   (b) Another student wrote the proportion $\frac{x}{x-4} = \frac{4}{8}$. Is this correct?



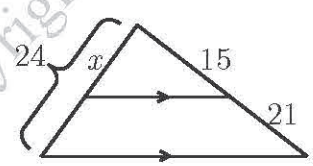
   (c) Write a simpler proportion that will give the correct answer.

3. Find the value of x in each of the figure below. (They are not drawn to scales.)

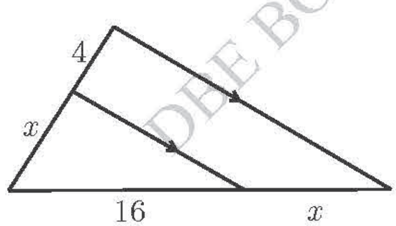
(a)



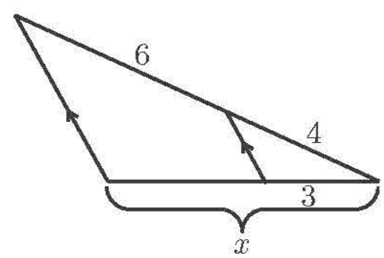
(b)



(c)



(d)

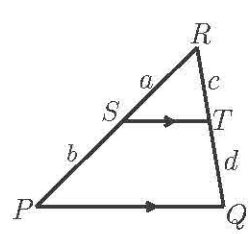


4. Given $\triangle PQR$ with $ST \parallel PQ$and lengths of segments are marked. Which of the following proportions are correct?

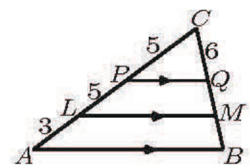
(a) $\frac{b}{a} = \frac{d}{c}$ (b) $\frac{a+b}{a} = \frac{c+d}{d}$

(c) $\frac{c}{d+c} = \frac{a}{b+a}$ (d) $\frac{a}{c} = \frac{b}{d}$

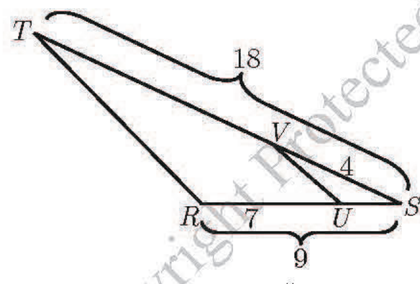
(e) $\frac{a}{b} = \frac{c}{d}$ (f) $\frac{a-b}{b} = \frac{c-d}{c}$.



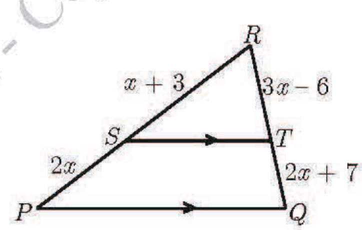
5. If $PQ \parallel LM \parallel AB$, and the lengths are as shown, how long are the segments MQ and BM?



6. If the segments in the figure have the lengths indicated, is $UV \parallel RT$? Justify your answer.

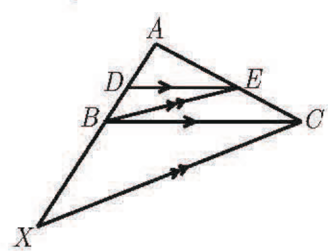


7. Given the figure as marked with $ST \parallel PQ$, find the lengths of the segments PS, SR, RT and TQ.



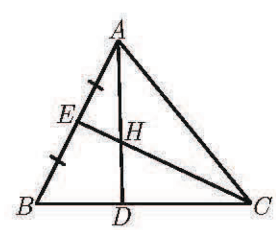
8. Given : $\dfrac{AD}{DB} = \dfrac{2}{1}$

   Prove : $\dfrac{AX}{XB} = \dfrac{3}{1}$



9. Given : $AE = EB$, $\dfrac{BD}{DC} = \dfrac{2}{3}$

   Find : The ratio $\dfrac{CH}{CE}$

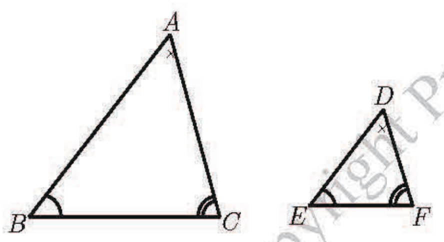


8.3 Basic Theorems on Similar Triangles

In this section we will study some theorems and properties of similar triangles.

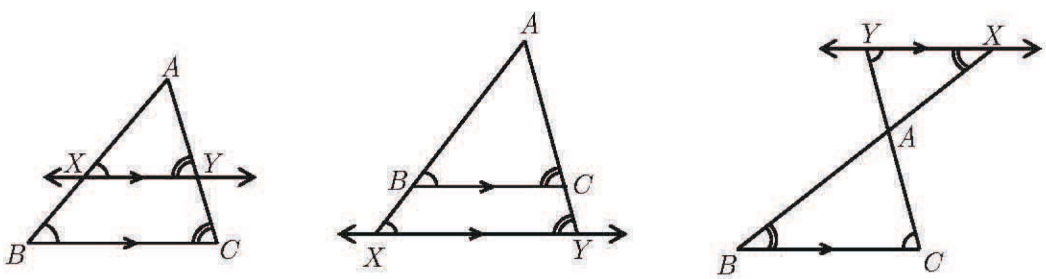
Theorem 2 (The AAA Similarity Theorem). If the angles of a triangle are equal to the angles of another triangle, then the two triangles are similar.

In $\triangle ABC$ and $\triangle DEF$, if $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ then $\triangle ABC \sim \triangle DEF$.



Corollary 2.1 (The AA Corollary). If two angles of a triangle are equal to two angles of another triangle, then the triangles are similar.

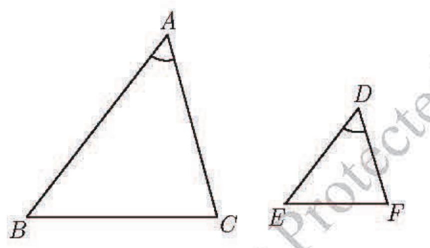
Corollary 2.2. If a line parallel to one side of a triangle determines a second triangle, then the second triangle will be similar to the original triangle.



In the above figures, lines XY are drawn parallel to BC forming another triangle AXY. We see that $\triangle ABC \sim \triangle AXY$, since indicated angles are equal.

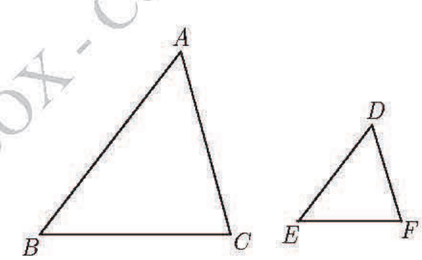
Theorem 3 (The SAS Similarity Theorem). If an angle of a triangle is equal to an angle of another triangle, and the sides including these angles are proportional, then the triangles are similar.

In $\triangle ABC$ and $\triangle DEF$, if $\angle A = \angle D$ and $\dfrac{AB}{DE} = \dfrac{AC}{DF}$ then $\triangle ABC \sim \triangle DEF$.



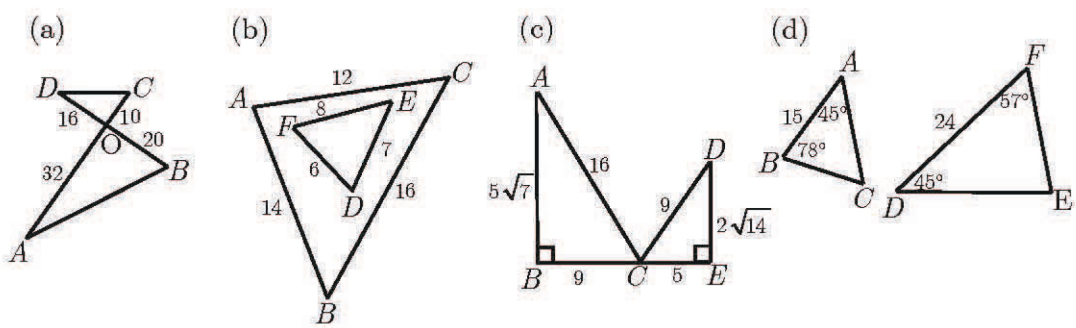
Theorem 4 (The SSS Similarity Theorem). If the corresponding sides of two triangles are proportional, then the triangles are similar.

In $\triangle ABC$ and $\triangle DEF$, if $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$ then $\triangle ABC \sim \triangle DEF$.



Example 2.

Determine whether the following pairs of triangle are similar or not. If they are similar, state why.



Solution

(a) Similar because $\angle AOB = \angle DOC$,

$$\frac{AO}{DO} = \frac{32}{16} = \frac{2}{1}, \qquad \frac{OB}{OC} = \frac{20}{10} = \frac{2}{1}$$

$$\therefore \frac{AO}{DO} = \frac{OB}{OC}$$

$$\therefore \triangle AOB \sim \triangle DOC(SAS).$$

(b) Similar because

$$\frac{DE}{AB} = \frac{7}{14} = \frac{1}{2}, \quad \frac{EF}{BC} = \frac{8}{16} = \frac{1}{2} \quad \frac{DF}{AC} = \frac{6}{12} = \frac{1}{2}.$$

$$\therefore \frac{DE}{AB} = \frac{EF}{BC} = \frac{DF}{AC}.$$

$$\therefore \triangle DEF \sim \triangle ABC(SSS).$$

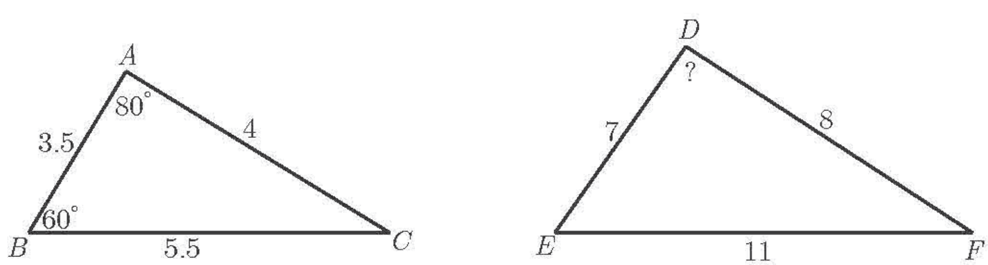
(c) Not similar.

(d) Similar because

$$\begin{aligned} \angle C &= 180^\circ - (45^\circ + 78^\circ) = 57^\circ \\ \angle E &= 180^\circ - (45^\circ + 57^\circ) = 78^\circ \\ \angle A &= \angle D, \angle B = \angle E, \angle C = \angle F \\ \therefore \triangle DEF &\sim \triangle ABC(AAA). \end{aligned}$$

Example 3.

Find the value of angle D.



Solution. In $\triangle ABC$ and $\triangle DEF$,

$$\frac{AC}{DF}=\frac{4}{8}=\frac{1}{2},\ \frac{AB}{DE}=\frac{3.5}{7}=\frac{1}{2},\ \frac{BC}{EF}=\frac{5.5}{11}=\frac{1}{2}$$

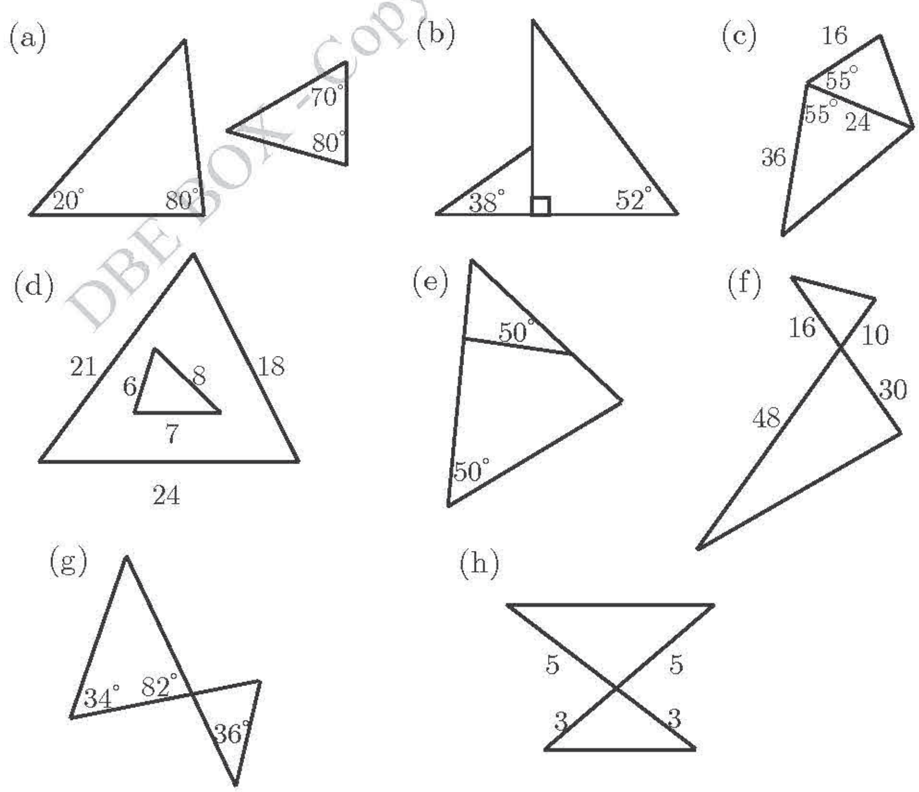
$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$$

$\triangle ABC \sim \triangle DEF(SSS)$

$\therefore \angle D = 80^\circ$.

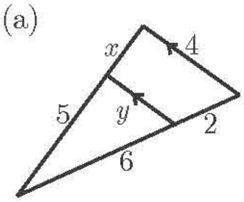
Exercise 8.3

1. Use the given information to tell whether each pair of triangles is similar or not. Give a reason for each answer.

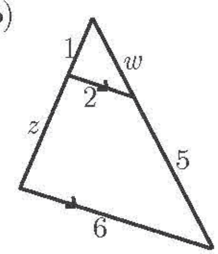


2. In each of the following triangles, the lengths of certain segments are marked. Find the values of x, y, z, w and v.

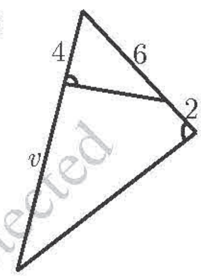
(a)



(b)

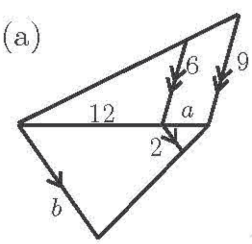


(c)

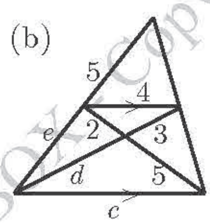


3. Find the marked lengths in each of the figures.

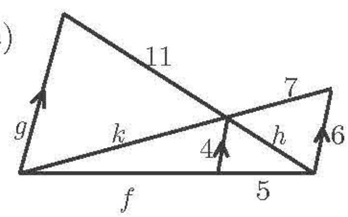
(a)



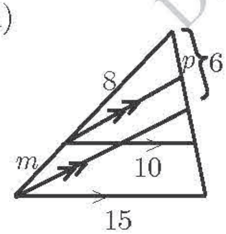
(b)



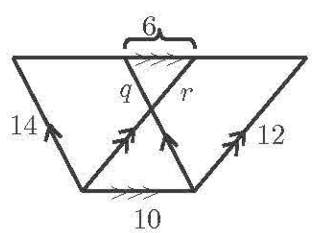
(c)



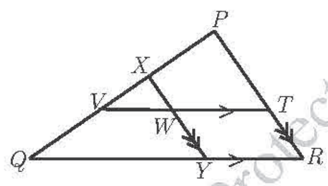
(d)



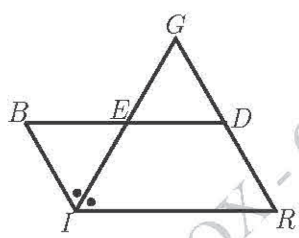
(e)



4. In the figure, $XY \parallel PR$ and $VT \parallel QR$. If $\frac{PT}{TR} = \frac{3}{2}$, $\frac{QY}{YR} = \frac{2}{1}$ and $PQ = 15$ cm, calculate
   (a) the lengths of PV, PX and XV.
   (b) the numerical values of $\frac{YW}{WX}$ and $\frac{VW}{QY}$.



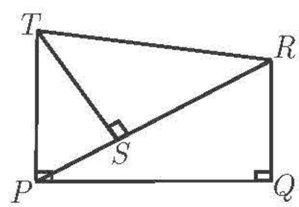
5.



Given : Parallelogram $BIRD$;
IG bisects $\angle BIR$.

Prove : $\frac{BE}{EI} = \frac{RG}{GI}$.

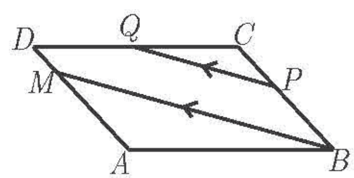
6.



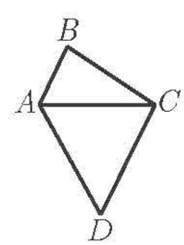
Given : $RQ \perp PQ, PQ \perp PT$,
$ST \perp PR$.

Prove : $ST \cdot RQ = PS \cdot PQ$.

7. Given : Parallelogram $ABCD$;
   $PQ \parallel MB$.
   Prove :$\triangle ABM \sim \triangle CQP$.



8. $\triangle ABC$ and $\triangle CAD$ are drawn on opposite sides of AC such that $AB : BC : CA = CA : AD : DC$. Prove that $DC \parallel AB$.



8.4 The Angle Bisector Theorem

An idea to be used in this section is the notion of dividing a segment internally or externally in a given ratio.

If B is a point on the line containing segment AC, then $\frac{AB}{BC}$ is the ratio in which B divides AC.

(1)

$$\frac{AB}{BC} = \frac{2}{4} = \frac{1}{2}$$

In this case B divides AC internally in the ratio 1 : 2.

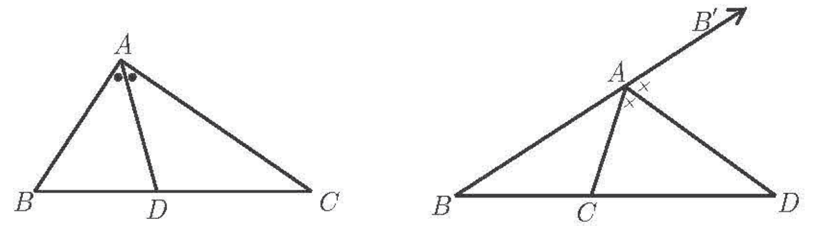
(2)

$$\frac{AB}{BC} = \frac{2}{4} = \frac{1}{2}$$

In this case B divides AC externally in the ratio 1 : 2.

For a given segment there are usually two points which divide the segment in the given ratio. One is an internal point, the other is an external point. This is shown in above figures.

Theorem 5 (The Angle Bisector Theorem - ABT). The bisector of an interior(exterior) angle of a triangle divides the opposite side internally(externally) into a ratio equal to the ratio of the other two sides of the triangle.



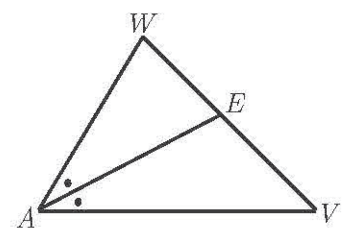
In $\triangle ABC$, if AD bisects $\angle BAC\,(\angle B'AC)$ then $\dfrac{AB}{AC} = \dfrac{BD}{DC}$.

Exercise 8.4

1. Which of the following proportions follow from the fact that AE bisects $\angle WAV$ in $\triangle WAV$?

   (a) $\dfrac{WE}{EV} = \dfrac{WA}{AV}$ (b) $\dfrac{WE}{EV} = \dfrac{VA}{AW}$

   (c) $\dfrac{WE}{WA} = \dfrac{EV}{AV}$ (d) $\dfrac{AV}{AW} = \dfrac{VE}{EW}$

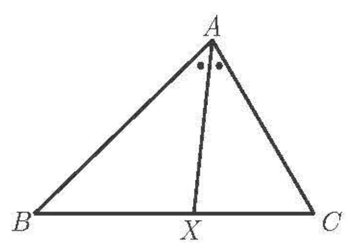


2. AX bisects $\angle CAB$. Complete the following statements:

   (a) $AC : AB = ...$

   (b) $AB : AC = ...$

   (c) $XC : XB = ...$

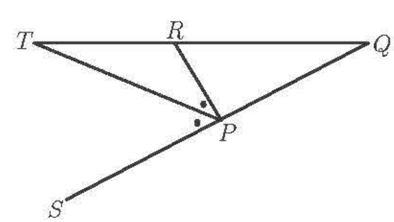


3. PT bisects $\angle RPS$. Complete the following statements:

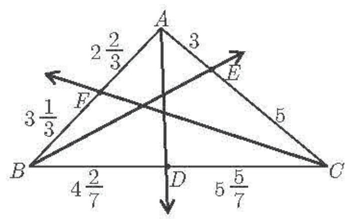
   (a) $PQ : PR = ...$

   (b) $TR : PR = ...$

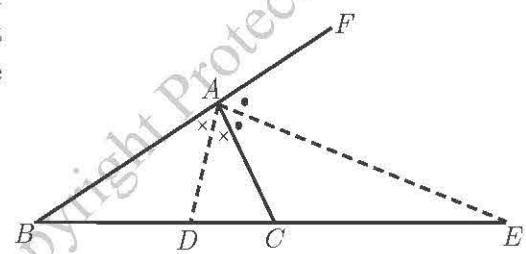
   (c) $QR : TR = ...$



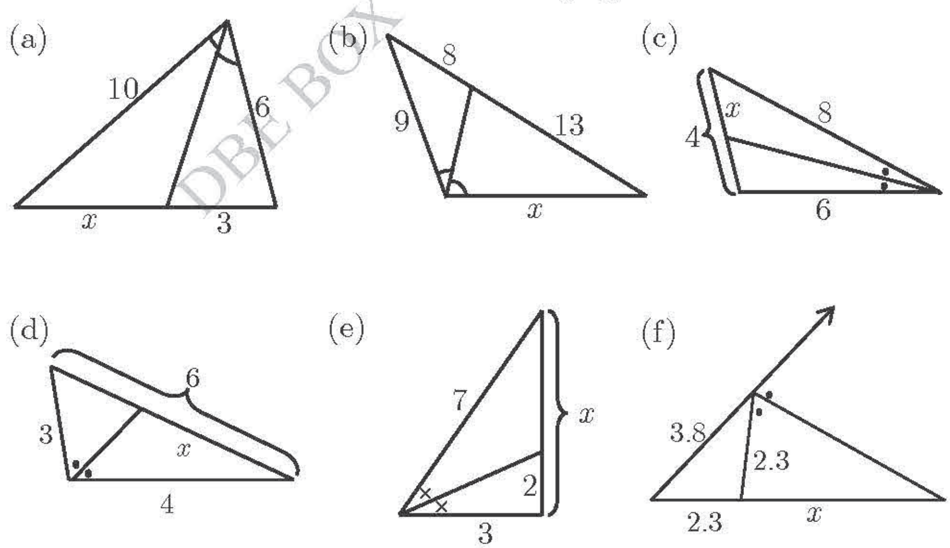
4. What can you say about the rays AD, BE and CF?



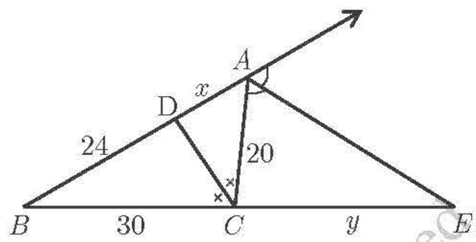
5. If AD and AE are bisectors of the interior and exterior angles at A of $\triangle ABC$, then which of the following are true?
   (a) $\angle DAE = 90°$
   (b) $BD : DC = BC : CE$
   (c) $BD : DC = BE : CE$
   (d) $AD : AE = DC : CE$



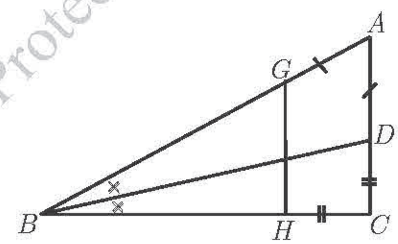
6. Find the value of x in each of the following figures.



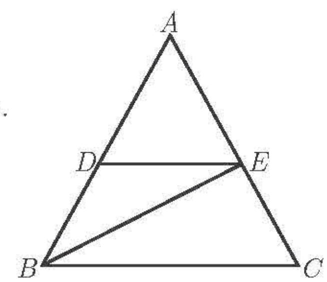
7. Find the unknown marked lengths in the figure.



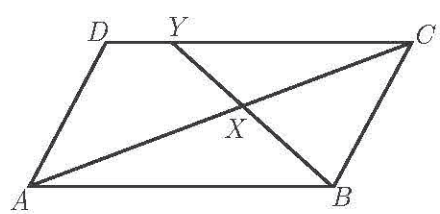
8. $AB = 12$ cm, $BC = 9$ cm, $CA = 7$ cm. BD bisects $\angle B$ and $AG = AD$, $CH = CD$. Calculate BG, BH. Does $GH \parallel AC$?



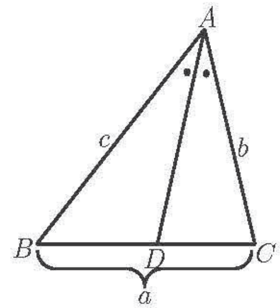
9. In $\triangle ABC, DE \parallel BC, AD = 2.7$ cm. $DB = 1.8$ cm and $BC = 3$ cm. Prove that BE bisects $\angle ABC$.



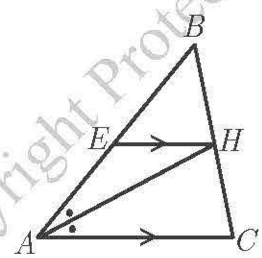
10. In a parallelogram $ABCD, AB = 3.6$ cm, $BC = 2.7$ cm. $AX = 3.2$ cm, $XC = 2.4$ cm. Prove that $\triangle BCY$ is isosceles.



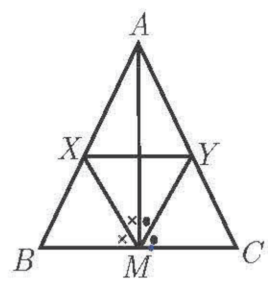
11. Calculate BD and DC in terms of a, b, c.



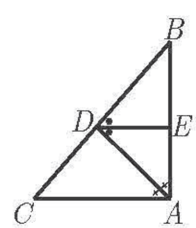
12. Given : AH bisects $\angle BAC$ in $\triangle ABC$.
$EH \parallel AC$.
Prove : $\frac{BE}{EA} = \frac{BA}{AC}$



13. Given : In $\triangle ABC, BM = MC$;
MX bisects $\angle AMB$
MY bisects $\angle AMC$.
Prove : $XY \parallel BC$.



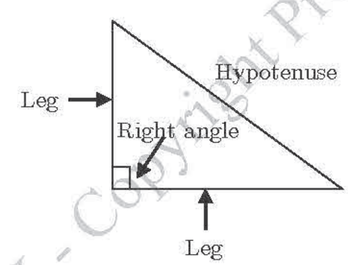
14. Given : In $\triangle ABC, \angle A = 2\angle C$,
AD bisects $\angle BAC$ and
DE bisects $\angle ADB$.
Prove : $\frac{BE}{EA} = \frac{BA}{AC}$.



8.5 The Pythagoras Theorem

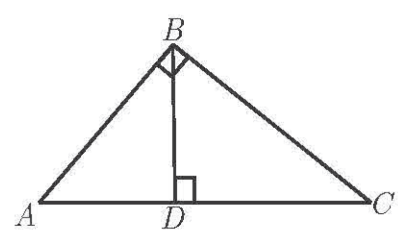
With our knowledge of similar triangles, we can prove the most famous theorem in Geometry, which is attributed to the Greek mathematician Pythagoras, who lived in the 6th century *B.C.* This theorem gives a relationship between the three sides of a right triangle or right-angled triangle. The first proof of this theorem is usually attributed to the Pythagoreans (a sect founded by Pythagoras).

Recall the definition that a triangle with a right angle is called a right triangle and the sides which determine the right angle are called legs of the right triangle, and the side opposite the right angle is called the hypotenuse.



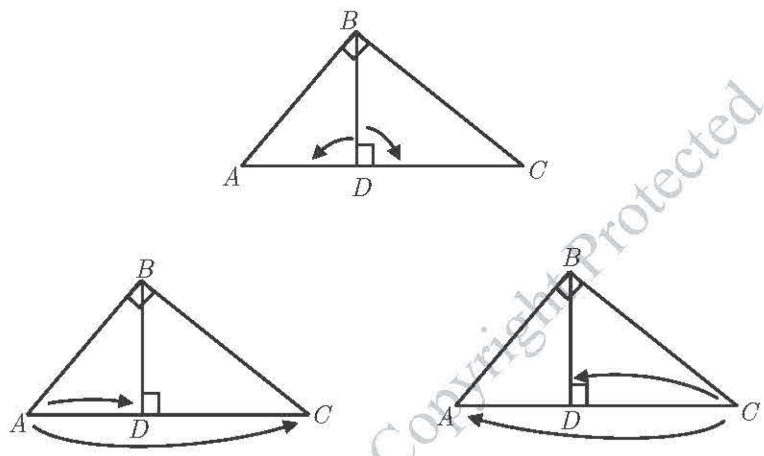
Theorem 6. The altitude to the hypotenuse of right triangle forms two triangles that are similar to each other and to the original triangle.

In a triangle ABC with $\angle ABC = 90°$, if BD is an altitude then $\triangle ADB \sim \triangle BDC \sim \triangle ABC$.



Corollary 6.1. The altitude to the hypotenuse of a right triangle is the **geometric mean** of the segments into which it separates the hypotenuse, and each leg of a right triangle is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to the leg.

In a triangle ABC with $\angle ABC = 90°$, if BD is an altitude then
(1) $BD^2 = AD \cdot DC$
(2) $AB^2 = AD \cdot AC$
(3) $BC^2 = CD \cdot CA$.



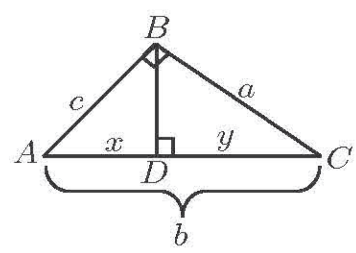
Now we will state and prove the most famous theorem in geometry.

Theorem 7 (Pythagoras Theorem). In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

Given : $\triangle ABC$ is a right triangle with $\angle B = 90°$
To Prove : $b^2 = c^2 + a^2$
Proof : Draw $BD \perp AC$.
Then $\triangle ADB \sim \triangle BDC \sim \triangle ABC$ (Theorem 6)
$c^2 = xb$ and $a^2 = yb$ (by Corollary 6.1)
By addition,

$$xb + yb = c^2 + a^2$$
$$b(x + y) = c^2 + a^2$$

Since $x + y = b$, we get $b(b) = c^2 + a^2$.
Thus $b^2 = c^2 + a^2$.



The converse of the Pythagoras Theorem provides a way of showing whether or not a triangle is a right triangle. It is stated here without proof.

Converse of Pythagoras Theorem: If a triangle has sides with lengths a, b, c and $a^2 + b^2 = c^2$, then the triangle is a right triangle.

8.6 Special Right Triangles

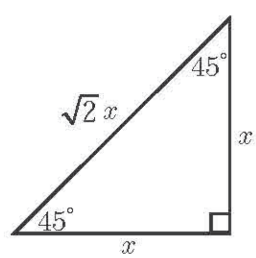
There are two special types of right triangles that are of particular interest. One is the isosceles right triangle; such a triangle is formed by two sides and a diagonal of a square (Fig. 8.1(a)). The other type is the right triangle with acute angles of measures 30° and 60° ; an altitude of an equilateral triangle determines two such triangles. (Fig. 8.1(b))

(a) (b)

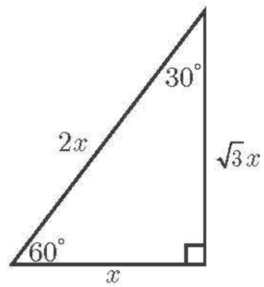
Fig. 8.1

The following theorems are based on Pythagoras Theorem, and therefore their proofs are left as exercises. There are frequent opportunities in geometry to apply these two theorems.

Theorem 8. In a 45°-45° right triangle, the length of hypotenuse is equal to the length of each leg times $\sqrt{2}$.

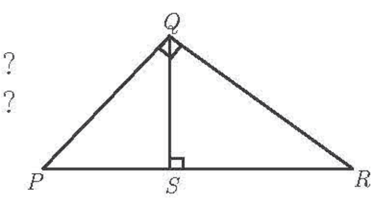


Theorem 9. In a 30°-60° right triangle, the leg opposite the 30° angle is one-half the length of the hypotenuse, and the other leg is equal to the length of hypotenuse times $\frac{\sqrt{3}}{2}$.

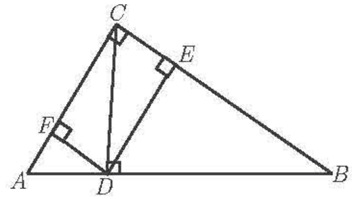


Exercise 8.5

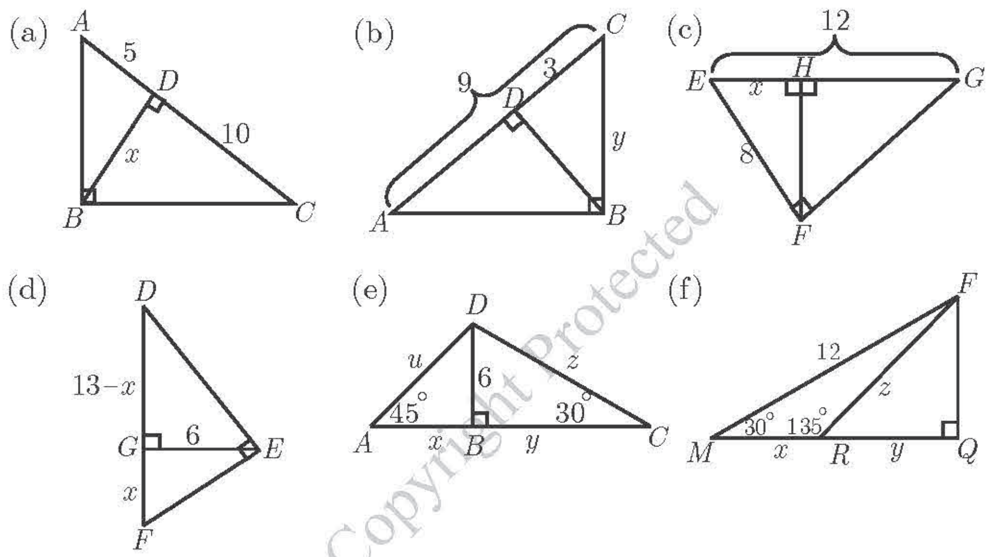
1. In the figure $\angle PQR = 90°$, $QS \perp PR$. Complete each of the following true statements.
   (a) $\triangle PQR \sim \triangle ? \sim \triangle ?$
   (b) QS is the geometric mean between ? and ?
   (c) QR is the geometric mean between ? and ?
   (d) $\frac{?}{PQ} = \frac{PQ}{?}$



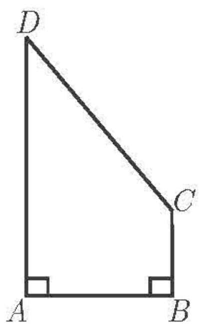
2. In the figure $CD \perp AB$ and $\angle C = 90°$. If $DE \perp BC, DF \perp CA$, write out all the triangles that are similar to $\triangle ABC$.



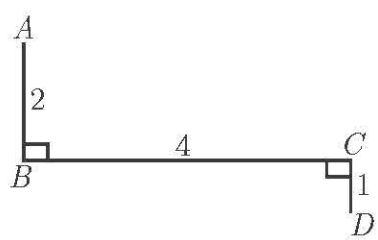
3. Find the length of each marked segment.



4. In the figure, if $AD = 10$ cm, $AB = 8$cm, $BC = 4$ cm, find the length of CD.

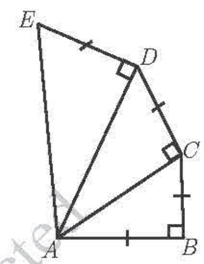


5. In the figure, find the distance of D from A.

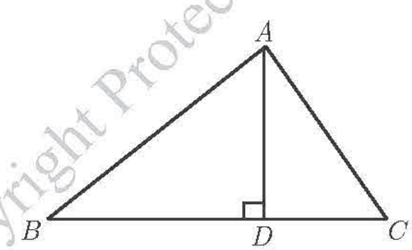


6. A parallelogram with sides 8 cm and 15 cm has a diagonal of 17 cm. Is it a rectangle?

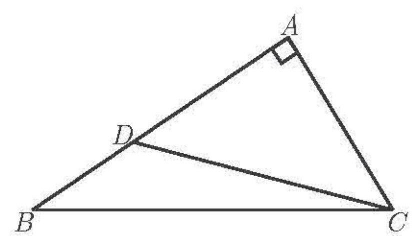
7. In the figure, $\triangle ABC, \triangle ACD, \triangle ADE$ are right triangles and $AB = BC = CD = DE$. Show that $AE = 2AB$.



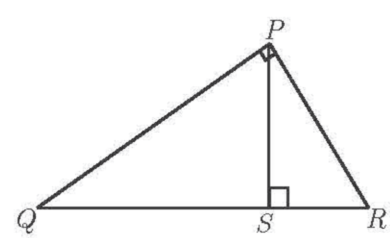
8. Given : $AD \perp BC$
   Prove : $AB^2 - AC^2 = BD^2 - DC^2$



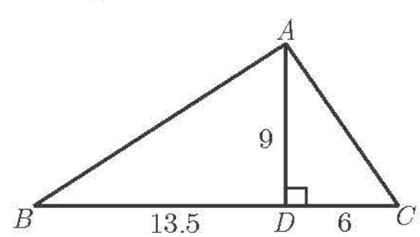
9. Given : $\angle BAC = 90°$
   D is any point on AB.
   Prove : $BC^2 + AD^2 = AB^2 + CD^2$



10. Given : $\angle QPR = 90°, PS \perp QR$.
    Prove : $\dfrac{1}{PS^2} = \dfrac{1}{PQ^2} + \dfrac{1}{PR^2}$



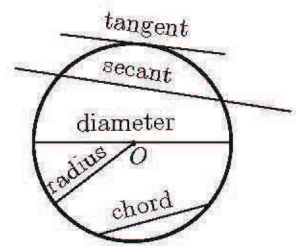
11. Given : $AD \perp BC, AD = 9$ cm,
    $BD = 13.5$ cm, $DC = 6$ cm.
    Prove : $\angle BAC = 90°$.



Chapter 9

Circles

In this chapter, properties of angles in a circle and properties of chords will be studied. We recall basic terms of a circle.

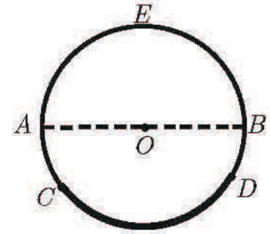


A **circle** is the set of all points that are at a fixed distance from a fixed point. The fixed point producing a circle is called the **centre** of the circle. A circle with centre O is called **circle** $\boldsymbol{O}$ and denoted by $\odot O$. Circles having the same centre are called **concentric circles**.

A **radius** is a segment joining the centre and a point on the circle. A segment joining two points on a circle is called a **chord** of the circle. A **diameter** is a chord passing through the centre of a circle. In a circle, a diameter is a longest chord. Circles having the same radius are called **congruent circles**.

A **secant** of a circle is a line that intersects the circle at two points. A line touching the circle at one point only is called a **tangent** to the circle. This touching point is called the **point of contact**.

An **arc** is a part of a circle. A **semicircle** is a half part of a circle. A **minor arc** is an arc shorter than a semicircle. A **major arc** is an arc longer than a semicircle.

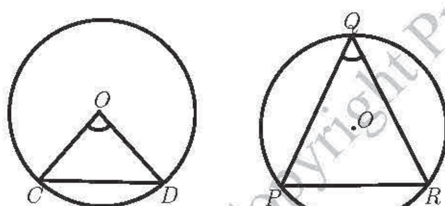


In the above figure, arc AEB is a semicircle, arc CD is a minor arc and arc CED is a major arc.

9.1 Angles in a Circle

Central Angles and Inscribed Angles

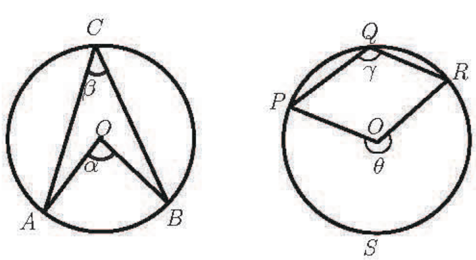
A **central angle** is an angle subtended by an arc (or a chord) of a circle at the centre. An **Inscribed angle** is an angle subtended by an arc (or a chord) of a circle at a point on the other arc.



In the above figure, $\angle COD$ is a central angle subtended by arc CD, and $\angle PQR$ is an inscribed angle subtended by arc PR.

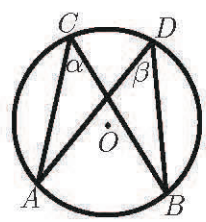
The following theorem and corollaries are properties of inscribed angles.

Theorem 1. The central angle is twice the inscribed angle subtended by the same arc.



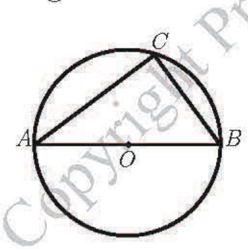
In $\odot O$, arc AB subtends central angle AOB and inscribed angle ACB. Then $\alpha = 2\beta$. Also, arc PSR subtends central angle POR and inscirbed angle PQR. Then $\theta = 2\gamma$.

Corollary 1.1. Inscribed angles subtended by the same arc are equal.



$\angle ACB$ and $\angle ADB$ are inscribed angles subtended by arc AB. Then $\alpha = \beta$.

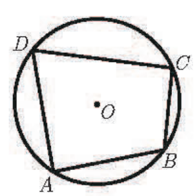
Corollary 1.2. An inscribed angle subtended by a diameter is a right angle.



In $\odot O, AB$ is a diameter. Then $\angle ACB = 90°$.

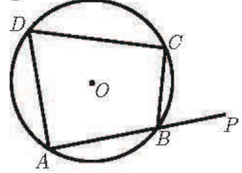
Cyclic Quadrilateral
A quadrilateral whose vertices lie on a circle is called a **cyclic quadrilateral**.

Theorem 2. Opposite angles of a cyclic quadrilateral are supplementary.



$ABCD$ is a cyclic quadrilateral. Then $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$.

Corollary 2.1. The exterior angle of a cyclic quadrilateral is equal to the opposite interior angle of the quadrilateral.



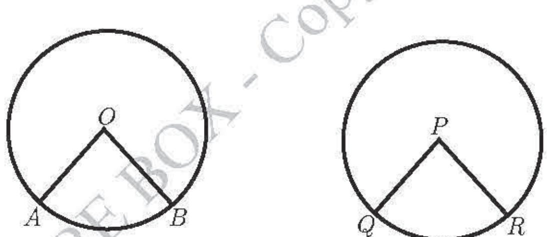
$ABCD$ is a cyclic quadrilateral. Then $\angle PBC = \angle D$.

The relations between central angles and arcs are stated in the following theorem.

Theorem 3. In the same circle or in congruent circles,

(i) arcs subtending equal central angles are equal,

(ii) equal arcs subtend equal central angles.



(i) Given: $\odot O$ and $\odot P$ are congruent and $\angle AOB = \angle QPR$.
To prove: arc AB = arc QR.
Proof: Place $\odot O$ on $\odot P$ so that O and P are coincide and OA falls along PQ.
Then $OA = PQ$. ($\because$ radii of congruent circles)
Hence A coincides with Q.
Since $\angle AOB = \angle QPR$, then OB falls along PR.
Since $OB = PR$, then B coincides with R.
Since every point on arc AB is equally distant from centre O and every point on arc QR is equally distant from centre P, then arc AB coincides with arc QR.
$\therefore$ arc AB = arc QR.

(ii) Given: $\odot O$ and $\odot P$ are congruent and arc AB = arc QR.
To prove : $\angle AOB = \angle QPR$.
Proof: Place $\odot O$ on $\odot P$ so that O and P are coincide and OA falls along PQ.
Then $OA = PQ$ ($\because$ radii of congruent circles)
Hence A coincides with Q.
Since arc AB = arc QR, then B coincides with R.
Therefore $\angle AOB = \angle QPR$.

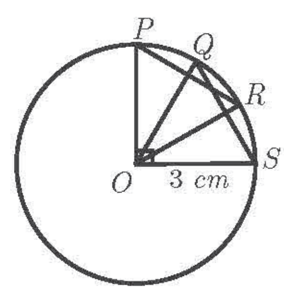
By Theorem 1 and Theorem 3, we get the following theorem.

Theorem 4. In the same circle or in congruent circles, two inscribed angles are equal if and only if the corresponding arcs are equal.

Example 1.

In the given figure, O is the centre of the circle, arc PQ = arc QR = arc RS. Find PR.

Solution



arc PQ = arc QR = arc RS (given)

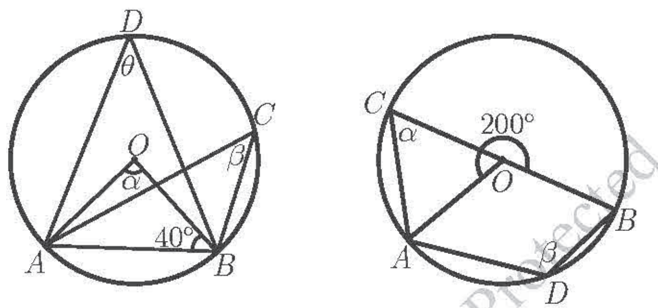
$$\therefore \angle POQ = \angle QOR = \angle ROS$$
$$\text{But } \angle POQ + \angle QOR + \angle ROS = 90^\circ$$
$$\therefore \angle POQ = \angle QOR = \angle ROS = 30^\circ$$
$$\angle POR = \angle POQ + \angle QOR = 60^\circ$$
$$\angle P = \angle R \quad (\because OR = OP = \text{radius})$$
$$\therefore \angle P = \angle R = \frac{180^\circ - 60^\circ}{2} = 60^\circ$$

So $\triangle POR$ is equilateral.
$\therefore PR = OR$

But $OR = OS$ (radii)
$\therefore PR = OS = 3$ cm

Example 2.

In $\odot O$, find the values of α, β and θ.



(a) (b)

Solution

(a) $\angle OAB = \angle OBA$ (radii)

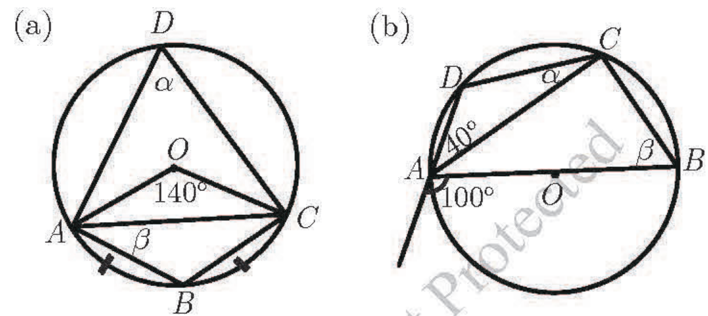
$\therefore \angle OAB = 40°$

$$\begin{aligned}
\therefore \alpha &= 180° - (40° + 40°) = 100° \\
\beta &= \frac{1}{2}\alpha \quad \text{(subtended by arc } AB\text{)} \\
\therefore \beta &= 50° \\
\theta &= \beta \quad \text{(subtended by arc} AB\text{)} \\
\therefore \theta &= 50°.
\end{aligned}$$

(b)
$$\begin{aligned}
\beta &= \frac{1}{2} \times 200° \quad \text{(subtended by arc } ACB\text{)} \\
\therefore \beta &= 100° \\
\alpha + \beta &= 180° \quad \text{(opposite angles of cyclic quadrilateral } ACBD\text{)} \\
\alpha &= 180° - 100° = 80°.
\end{aligned}$$

Example 3.

In the diagram below, O is the centre of the circle. Find the values of α and β.



Solution

(a) $\alpha = \frac{1}{2}\angle AOC$ (subtended by arc ABC)

$\alpha = 70°$

$\angle B + \alpha = 180°$ (opposite angles of cyclic quadrilateral $ABCD$)

$\therefore \angle B = 180° - \alpha = 110°$

$\beta = \angle BCA$ (arcAB = arcBC)

But $\beta + \angle B + \angle BCA = 180°$

$\therefore \beta = \frac{180° - \angle B}{2} = \frac{180° - 110°}{2} = 35°$

(b) $\angle ACB = 90°$ (angle subtended by diameter AB)

$\alpha + \angle ACB = 100°$ (exterior angle and opposite interior angle)

$\alpha = 100° - 90° = 10°$

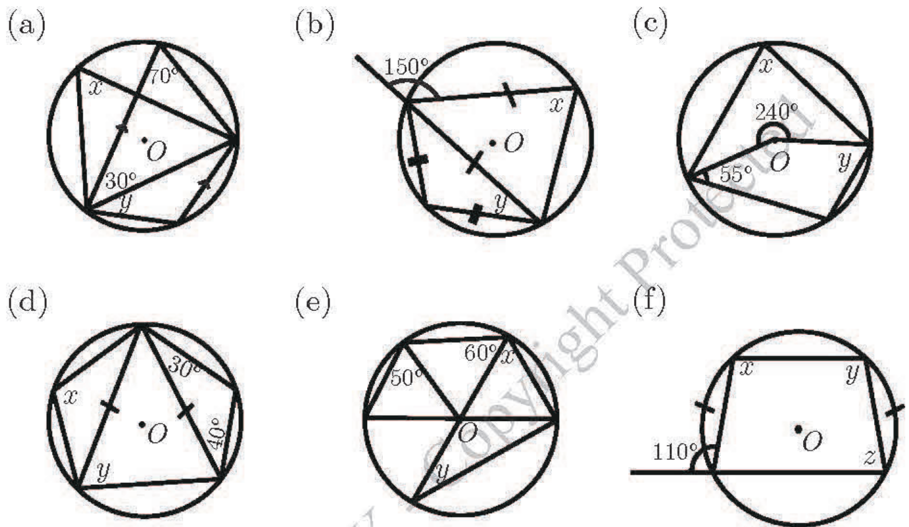
$\angle D + \angle DAC + \alpha = 180°$

$\therefore \angle D = 180° - (40° + \alpha) = 130°$

$\angle D + \beta = 180°$ (opposite angles of cyclic quadrilateral $ABCD$)
$\beta = 180° - \angle D = 50°$.

Exercise 9.1

1. The point O is the centre of the given circle. Find the values of x, y and z.



2. ABC is an acute triangle inscribed in $\odot O$, and OD is the perpendicular drawn O to BC. Prove that $\angle BOD = \angle BAC$.

3. In $\odot O$, two chords AB and CD intersect in the circle at P. Show that $\angle APD = \frac{1}{2}(\angle AOD + \angle BOC)$.

4. Two circles intersect at M, N. From M, diameters MA, MB are drawn in each circle. If A, B are joined to N, prove that ANB is a straight line.

5. OA and OB are two radii of a circle meeting at right angles. From A, B two parallel chords AX, BY are drawn. Prove that $AY \perp BX$.

6. Two circles intersect at R and S. Two straight lines ARB and CSD are drawn meeting one circle at A, C and the other at B, D. Prove that $AC \parallel BD$. If $AB \parallel CD$, show that $AB = CD$.

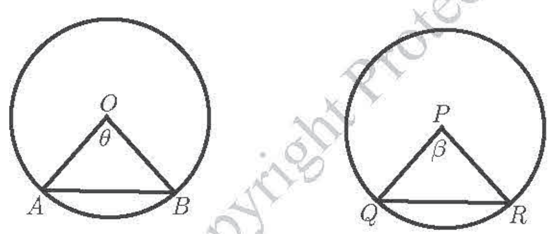
9.2 Properties of Chords

In this section, the relations between chords and central angles, symmetrical properties of chords, and properties of lengths of segments formed by chords and secants will be studied.

Theorem 5. In the same circle or in congruent circles,

(i) equal chords subtend equal central angles,

(ii) equal central angles cut off equal chords.

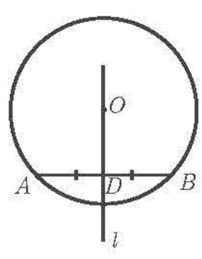


$\odot O$ and $\odot P$ are congruent. $AB = QR$ if and only if $\theta = \beta$.

From the above theorem, we can conclude that in the same circle or in congruent circles, two central angles are equal if and only if two corresponding minor arcs are equal if and only if two corresponding chords are equal.

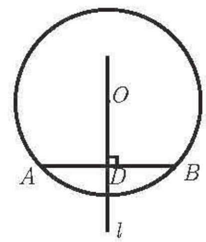
The following four theorems are the symmetric properties of chords.

Theorem 6. If a line passing through the centre of a circle bisects a chord of the circle, then the line is perpendicular to the chord.



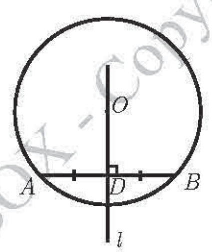
l is the line passing through the centre O and bisects chord AB at D. Then $l \perp AB$.

Theorem 7. If a line passing through the centre of a circle is perpendicular to a chord of the circle, then the line bisects the chord.



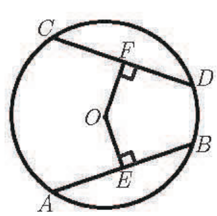
l is the line passing through the centre O and is perpendicular to chord AB at D. Then $AD = BD$.

Theorem 8. The perpendicular bisector of a chord of a circle passes through the centre.



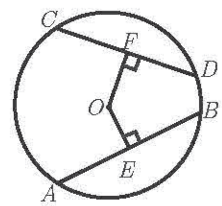
l is the perpendicular bisector of chord AB. Then l passes through the centre O.

Theorem 9. In the same circle or in congruent circles, chords are equal if and only if they are equidistant from the centre of the circle.



In $\odot O$, $AB = CD$ if and only if $OE = OF$.

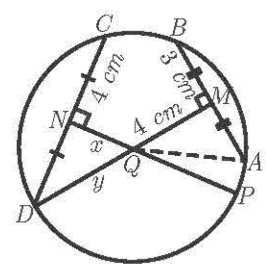
Theorem 10. Of any two chords of a circle, the greater chord is nearer to the centre, and conversely, the chord nearer to the centre is larger.



In $\odot O$, $AB > CD$ if and only if $OE < OF$.

Example 4.

In the given figure, find the values of x and y.



Solution

DM is the perpendicular bisector of AB.
Therefore the centre is on DM.
PN is the perpencicular bisector of CD.
Therefore the centre is also on PN.
DM and PN intersect at Q.
$\therefore Q$ is the centre of the given circle.
Join AQ.

$$\begin{aligned} AM &= MB \quad \text{(given)} \\ \therefore AM &= 3 \text{ cm} \end{aligned}$$

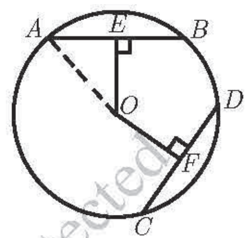
$$\begin{aligned} AQ^2 &= AM^2 + MQ^2 = 3^2 + 4^2 = 25 \\ \therefore AQ &= 5 \text{ cm} \\ y &= AQ \quad \text{(radii)} \\ \therefore y &= 5 \text{ cm} \end{aligned}$$

$$\begin{aligned} DN &= NC \quad \text{(given)} \\ \therefore DN &= 4 \text{ cm} \\ y^2 &= x^2 + DN^2 \\ \therefore x &= \sqrt{y^2 - 4^2} = \sqrt{5^2 - 4^2} = 3 \text{ cm}. \end{aligned}$$

Example 5.

Given $\odot O$, $OE = OF = 6$, $AE = x + 2$ and $CD = 3x - 2$. Find the radius of the circle.

Solution



$OE \perp AB$

$\therefore AB = 2AE = 2(x + 2) = 2x + 4$

Since $OE = OF$, then $AB = CD$.

$\therefore 2x + 4 = 3x - 2$

$x = 6.$

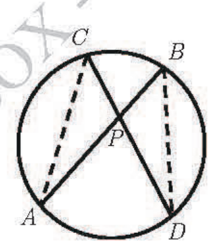
$\therefore AE = 6 + 2 = 8$

Join OA.

$\therefore$ radius $= OA = \sqrt{OE^2 + AE^2} = \sqrt{6^2 + 8^2} = 10.$

When two chords intersect each other inside or outside the circle, the product properties of the lengths of these line segments are as follows:

Theorem 11. If two chords of a circle intersect in the circle, the product of the lengths of segments of one chord is equal to the product of the lengths of segments of the other chord.



Given: Chord AB and chord CD intersect at a point P in the circle.

To prove: $PA \cdot PB = PC \cdot PD$

Proof: Join AC, BD.

In $\triangle APC$ and $\triangle DPB$,

$\angle A = \angle D$ (subtended by arc BC)

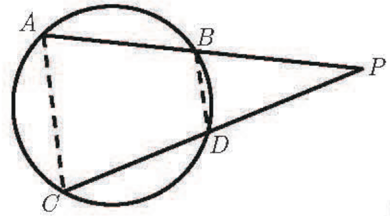
$\angle C = \angle B$ (subtended by arc AD)

$\therefore \triangle APC \sim \triangle DPB$ (AA Corollary)

$$\therefore \frac{PA}{PD} = \frac{PC}{PB}$$

$\therefore PA \cdot PB = PC \cdot PD.$

Theorem 12. If two secants are drawn to a circle from an external point, the product of the lengths of one secant and its external segment is equal to the product of the lengths of the other secant and its external segment.



Given: PBA and PDC are two secant segments.

To prove: $PA \cdot PB = PC \cdot PD$

Proof: Join AC, BD.

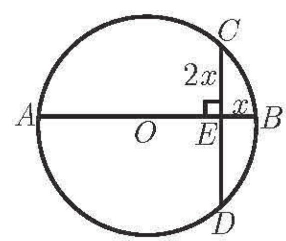
In $\triangle DPB$ and $\triangle APC$,

$$\begin{aligned} \angle P &= \angle P \\ \angle PDB &= \angle PAC \quad \text{(exterior and opposite interior)} \\ \therefore \triangle DPB &\sim \triangle APC \quad \text{(AA Corollary)} \\ \therefore \frac{PD}{PA} &= \frac{PB}{PC} \\ \therefore PA \cdot PB &= PC \cdot PD. \end{aligned}$$

Example 6.

Given $\odot O$, $AE = 12$, $BE = x$ and $CE = 2x$. Find radius of the circle.

Solution



$OB \perp CD$

$\therefore ED = CE = 2x$

AB and CD intersect at E.

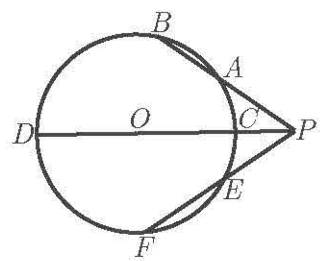
$$\begin{aligned} \therefore AE \cdot EB &= CE \cdot ED \\ 12 \cdot x &= 2x \cdot 2x \\ x &= 3 \end{aligned}$$

$AB = AE + EB = 12 + x = 15$

$\therefore$ radius $= OA = \frac{1}{2}AB = 7.5$.

Example 7.

Given $\odot O$, radius $= 6$, $PC = 3$, $PA = 5$ and $EF = 6$. Find AB and PE.



Solution

$DC = \text{diameter} = 12$
$PD = PC + CD = 3 + 12 = 15$

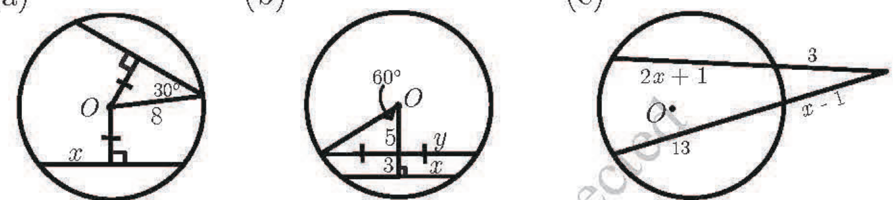
$$\begin{aligned} PA \cdot PB &= PC \cdot PD \\ 5(5 + AB) &= 3 \times 15 \\ 5 + AB &= 9 \\ AB &= 4. \end{aligned}$$

$$\begin{aligned} PE \cdot PF &= PC \cdot PD \\ PE(PE + 6) &= 3 \times 15 \\ PE^2 + 6PE - 45 &= 0 \\ PE^2 + 6PE + 9 &= 45 + 9 \\ (PE + 3)^2 &= 54 \\ PE + 3 &= \sqrt{54} \\ \therefore PE &= \sqrt{54} - 3 = 3\sqrt{6} - 3. \end{aligned}$$

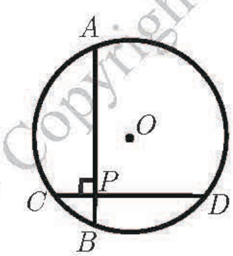
Exercise 9.2

1. In the following figures, O is the centre of circles. Find the values of x and y.

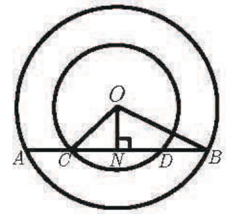
(a) (b) (c)



2. In $\odot O$, chords AB is perpendicular to CD at P, $AB = 16$, $CP = 4$, $PD = 10$. Find the radius.



3. In the figure, O is the centre of the concentric circles and $ON \perp AB$. If $OC = 10$, $ON = 8$ and $OB = 17$, find AC.



4. Prove Theorem 10: Of any two chords of a circle, the greater chord is nearer to the centre, and conversely, the chord nearer to the centre is larger.

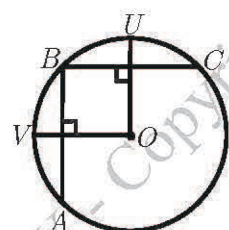
5. Let P be a point inside a circle. AB is the diameter passes through P and CPD is the chord perpendicular to AB. Show that CD is the shortest of all chords passing through P.

6. Through a point P in a circle, the longest chord that can be drawn is 10 cm long and the shortest chord is 6 cm long. What is the radius of the circle and how far is P from the centre?

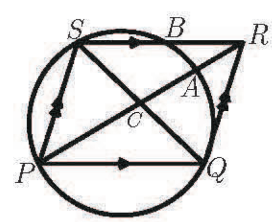
7. In $\odot O$, chords AB and CD are equal and intersect in the circle at E such that $AE < EB$ and $CE < ED$. Show that $\triangle BDE$ is isosceles with base BD.

8. In $\odot O$, congruent chords AB and CD are produced to meet at P. Prove that $\triangle PAC$ is isosceles.

9. In $\odot O$, AB and BC are equal chords, $OV \perp AB$, and $OU \perp BC$. Prove that B is the midpoint of arc VU.



10. In parallelogram $PQRS$, $PQ = 5$ cm, $PR = 8$ cm, $QS = 6$ cm. Calculate the lengths of AR and BR.



11. Chords AB and CD intersect at E and $AE = EB$. A semicircle is drawn with diameter CD. EF, perpendicular to CD, meets this semicircle at F. Prove that $AE = EF$.

12. $\odot O$ and $\odot P$ intersect at A and B. Show that OP is the perpendicular bisector of the common chord AB.

Chapter 10

Trigonometry

The word **trigonometry** is derived from the words "tri"(meaning three), "gon"(meaning sides) and "metry"(meaning measure). Thus trigonometry deals with the measurement of sides and angles of a triangle.

It has been widely used in Astronomy, Surveying, Geography, Physics, Navigation etc. The captain of a ship employs trigonometry to calculate the distance from the far off island, sea shores, cliffs and other ships on the high seas.

To study trigonometry, the students should already be acquainted with the theorems on similar triangles. Only then will they find it easy to understand the definition of trigonometric ratios. To begin with, we shall only consider the acute angles in this chapter. There is more to learn in trigonometry than measuring triangles. In fact, trigonometry generally deals with angles of all sizes with measurement not necessarily confined to the angles of triangles. In every branch of Higher Mathematics, whether Pure or Applied, a knowledge of trigonometry is of great value.

10.1 Angles

In trigonometry an angle is determined by rotating a ray about its endpoint from an initial position to terminal position.

Consider a line OP which is free to rotate in the XY-plane. O is taken to be the origin about which a line OP rotates.

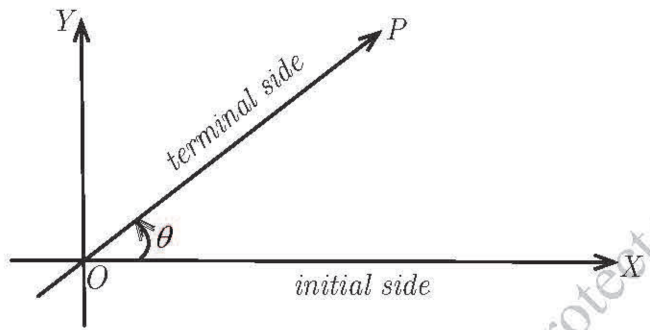


Figure 10.1

When the line OP is rotated, it is possible to vary the size of the angle θ between OP and OX (see Fig. 10.1). Angles measured from the X-axis (i.e., OX) in an anticlockwise direction are positive angles. Angles measured from the X-axis in a clockwise direction are negative angles.

Therefore in Fig. 10.2, the angle α (i.e., $\angle P_1OX$) is positive while the angle β (i.e., $\angle P_2OX$) is negative.

One complete revolution of the line OP from OX makes an angle of 360°. The range of values of θ between 0° and 360° in each quadrant is as shown in Fig. 10.3.

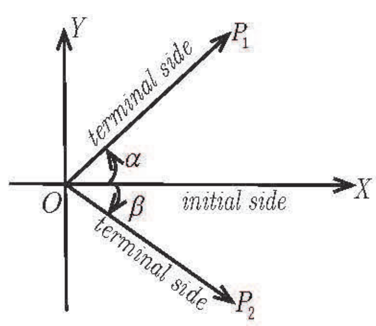


Figure 10.2

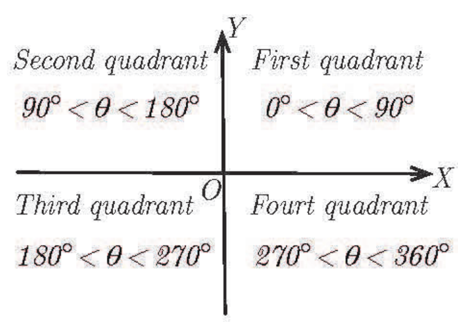


Figure 10.3

10.2 The Relation between Degree and Radian Measure

Two kinds of units commonly used for measuring angles are radian measure and degree measure. The radian measure is employed almost exclusively in advanced mathematics and in many branches of science. In this chapter first we introduce the concept of radian and study the relation between degrees and radians.

Consider a circle with centre O and radius r units as shown in Fig. 10.4. Let arc AB be an arc on the circle of length equal to r. We define the magnitude of angle AOB which the arc AB subtends at the centre as one radian.

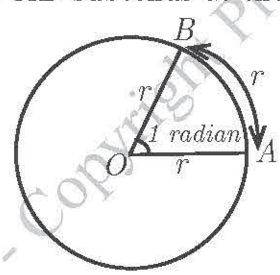


Figure 10.4

Since the circumference of a circle is equal to $2\pi r$, it subtends a central angle of 2π radians. That is there are 2π radians in a complete rotation of $360°$. Therefore 2π radians $= 360°$ and hence π radians $= 180°$ which is a fundamental relation between radians and degrees.
We have

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \approx 57°19'$$

$$1° = \frac{\pi}{180} \text{ radians} \approx 0.01764 \text{ radians.}$$

Example 1.

Express the following in radian measures.
(a)$60°$ (b)$135°$

Solution
(a) $60° = 60 \times \frac{\pi}{180}$ radians $= \frac{\pi}{3}$ radians
(b) $135° = 135 \times \frac{\pi}{180}$ radians $= \frac{3\pi}{4}$ radians.

Example 2.

Express the following in degree measures.

(a) $\frac{\pi}{4}$ radians (b) $\frac{4\pi}{5}$ radians

Solution

(a) $\frac{\pi}{4}$ radians $= \frac{\pi}{4} \times \frac{180}{\pi}$ degrees $= 45°$

(b) $\frac{4\pi}{5}$ radians $= \frac{4\pi}{5} \times \frac{180}{\pi}$ degrees $= 144°$.

Note: Usually when the units of an angle are not specified, it is understood that the angle is expressed in radians.

10.3 Arc Length and Area of a Sector of a Circle

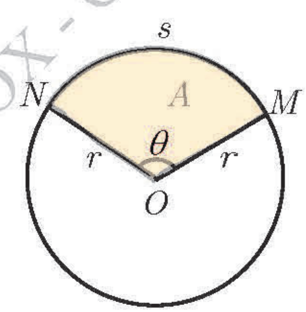


Figure 10.5

Let the arc MN subtend an angle of magnitude θ radians at the centre of a circle of radius r as shown in Fig. 10.5. Clearly the length "s" of arc MN is proportional to the angle θ and we have

$$\frac{\text{length of arc } MN}{\text{length of the circumference}} = \frac{\text{angle subtended by arc } MN}{\text{angle subtended by circumference}}$$

i.e., $$\boxed{\frac{s}{2\pi r} = \frac{\theta}{2\pi} \quad \text{(or)} \quad \theta(\text{in radians}) = \frac{s}{r}}$$

Furthermore, as the area of the sector MON (the shaded region shown in Fig. 10.5) is also, proportional to the angle θ, we have

$$\frac{\text{area of sector } MON}{\text{area of circle}} = \frac{\theta}{2\pi}$$

i.e.

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

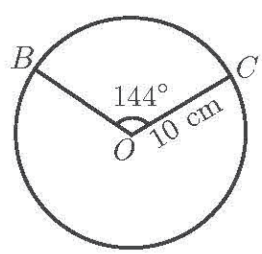
(or)

$$\boxed{A = \frac{1}{2}r^2\theta}$$

where θ is given in radian measures.

Example 3.

An arc BC subtends an angle of 144° at the centre O of a circle of radius 10cm. Find the length of arc BC and the area of the sector BOC.



Solution

$\theta = 144° = 144 \times \frac{\pi}{180} = \frac{4\pi}{5}$ radians and $r = 10$ cm.

The length of arc $BC = r\theta = 10 \times \frac{4\pi}{5} = 8\pi$ cm.

The area of the sector $BOC = \frac{1}{2}r^2\theta = \frac{1}{2} \times 10^2 \times \frac{4\pi}{5} = 40\pi\ cm^2$.

Exercise 10.1

1. Convert each of the following to radians.

   (a) $120°$ (b) $90°$ (c) $72°$ (d) $225°$

   (e) $150°$ (f) $108°$ (g) $160°$ (h) $390°$

2. Convert each of the following to degrees.

   (a) $\frac{\pi}{5}$ (b) $\frac{3\pi}{4}$ (c) $\frac{5\pi}{6}$ (d) π

   (e)$\frac{8\pi}{9}$ (f)$\frac{12\pi}{5}$ (g) $\frac{\pi}{3}$ (f) $\frac{7\pi}{3}$

3. A central angle θ subtends an arc of $\frac{11\pi}{2}$ cm on a circle of radius 6 cm. Find the measure of θ in radians and the area of a sector of a circle which has θ is its central angle.

4. The area of a sector of a circle is 143 cm^2 and the length of the arc of a sector is 11 cm. Find the radius of the circle.

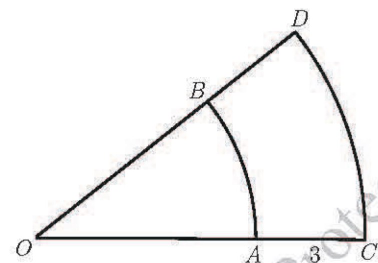
5. A sector cut from a circle of radius 3 cm has a perimeter of 16 cm. Find the area of this sector.

6. A piece of wire of fixed length L cm, is bent to form the boundary a sector of a circle. The circle has radius r cm and the angle of the sector is $\theta = (\frac{32}{r} - 2)$ radians. Find the wire of fixed length L and show that the area of the sector, A cm^2 is given by $A = 16r - r^2$.

7. A race is run at a uniform speed on a circular course. In each minute, a runner traverses an arc of a circle which subtends $2\frac{6}{7}$ radians at the centre of the course. If each lap is 792 yards, how long does the runner take to run a mile?

8. The large hand of a clock is 28 inches long; how many inches does its extremity move in 20 minutes?

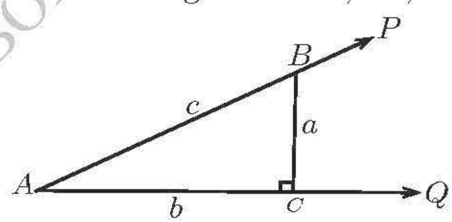
9. The figure shows two sectors in which the arcs AB and CD are arcs of concentric circles, centre O. If $\angle AOB = \frac{2}{3}$ radians, $AC = 3$ cm and the area of a sector AOB is 12 cm^2, calculate the area and the perimeter of $ABDC$.



10.4 Six Trigonometric Ratios

Consider the acute angle PAQ, that is an angle whose measure in degrees is less than 90°. On one arm AP of the angle PAQ, select a point B and draw BC perpendicular to AQ at C. Thus a right triangle ACB is formed.

Denote the lengths of the segments BC, AC, AB by the letters a, b, c



respectively.

We say that BC is the side which is *opposite* to angle A; AC is the side which is *adjacent* to angle A; AB is the *hypotenuse.* With reference to the angle A the following definitions are employed.

The ratio $\frac{BC}{AB}$ or $\frac{\text{opposite side of}\angle A}{\text{hypotenuse}}$ is called the **sine** of angle A (or $\sin A$).

The ratio $\frac{AC}{AB}$ or $\frac{\text{adjacent side of}\angle A}{\text{hypotenuse}}$ is called the **cosine** of angle A (or $\cos A$).

The ratio $\frac{BC}{AC}$ or $\frac{\text{opposite side of}\angle A}{\text{adjacent side of}\angle A}$ is called the **tangent** of angle A (or $\tan A$).

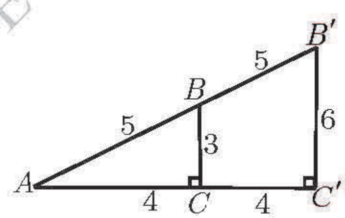
The ratio $\frac{AC}{BC}$ or $\frac{\text{adjacent side of}\angle A}{\text{opposite side of}\angle A}$ is called the **cotangent** of angle A (or $\cot A$).

The ratio $\frac{AB}{AC}$ or $\frac{\text{hypotenuse}}{\text{adjacent side of}\angle A}$ is called the **secant** of angle A (or $\sec A$).

The ratio $\frac{AB}{BC}$ or $\frac{\text{hypotenuse}}{\text{opposite side of}\angle A}$ is called the **cosecant** of angle A (or $\csc A$).

Since the measure of the angles are in degrees, then $\angle A = \alpha$ means that the measure of angle A is α degrees. Thus if $\angle A = \alpha$, then we may write $\sin A = \sin\alpha$, $\cos A = \cos\alpha$ and $\tan A = \tan\alpha$. Note the six ratios do not depend on the size of the triangle. The following example illustrates this.

Example 4.
(a) Using the right triangle ABC, find $\sin A$, $\cos A$, $\tan A$.
(b) Using the right triangle $AB'C'$, find $\sin A$, $\cos A$, $\tan A$.
Solution



(a) From the right triangle ABC,

$$\begin{aligned}
\sin A &= \frac{\text{opposite side of } \angle A}{\text{hypotenuse}} = \frac{BC}{AB} = \frac{3}{5}, \\
\cos A &= \frac{\text{adjacent side of } \angle A}{\text{hypotenuse}} = \frac{AC}{AB} = \frac{4}{5}, \\
\tan A &= \frac{\text{opposite side of } \angle A}{\text{adjacent side of } \angle A} = \frac{BC}{AC} = \frac{3}{4}.
\end{aligned}$$

(b) From the right triangle $AB'C'$,

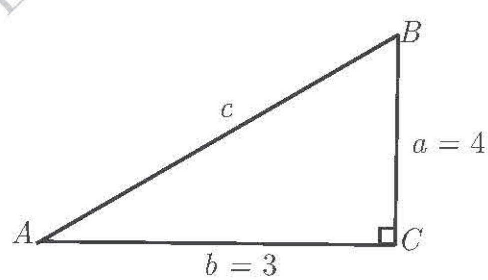
$$\begin{aligned}\sin A &= \frac{\text{opposite side of } \angle A}{\text{hypotenuse}} = \frac{B'C'}{AB'} = \frac{6}{10} = \frac{3}{5},\\ \cos A &= \frac{\text{adjacent side of } \angle A}{\text{hypotenuse}} = \frac{AC'}{AB'} = \frac{8}{10} = \frac{4}{5},\\ \tan A &= \frac{\text{opposite side of } \angle A}{\text{adjacent side of } \angle A} = \frac{B'C'}{AC'} = \frac{6}{8} = \frac{3}{4}.\end{aligned}$$

We see that $\frac{BC}{AB} = \frac{B'C'}{AB'} = \frac{3}{5}$, $\frac{AC}{AB} = \frac{AC'}{AB'} = \frac{4}{5}$, $\frac{BC}{AC} = \frac{B'C'}{AC'} = \frac{3}{4}$.

Thus $\sin A, \cos A$ and $\tan A$ do not depend on the size of the triangle.

Example 5.

ABC is a right triangle in which C is the right angle. If $a = 4$ and $b = 3$, find c, $\cos A$ and $\sec B$.



Solution

By Pythagoras theorem, $c^2 = a^2 + b^2 = 4^2 + 3^2 = 25$.

$$\therefore c = 5$$

$$\cos A = \frac{b}{c} = \frac{3}{5}$$

$$\sec B = \frac{c}{a} = \frac{5}{4}.$$

Example 6.

Given right triangle ABC with $\angle C = 90°$ and $\tan A = \frac{5}{12}$, find $\sin A$ and $\cos A$.

Solution

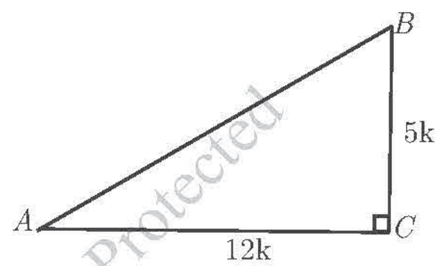
Since $\tan A = \frac{5}{12} = \frac{BC}{AC}$, we can take

$BC = 5k$ and $AC = 12k$, where k =constant.

By Pythagoras theorem,

$AB^2 = BC^2 + AC^2 = (5k)^2 + (12k)^2 = 169k^2$

$\therefore AB = 13k.$

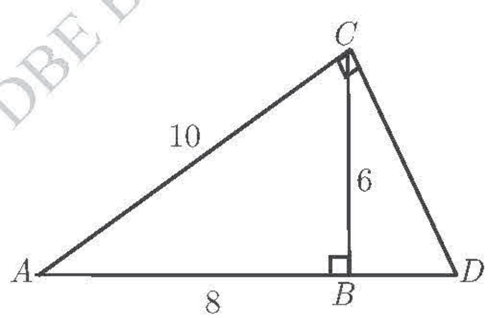


Hence, $\sin A = \frac{BC}{AB} = \frac{5k}{13k} = \frac{5}{13}$

$\cos A = \frac{AC}{AB} = \frac{12k}{13k} = \frac{12}{13}.$

Example 7.

Given right triangle ABC in which $\angle B = 90°$, CD is drawn perpendicular to CA and meets AB produced at D. If $BC = 6$, $AB = 8$, $AC = 10$, find CD and AD.



Solution

From the right triangle ABC, $\tan A = \frac{BC}{AB}$

From the right triangle ACD, $\tan A = \frac{CD}{AC}$

$$\therefore \frac{CD}{AC} = \frac{BC}{AB}$$

$$CD = \frac{BC}{AB} \times AC = \frac{6}{8} \times 10 = \frac{15}{2} = 7.5$$

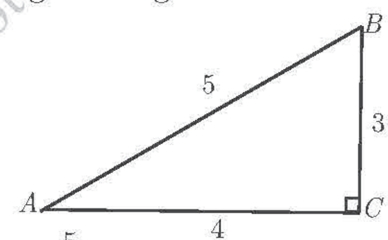
$$\frac{AD}{AC} = \sec A = \frac{AC}{AB}$$

$$AD = \frac{AC^2}{AB} = \frac{10^2}{8} = \frac{100}{8} = \frac{25}{2} = 12.5.$$

Exercise 10.2

1. Find the following trigonometric ratios for a right triangle with sides as indicated,

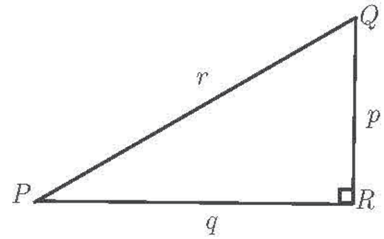
   $\sin A,\ \cos A,\ \tan A,\ \cot A,\ \sec A,\ \csc A,$
   $\sin B,\ \cos B,\ \tan B,\ \cot B,\ \sec B,\ \csc B.$



2. Given a right $\triangle ABC$ with $\angle C = 90°$ and $\cos A = \frac{5}{13}$, determine the values of $\tan A$ and $\sin A$.

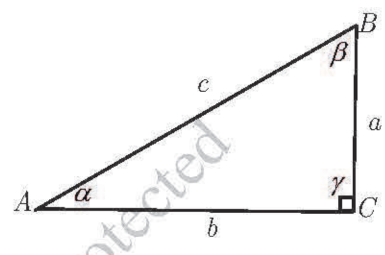
3. Given a right $\triangle PQR$ with $\angle R = 90°$; if $PQ = 13,\ QR = 5,\ PR = 12$, find the values of $\tan P$ and $\sec Q,\ \csc Q,\ \sin P$.

4. Calculate each of the following for a right $\triangle PQR$ with $\angle R = 90°$.
   (i) $(\cos P)(\sec P)$ (ii) $(\tan Q)(\cot Q)$ (iii) $(\sin P)(\csc P)$
   (iv) $\sec^2 Q - \tan^2 Q$ (v) $\sin^2 P + \cos^2 P$ (vi) $\csc^2 Q - \cot^2 Q$



5. $PQRS$ is a quadrilateral in which $\angle PSR = 90°$. If the diagonal PR is at right angles to RQ, and $RP = 20,\ RQ = 21,\ RS = 16$, find $\sin \angle PRS,\ \tan \angle RPS,\ \cos \angle RPQ$, and $\csc \angle PQR$.

10.5 Relations between the Trigonometric Ratios



Let $\triangle ABC$ be a right triangle with $\angle C = 90°$.

(i) Then $\sin A = \frac{a}{c}$, and $\csc A = \frac{c}{a}$.
$\therefore \sin A \times \csc A = \frac{a}{c} \times \frac{c}{a} = 1$.
Thus $\sin A$ and $\csc A$ are reciprocals.
$\therefore \sin A = \frac{1}{\csc A}$ and $\csc \mathrm{A} = \frac{1}{\sin A}$.

(ii) $\cos A = \frac{b}{c}$, and $\sec A = \frac{c}{b}$.
$\therefore \cos A \times \sec A = \frac{b}{c} \times \frac{c}{b} = 1$.
Thus $\cos A$ and $\sec A$ are reciprocals.
$\therefore \cos A = \frac{1}{\sec A}$ and $\sec A = \frac{1}{\cos A}$.

(iii) $\tan A = \frac{a}{b}$, and $\cot A = \frac{b}{a}$.
$\therefore \tan A \times \cot A = \frac{a}{b} \times \frac{b}{a} = 1$.
Thus $\tan A$ and $\cot A$ are reciprocals.
$\therefore \tan A = \frac{1}{\cot A}$ and $\cot A = \frac{1}{\tan A}$.

(iv) $\tan A = \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{\sin A}{\cos A}$, and $\cot A = \frac{b}{a} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{\cos A}{\sin A}$.

(v) $\sin^2 A + \cos^2 A = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1$.
Similarly, we can prove that

$$\begin{aligned} 1 + \tan^2 A &= \sec^2 A \\ \text{and} \quad 1 + \cot^2 A &= \csc^2 A. \end{aligned}$$

(vi) Let $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma = 90°$.

From the figure, we see that $\alpha + \beta = 90°$.

$$\therefore \beta = 90° - \alpha.$$

$$\therefore \sin(90° - \alpha) = \sin\beta = \frac{b}{c} = \cos\alpha \text{ and}$$

$$\cos(90° - \alpha) = \cos\beta = \frac{a}{c} = \sin\alpha.$$

Similarly, we can prove that

$$\begin{aligned} \tan(90° - \alpha) &= \cot\alpha \\ \cot(90° - \alpha) &= \tan\alpha \\ \sec(90° - \alpha) &= \csc\alpha \\ \csc(90° - \alpha) &= \sec\alpha. \end{aligned}$$

Example 8.

Prove that $\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \tan\theta$.

Solution

$$\begin{aligned} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} &= \frac{\sin\theta}{\sqrt{\cos^2\theta}} \quad (\because \sin^2\theta + \cos^2\theta = 1) \\ &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta. \end{aligned}$$

Example 9.

Verify the identity $(1-\sin\theta)(1+\sin\theta) = \dfrac{1}{1+\tan^2\theta}$.

Solution

$$\begin{aligned} (1-\sin\theta)(1+\sin\theta) &= 1-\sin^2\theta \\ &= \cos^2\theta \quad (\because \sin^2\theta + \cos^2\theta = 1) \\ &= \frac{1}{\sec^2\theta} \quad (\because \cos^2\theta = \frac{1}{\sec^2\theta}) \\ &= \frac{1}{1+\tan^2\theta}. \end{aligned}$$

Example 10.

Find the value of acute angle α when $\cos 3\alpha = \sin 2\alpha$.
Solution

$$\begin{aligned} \cos 3\alpha &= \sin 2\alpha \\ \sin(90^\circ - 3\alpha) &= \sin 2\alpha \\ \therefore 90^\circ - 3\alpha &= 2\alpha \\ 5\alpha &= 90^\circ \\ \alpha &= 18^\circ. \end{aligned}$$

Exercise 10.3

Prove the following identities.

1. $\cot\theta\sqrt{1-\cos^2\theta} = \cos\theta$.
2. $\dfrac{\tan^2\theta + 1}{\tan\theta\csc^2\theta} = \tan\theta$.
3. $(1-\sin^2\theta)(1+\cot^2\theta) = \cot^2\theta$.
4. $\tan^2\theta - \cot^2\theta = \sec^2\theta - \csc^2\theta$.
5. $\sin\theta\sec\theta\sqrt{\csc^2\theta - 1} = 1$.
6. $\sec^2\theta + \csc^2\theta = \sec^2\theta\csc^2\theta$.
7. $(1+\tan^2\theta)(1-\sin^2\theta) = 1$.
8. $(1+\tan\theta)^2 + (1-\tan\theta)^2 = 2\sec^2\theta$.
9. $\sec^2\theta\cot^2\theta - 1 = \cot^2\theta$.
10. $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$.
11. $\dfrac{1}{\sin^2\theta} - \dfrac{1}{\tan^2\theta} = 1$.
12. $(\tan\theta + \sec\theta)^2 = \dfrac{1+\sin\theta}{1-\sin\theta}$.

13. $\sin^4\theta - \cos^4\theta = 1 - 2\cos^2\theta$.

14. $\dfrac{\tan^2\theta + 1}{\tan^2\theta} = \csc^2\theta$.

15. $\sin^2\theta\tan\theta + \cos^2\theta\cot\theta + 2\sin\theta\cos\theta = \tan\theta + \cot\theta$.

16. Find the value of acute angle α in each of the following equations:
(a) $\cos 2\alpha = \sin 7\alpha$ (b) $\tan 3\alpha = \cot 2\alpha$ (c) $\sec\alpha = \csc 5\alpha$

17. Prove the identity $\cos(90° - \alpha)\tan(90° - \alpha) = \cos\alpha$.

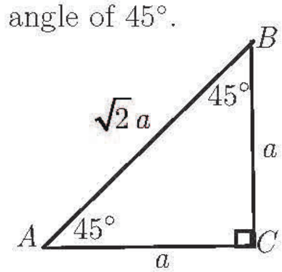
18. Prove the identity $\sin(90° - \alpha)\sec(90° - \alpha) = \cot\alpha$.

10.6 Value of the Trigonometric Ratios for Some Special Angles

Trigonometric Ratios for an Angle of 45°

In a 45°-45° right triangle by considering one with the two equal sides of length a units each, the hypotenuse is $\sqrt{2}a$.

We can now write the six trigonometric ratios for an angle of 45°.



$$\sin 45° = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{a}{a} = 1$$

The other three ratios are the reciprocals of these; thus $\csc 45° = \sqrt{2}$, $\sec 45° = \sqrt{2}$, $\cot 45° = 1$; or they may be read off from the figure.

Trigonometric Ratios for an Angle of 30° and 60°

In a 30°-60° right triangle with the shorter leg of a units in length, the hypotenuse is $2a$ and the length of the other leg is $\sqrt{3}a$.

We can now write the six trigonometric ratios for an angle of 30° and 60°.

$$\sin 30° = \frac{a}{2a} = \frac{1}{2}$$

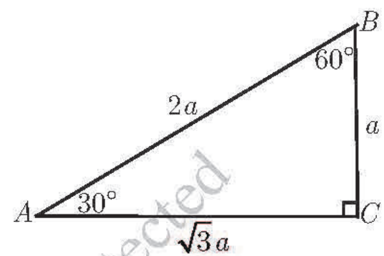
$$\cos 30° = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{a}{\sqrt{3}a} = \frac{\sqrt{3}}{3}$$

Again, $\sin 60° = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$

$$\cos 60° = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60° = \frac{\sqrt{3}a}{a} = \sqrt{3}$$



The other three ratios may be read off from the figure.

Table summarizing the trigonometric ratios for special angles.

Table 10.1

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

Example 11.

Find the values of $\csc^3 60°$ and $\sec 30° \sin 60° \cos 45°$.

Solution

$$\csc^3 60° = (\csc 60°)^3 = \left(\frac{2\sqrt{3}}{3}\right)^3 = \frac{2\sqrt{3}}{3} \times \frac{2\sqrt{3}}{3} \times \frac{2\sqrt{3}}{3} = \frac{8\sqrt{3}}{9}$$

$$\sec 30° \sin 60° \cos 45° = \frac{2\sqrt{3}}{3} \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

Example 12.

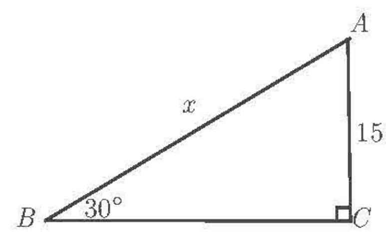
Find the value of $2\tan 45^\circ + 2\sin^3 30^\circ - 4\cos^4 30^\circ + 3\tan^2 30^\circ$.

Solution

$$\begin{aligned}
&2\tan 45^\circ + 2\sin^3 30^\circ - 4\cos^4 30^\circ + 3\tan^2 30^\circ \\
&= (2\times 1) + 2\left(\frac{1}{2}\right)^3 - 4\left(\frac{\sqrt{3}}{2}\right)^4 + 3\left(\frac{\sqrt{3}}{3}\right)^2 \\
&= 2 + \frac{1}{4} - \frac{9}{4} + 1 \\
&= 1
\end{aligned}$$

Example 13.

Find the side marked x from the given triangle.



Solution

To find x, we must choose a ratio that includes AB and AC. Thus

$$\begin{aligned}
\csc B &= \frac{AB}{AC} \\
\csc 30^\circ &= \frac{x}{15} \\
x &= 15\csc 30^\circ \\
&= 15 \times 2 = 30
\end{aligned}$$

Exercise 10.4

1. Draw a right triangle and find $\angle A$.

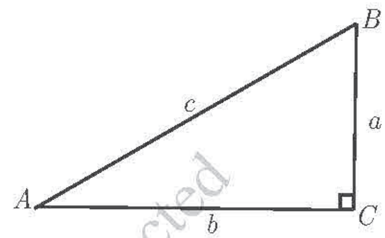
   (a) $\sin A = \frac{1}{2}$ (b) $\cos A = \frac{\sqrt{3}}{2}$

   (c) $\tan A = \sqrt{3}$ (d) $\cot A = 1$

   (e) $\sec A = \sqrt{2}$ (f) $\csc A = 2$

For each of the right triangles ABC, find the indicated sides.

2. $\angle A = 30°$, $c = 30$, find a.
3. $\angle A = 60°$, $a = 15$, find b.
4. $\angle B = 45°$, $a = 16$, find c.
5. $\angle B = 30°$, $b = 8$, find c.



6. A ladder is placed along a wall such that it upper end is touching the top of the wall. The foot of the ladder is 5 ft away from the wall and the ladder is making an angle of 60° with the level of the ground. Find the height of the wall.

Find the numerical value of:

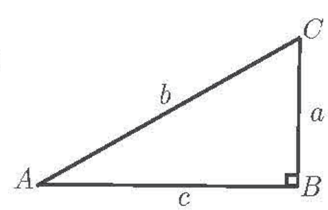
7. $\cot^3 45° + 4\sin^3 30°$
8. $\tan 60° \cot 30° + 4\sec^2 30°$
9. $\tan^2 45° + \sin 30° - \cos^2 30° + 2\tan^2 60°$
10. $\frac{1}{2}\sec^2 30° + \csc^2 45° - 2\tan^2 30°$

10.7 Solution of Right Triangles

Every triangle has six parts, namely, three sides and three angles. In the solution of right triangles there are really only two cases to be considered:

Case (1) To solve a right triangle when two sides are given.

Let $\triangle ABC$ be a right triangle with $\angle B = 90°$. Suppose that any two sides are given. Then the third side may be found from the equation $b^2 = a^2 + c^2$. Also $\sin A = \dfrac{a}{b}$, and $\angle C = 90° - \angle A$; whence $\angle A$ and $\angle C$ may be obtained.



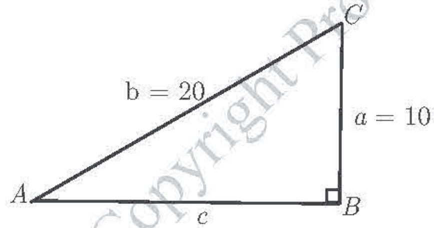
Case(2) To solve a right triangle one side and one acute angle are given.

Let $\triangle ABC$ be a right triangle with $\angle B = 90°$.
Suppose that one side c and one acute angle A are given.
Then $\angle C = 90° - \angle A$, $\dfrac{b}{c} = \sec A$, and $\dfrac{a}{c} = \tan A$;
whence $\angle C$, b and a may be obtained.

Example 14.

Solve the triangle ABC with $\angle B = 90°$, $a = 10$, $b = 20$.

Solution



Here $c^2 = b^2 - a^2 = 400 - 100 = 300$.

$$c = 10\sqrt{3}$$

$$\sin A = \frac{a}{b} = \frac{10}{20} = \frac{1}{2}$$

$$\therefore \angle A = 30°$$

$\angle C = 90° - \angle A = 90° - 30° = 60°$.

Alternative solution

$$\cos C = \frac{a}{b} = \frac{10}{20} = \frac{1}{2}$$

$$\therefore \angle C = 60°$$

$\angle A = 90° - \angle C = 90° - 60° = 30°$

$$\cos A = \frac{c}{b}$$

$$\cos 30° = \frac{c}{20}$$

$\therefore c = 20 \cos 30° = 20 \times \dfrac{\sqrt{3}}{2} = 10\sqrt{3}$.

Example 15.

Solve the triangle ABC with $\angle B = 90°, \angle A = 30°,\ c = 6.$
Solution

Here $\angle C = 90° - \angle A = 90° - 30° = 60°.$

$$\frac{a}{c} = \tan A$$

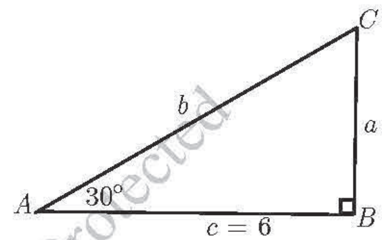
$$\frac{a}{6} = \tan 30°$$

$$\therefore a = 6\tan 30° = 6 \times \frac{\sqrt{3}}{3} = 2\sqrt{3}.$$

$$\frac{b}{c} = \sec A$$

$$\frac{b}{6} = \sec 30°$$

$$\therefore b = 6\sec 30° = 6 \times \frac{2}{\sqrt{3}} = 4\sqrt{3}.$$



Example 16.

In $\triangle ABC$ the angles A and C are equal to $30°$ and $120°$ respectively, and the side $AC = 20$ ft, find the length of the perpendicular from B upon AC produced.

Solution Draw BD perpendicular to AC produced.

Then $\theta = 180° - 120° = 60°$

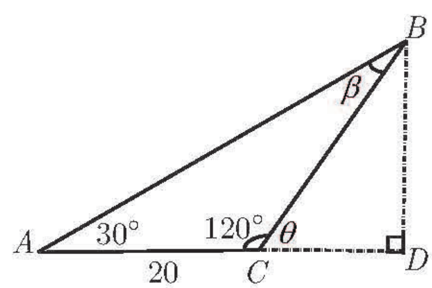
and $\beta = 180° - (30° + 120°) = 30°.$

$\therefore \angle A = \beta = 30°$ and $AC = BC = 20.$

In right $\triangle CDB$, $\quad \dfrac{BD}{BC} = \sin\theta$

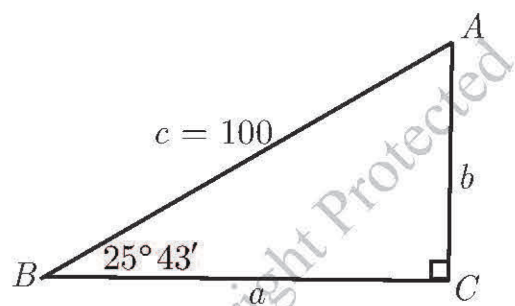
$$\frac{BD}{20} = \sin 60°$$

$$BD = 20\sin 60° = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3} \text{ ft.}$$



Example 17.

Solve the triangle ABC with $\angle C = 90°, \angle B = 25°43'$ and $C = 100$. Using as much of the information below as necessary.
$[\cos 25°43' = 0.9010, \tan 25°43' = 0.4817]$.



Solution

$\angle A = 90° - \angle B = 90° - 25°43' = 64°17'$

$$\begin{aligned}
\frac{a}{c} &= \cos B \\
a &= c\cos B = 100 \times \cos 25°43' \\
&= 100 \times 0.9010 \quad = 90.10 \\
\frac{b}{a} &= \tan B \\
b &= a\tan B = 90.10 \times \tan 25°43' \\
&= 90.12 \times 0.4817 \quad = 43.41.
\end{aligned}$$

Exercise 10.5

1. Solve the triangles:
   (a) $\angle A = 90°, \quad a = 4, \quad c = 2\sqrt{3}$.
   (b) $\angle B = 90°, \quad c = 6, \quad b = 12$.
   (c) $\angle C = 90°, \quad \angle A = 30°, \quad a = 6\sqrt{3}$.
   (d) $\angle A = 30°, \quad \angle B = 60°, \quad b = 10\sqrt{3}$.

2. Given $\triangle ABC$ with $\angle A = 30°$, $\angle B = 135°$ and $AB = 100$, find the length of the perpendicular from C to AB produced.

3. If BD is perpendicular to the base AC of a triangle ABC, find a and c, given $\angle A = 30°$, $\angle C = 45°, BD = 10$.

4. In the triangle ABC, the angles B and C are equal to $45°$ and $120°$ respectively; if $a = 40$, find the length of the perpendicular from A on BC produced.

5. Solve the isosceles triangle. (Draw the perpendicular from the vertex to the base.)

   (a) $b = c = 12, \quad \angle B = 30°$.

   (b) $a = b = 9, \quad \angle A = 60°$.

   (c) $b = c = 10, \quad \angle A = 120°$.

6. In $\triangle ABC$, $\angle A = 60°$, $AC = 12$ and $AB = 20$. Find the length of the perpendicular drawn from C to AB. Also find the area of the triangle ABC.

7. Find the area of the triangle ABC, given $AB = 30$, $BC = 16$ and $\angle B = 30°$.

8. Solve the triangle ABC with $\angle B = 90°, \angle A = 36°$ and $c = 100$. Using as much of the information below as necessary.

   $[\tan 36° = 0.7265, \sec 36° = 1.2361]$

9. Solve the triangle ABC with $\angle A = 90°, c = 37$ and $a = 100$. Using as much of the information below as necessary.

   $[\sin 21°43' = 0.3700, \cos 21°43' = 0.9290]$

10.8 Angle of Elevation and Angle of Depression

Let OA be a horizontal line in the same vertical plane as an object B. Let O and B be joined.

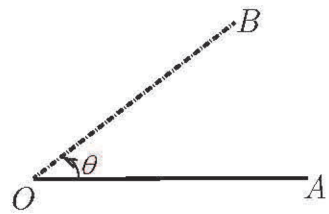


Figure 10.6(a)

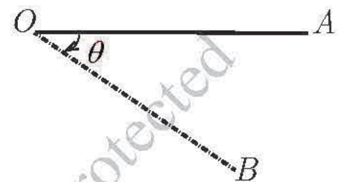


Figure 10.6(b)

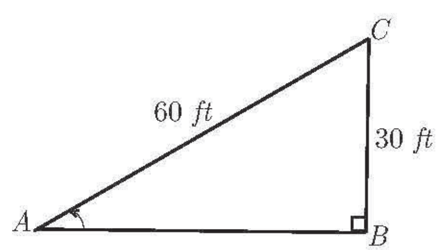
In Fig. 10.6(a), where the object B is above the horizontal line OA, θ is called **the angle of elevation** of the object B as seen from the point O.

In Fig. 10.6(b), where the object B is below the horizontal line OA, θ is called **the angle of depression** of the object B as seen from the point O.

Example 18.

A kite is at the end of a 60 ft string that is taut. It is 30 ft above the ground. What is the angle of elevation of the kite?

Solution



$$\sin A = \frac{30}{60} = \frac{1}{2}$$

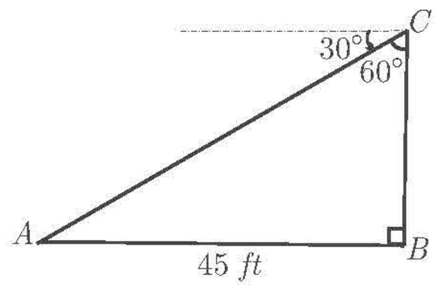
$$\angle A = 30^\circ$$

Therefore, the angle of elevation of the kite is 30°.

Example 19.

From the top of a house the angle of depression to a point on the ground is 30°. The point is 45 ft from the base of the building. How high is the building?

Solution



The angle of depression is not within the triangle. However, the complement of the angle 60°, is the measure of $\angle B$ inside the triangle.

Thus $\cot 60° = \dfrac{BC}{AC}$

$$BC = AC \cot 60°$$
$$= 45 \times \frac{1}{\sqrt{3}}$$
$$= 15\sqrt{3} \text{ ft}$$

Exercise 10.6

1. A mountain railway runs for 400 yards at a uniform slope of 30° with the horizontal. What is the horizontal distance between its two ends?

2. A vertical mast is secured from its top by straight cables 600 ft long fixed into the ground. Each cable makes angle of 60° with the ground. What is the height of the mast?

3. A kite flying at a height of 60 yards is attached to a string inclined at 45° to the horizontal. What is the length of the string? Assume there is no slack in the string.

4. An observer, 6 ft tall, is 20 yards away from a tower 22 yards high. Determine the angle of elevation from his eye to the top of the tower.

5. Two observers are on the opposite sides of a tower. They measure the angles of elevation to the top of the tower as 30° and 45° respectively. If the height of the tower is 40 yards, find the distance between them.

6. Find the angle of elevation of the sun when the shadow of a pole 12 ft high is $4\sqrt{3}$ ft long.

7. Two masts are 60 ft and 40 ft high, and the line joining their tops makes an angle of 30° with the horizon, find the distance between them.

8. From the foot of a tower, the angle of elevation to the top of a 60 ft column is 60°. The angle of elevation from the top of the tower to the top of the column is 30°. Find the height of the tower.