Preface

This text book is the Grade 10 High School Mathematics written in accordance with the new Curriculum. This volume is a continuation of the first one which is already prescribed for Grade 10 students. Most of the chapters in this text spirally develop the mathematical concept and skill, and try to instil in students a deeper and thorough understanding of basic ideas in mathematics.

The general aims of teaching mathematics at high school level are specified as follows: At the end of Secondary Level, students should be able

- ➢ to gain basic mathematical knowledge and understanding
- ➢ to acquire necessary mathematical skill
- ➢ to apply mathematical knowledge and skill in real-life situations, and
- ➢ to interest in, and appreciation of mathematics, together with the development of maths-related values.

Chapter 1 introduces basic ideas of coordinate geometry. This chapter expresses the properties of mid-point, properties of slopes, distance between two points and equation of a line.

Chapter 2 emphasizes the basic properties of exponents, radicals and solving the exponential equations. The main purpose is to use the properties of the exponents and radicals correctly.

Chapter 3 emphasizes the basic properties of logarithms, change of base and using common logarithms. Ideas mainly developed in this chapter are the algebra of logarithm of a positive number.

Chapter 4 is concerned with the functions. The study of functions aim to develop an understanding of functions as mappings and to recognize functions as relations between sets. Also the basic properties of composite functions and inverse functions are included.

Chapter 5 deals with the methods for solving a quadratic equation are discussed. Also discrimination of quadratic and the graphical solution of quadratic inequations are expressed.

Chapter 6 introduces basic ideas in absolute value of linear function and inequality of absolute.

Chapter 7 is concerned with the probability of an event. The uses of tree diagrams and table of outcomes are emphasized in calculating the probability. Method of calculating expected frequency is involved.

Chapter 8 and 9 attempt to stress the formal structure of geometry and integrate geometry with arithmetic and algebra. Emphasis is laid down upon the use of precise language in the statements of definitions, postulate, and theorems. Chapter 8 is about basis ideas of similar triangles, angle bisector theorem, and extension of the Pythagoras Theorem.

Chapter 9 deals with the definitions of circle, and properties of chords.

Chapter 10 which is the last chapter of this text, attempts to broaden the student's understanding of geometric properties, and interrelations between sides and angles of plane figures. A general feature of an angle is first presented followed by relation between degree and radian measures. Then, the definitions of six trigonometric ratios are extended to include all angles.

After learning this course, students will develop and practise higher order thinking skills: comprehension, analysis, synthesis and evaluation. They will be able to participate actively in all lessons through the **5 C's** as important **21st century skills for learning**:

- **Collaboration** - in lesson students will be working in groups, to share ideas with their classmates and to find the solution together
- **Communication** - students will develop verbal and non-verbal communication skills in group works
- **Critical thinking and problem solving** - students will be given interesting problems to solve-finding and explaining solutions, looking for correcting errors
- **Creativity and innovation** - thinking 'outside the box' is an important 21st century skill. Students will be encouraged to explore new ideas and solve problems in new ways.
- **Citizenship** - students will join the school community and develop fairness and conflict resolution skills.

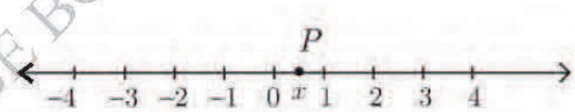
An important feature of these texts for high school level is that ideas, concepts, principles and methods are integrated within each branch of mathematics and across the branches.

Finally, students are encouraged to work through the text, to acquire basic mathematical knowledge and skill, to apply them to real-life situation and to develop mathematical thinking and reasoning.

Chapter 1

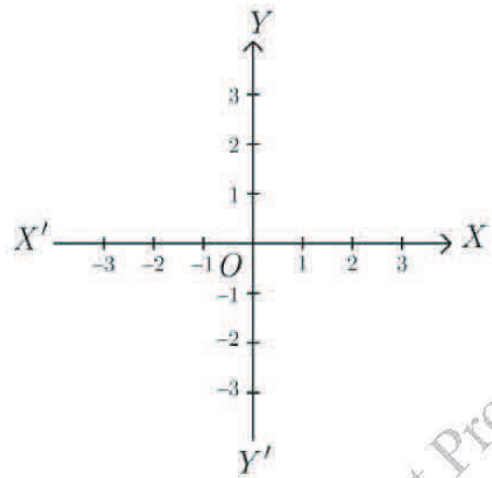
Introduction to Coordinate Geometry

René Descarts' greatest contribution to mathematics was the discovery of coordinate systems and their applications to problems of geometry. Coordinate system used in this book is referred to as Cartesian coordinate system.
We have seen how coordinate systems work on a line.

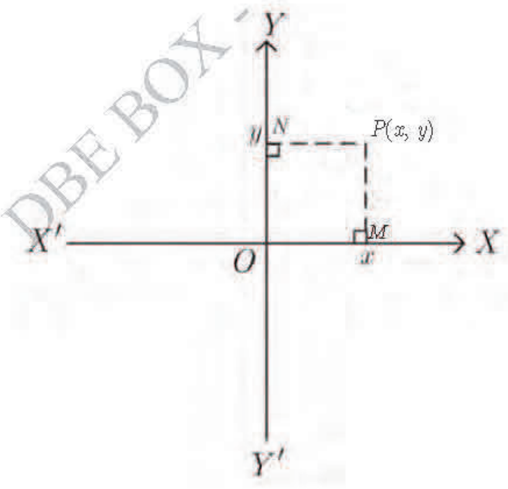


Once we have set up a coordinate system on a line, every number corresponds to a point on the line and every point on the line corresponds to a number. We shall now extend this idea to the points in a plane, a point will correspond not to a single number, but to an ordered pair of numbers. The scheme works like this.

First we take two perpendicular lines, a horizontal line $X'OX$(the $\boldsymbol{X}$**-axis**) and a vertical line $Y'OY$(the $\boldsymbol{Y}$**-axis**), they intersect at zero point in the $\boldsymbol{XY}$**-plane**. The zero point which is the intersection of these two lines is called **the origin**, normally labeled O. On the X-axis, values to the right are positive and those to the left are negative. On the Y-axis, values above the origin are positive and those below are negative.

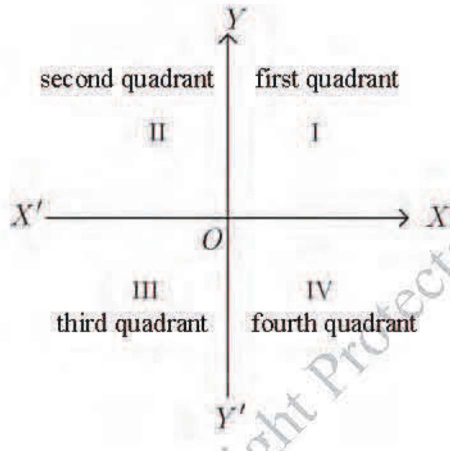


We can now describe any point P of the plane by an ordered pair of numbers, as follows. Draw PM and PN, perpendicular to the X-axis and Y-axis. Let x be the coordinate of M on the line $X'OX$ and y be the coordinate of N on the line $Y'OY$.

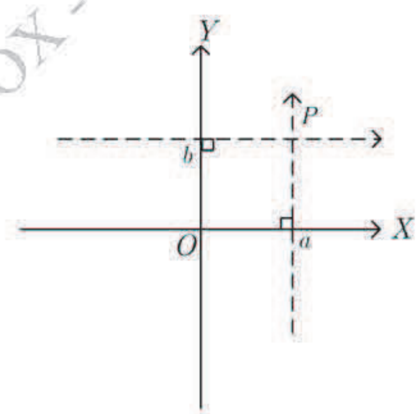


The number x and y are called the x-coordinate and y-coordinate of P respectively. In short, we indicate that P has these coordinates by writing $P(x, y)$. In particular, the origin O has coordinates $(0, 0)$.

Just as single line separates the plane into two parts (each of which is a half-plane), so the two axes separate the XY-plane into four parts, called quadrants. The four quadrants are identified by the Roman numerals, I, II, III, IV.



We have shown that under the scheme, every point P determines an ordered pair of real numbers. Does it work in reverse? That is, does every ordered pair (a, b) of real numbers determine a point? It is easy to see that the answer is "Yes".



Mark a point on the X-axis, so that the x-coordinate of that point is a. Draw a perpendicular line passes through that point. Then draw another perpendicular line at the point in which the y-coordinate is b. The point where these perpendiculars intersect is the point with coordinates (a, b).

Thus we have a one-to-one correspondence between the points of the plane and the ordered pairs of real numbers. Such a correspondence is called a **rectangular coordinate system**.

To describe such a coordinate system, we need to choose
(i) a line $X'OX$ to be the X-axis,
(ii) a line $Y'OY$ perpendicular to $X'OX$ to be the Y-axis, and
(iii) a positive direction on each of the axes.

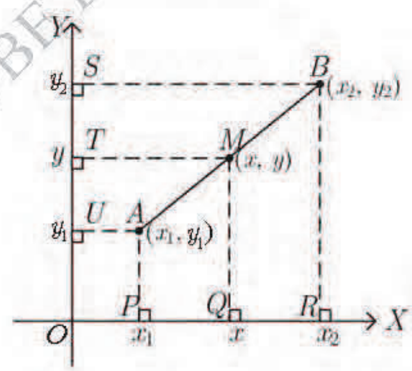
Once we have made these choices, the coordinate systems on both axes are determined, and they in turn determine the coordinates of all points of the plane. This plane is referred as XY-plane. With reference to a coordinate system, every point P determines an ordered pair (a, b) and every ordered pair (a, b) determines a point.

1.1 Midpoint and Length of a Line Segment

Every line segment has a midpoint. Both the midpoint and length of a line segment can be found by using the coordinates of the endpoints.

Midpoint of a Line Segment in XY-plane
The midpoint M of a line segment is the halfway point between the two endpoints.

To find the coordinates of the midpoint of a non-horizontal, non-vertical line segment joining the two given points in the XY-plane.



Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points and $M(x, y)$ be the midpoint of AB. We will now find the coordinates of M in terms of x_1, x_2, y_1 and y_2.

Draw the perpendicular lines AP, MQ, BR to the X-axis as shown. Draw the perpendicular lines AU, MT, BS to Y-axis. Clearly AU, MT, BS are horizontal lines while AP, MQ, BR are vertical lines. The points M and Q

have the same x-coordinates and M and T have the same y-coordinates.

Q is the midpoint of PR, the x-coordinate of Q is $\dfrac{x_1 + x_2}{2}$.

T is the midpoint of SU, the y-coordinate of T is $\dfrac{y_1 + y_2}{2}$.

$\therefore$ x-coordinate of midpoint $M = x = \dfrac{x_1 + x_2}{2}$, and

y-coordinate of midpoint $M = y = \dfrac{y_1 + y_2}{2}$.

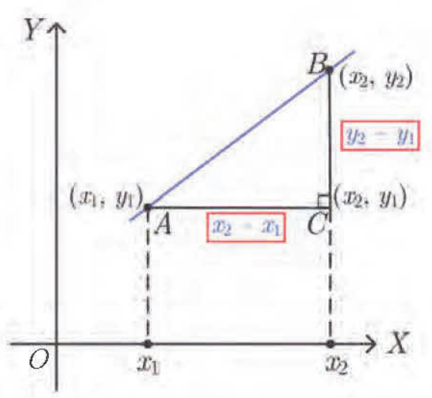
Midpoint Formula:
The coordinates of midpoint of $A(x_1, y_1)$ and $B(x_2, y_2)$ are
$$M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 1.
Find the coordinates of the midpoint of PQ with endpoints $P(2, 5)$ and $Q(6, 1)$.

Solution

If M is the midpoint of PQ, then
$$M = \left(\frac{2+6}{2}, \frac{5+1}{2}\right) = (4, 3).$$

The Length of a Line Segment in XY-plane



Let AB be a non-horizontal, non-vertical line segment with $A(x_1, y_1)$ and $B(x_2, y_2)$. Complete a right-angled triangle ABC as shown.
The line segment is the hypotenuse of a right-angled triangle. You may use Pythagoras' theorem to find the length of a straight line segment.

Horizontal distance $AC = x_2 - x_1$
Vertical distance $BC = y_2 - y_1$

By Pythagoras' theorem:

$AB^2 = AC^2 + BC^2$
$AB = \sqrt{AC^2 + BC^2}$

Therefore, the length of line segment AB is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Distance Formula:
The distance between two points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 2.
Find the length of PQ with endpoints $P(1, 5)$ and $Q(4, 1)$.
Solution
Let $(1, 5) = (x_1, y_1)$ and $(4, 1) = (x_2, y_2)$.

$$\begin{aligned} \text{Length of } PQ &= \sqrt{(4-1)^2 + (1-5)^2} \\ &= \sqrt{(3)^2 + (-4)^2} \\ &= \sqrt{9 + 16} = \sqrt{25} = 5. \end{aligned}$$

Example 3.
A is the point $(5, -3)$ and B is the point $(-2, 1)$.
(a) Find the midpoint of AB. (b) Find the length of AB.
Solution
(a) Let $(5, -3) = (x_1, y_1)$ and $(-2, 1) = (x_2, y_2)$.

$$\text{Midpoint of } AB = \left(\frac{5 + (-2)}{2}, \frac{-3 + 1}{2}\right) = \left(\frac{3}{2}, \frac{-2}{2}\right) = (1.5, -1)$$

(b)
$$\text{Length of } AB = \sqrt{(-2-5)^2 + (1-(-3))^2} = \sqrt{49 + 16} = \sqrt{65}$$

Example 4.

The point $M(a, 4)$ is the midpoint of the line segment with endpoints at $A(1, 3)$ and $B(5, b)$. Find the value of a and of b.

Solution

Let (x_1, y_1) be $(1, 3)$ and (x_2, y_2) be $(5, b)$.

$$\begin{aligned} \text{Midpoint of } AB &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ (a, 4) &= \left(\frac{1 + 5}{2}, \frac{3 + b}{2}\right). \end{aligned}$$

Equating the x-coordinates and y-coordinates respectively, we get

$$\begin{aligned} a = \frac{1 + 5}{2} \quad &, \quad 4 = \frac{3 + b}{2} \\ a = 3 \quad &, \quad b = 5. \end{aligned}$$

Example 5.

The distance between two points $R(9, a)$ and $S(a + 1, 2)$ is 6. Find the two possible values of a.

Solution

Let (x_1, y_1) be $(9, a)$ and (x_2, y_2) be $(a + 1, 2)$.

Using distance formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and $RS = 6$, we get

$$\begin{aligned} \sqrt{(a + 1 - 9)^2 + (2 - a)^2} &= 6 \\ (a - 8)^2 + (2 - a)^2 &= 6^2 \\ a^2 - 16a + 64 + 4 - 4a + a^2 &= 36 \\ 2a^2 - 20a + 32 &= 0 \\ a^2 - 10a + 16 &= 0 \\ (a - 8)(a - 2) &= 0 \\ a - 8 = 0 &\text{ or } a - 2 = 0 \\ a = 8 &\text{ or } a = 2. \end{aligned}$$

Example 6.

In a parallelogram $ABCD$, three vertices are $A(-3,1), B(2,4)$ and $C(3,1)$.
(a) Find the midpoint of the diagonal AC. (b) Find the coordinates of D.

Solution

(a) Midpoint of $AC = \left(\dfrac{-3+3}{2}, \dfrac{1+1}{2}\right) = (0,1)$.

(b) Let the coordinates of D be (x,y).

$$\begin{aligned} \text{Midpoint of } AC &= \text{Midpoint of } BD \qquad (\because ABCD \text{ is a prallelogram.}) \\ (0,1) &= \left(\frac{2+x}{2}, \frac{4+y}{2}\right) \end{aligned}$$

Equating the x-coordinates and y-coordinates respectively, we have

$$\begin{aligned} \frac{2+x}{2} &= 0 \quad \text{and} \quad \frac{4+y}{2} = 1 \\ 2+x &= 0 \quad \text{and} \quad 4+y = 2 \\ x &= -2 \quad \text{and} \quad y = -2. \end{aligned}$$

$\therefore D(x,y) = (-2,-2)$.

Exercise 1.1

1. Draw a set of coordinate axes. Locate the points, $A(2,3)$, $B(2,-4)$ and $C(-4,3)$. Lable each point with its coordinates. Determine whether each of the line segments AB, BC and CA is horizontal or vertical.

2. Find the missing coordinates in the following table if M is the midpoint of points P and Q.

P	Q	M
$(2,6)$		$(3,3)$
$(3,2)$	$(-3,-1)$	
	$(0,-1)$	$(-3,2)$
$(1,5)$		$(2.5,3.5)$

3. Find the coordinates of the midpoint and the length of the line segment joining these pairs of points.
(a) $(0,0)$ and $(4,-4)$ (b) $(1,5)$ and $(3,1)$ (c) $(-3,-3)$ and $(0,0)$
(d) $(-1,3)$ and $(5,1)$ (e) $(-1,6)$ and $(2,-2)$ (f) $(-3,-4)$ and $(3,-1)$

4. If $(1,0)$ is the midpoint of the line passing through the points $A(-5,2)$ and $B(x,y)$, find the value of x and of y.

5. Calculate the perimeter of given polygons correct to one decimal place.
   (a) A triangle with vertices $P(-2,3), Q(5,-4)$ and $R(1,8)$.
   (b) A parallelogram with vertices $A(-10,1), B(6,-2), C(14,4)$ and $D(-2,7)$.
   (c) A trapezium with vertices $E(-6,-2), F(1,-2), G(0,4)$ and $H(-5,4)$.

6. A circle has centre (2, 1). Find the coordinates of the endponit of a diameter if one endpoint is $(7,1)$.

7. $\triangle KLM$ has vertices $K(-5,18), L(10,-2)$ and $M(-5,-10)$.
   (a) Find the length of each side.
   (b) Find the perimeter of $\triangle KLM$.
   (c) Find the area of $\triangle KLM$.

8. Prove that the triangle whose vertices are $P(2,3)$, $Q(-1,-1)$, $R(3,-4)$ is isosceles.

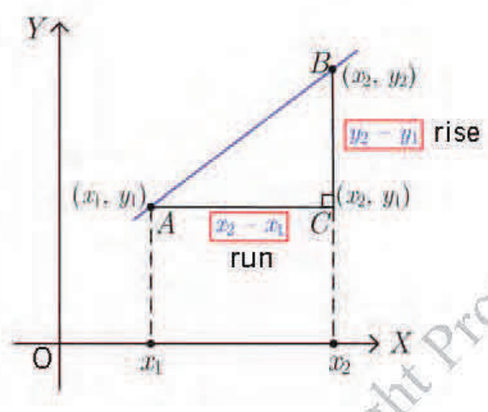
9. A triangle has vertices $E(0,7), F(5,-5)$ and $G(10,7)$. Find the length of the altitude to the shortest side.

10. The vertices of a quadrilateral are $A(4,-3), B(7,10), C(-8,2)$ and $D(-1,-5)$. Find the length of each diagonal.

11. The distance between the two points $M(15,a)$ and $N(a,-5)$ is 20. Find the value of a.

12. In a parallelogram $PQRS$, three of vertices are $P(1,1)$, $Q(2,6)$ and $R(5,3)$. Find the midpoint of PR and use it to find the fourth vertex S. Find also the lengths of the diagonals.

1.2 Slope of a Straight Line



The slope of the line passing through any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is a measure of the steepness of the line AB. Simply it is the ratio of the vertical change (rise) divided by the horizontal change (run). To determine rise and run, select any two points on the line. The horizontal distance between these two points is called the run and the vertical distance is called the rise.

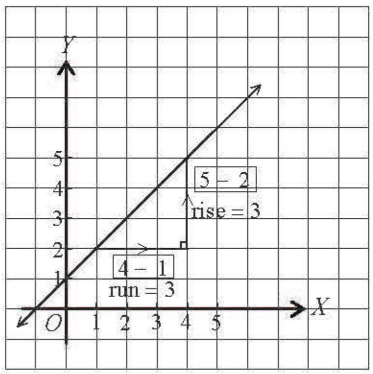
$$\text{slope m} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}} = \frac{BC}{AC} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad \frac{y_1 - y_2}{x_1 - x_2}.$$

Slope Formula:
The slope of the line passing through any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\text{Slope m} = \frac{y_2 - y_1}{x_2 - x_1}.$$

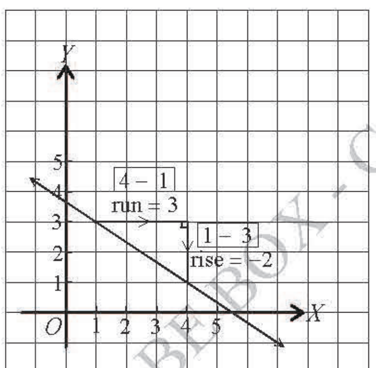
For example, to find the slope of a slanting line, select any two points on a line to determine the rise and run.

(i) Positive Slope



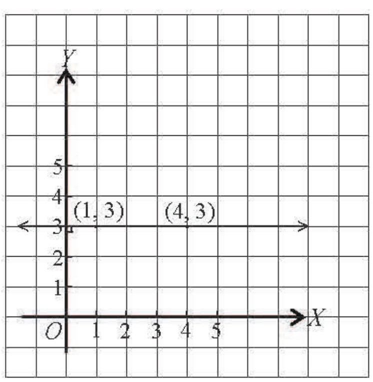
create a right-angled triangle to determine rise and run
increase in x, run = 3
increase in y, rise = 3
slope $= \frac{\text{rise}}{\text{run}} = \frac{3}{3} = 1$

(ii) Negative Slope



increase in x, run = 3
decrease in y, rise = −2
slope $= \frac{\text{rise}}{\text{run}} = \frac{-2}{3} = -\frac{2}{3}$

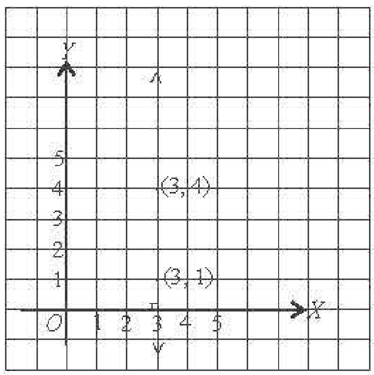
(iii) Zero Slope



increase in x, run = 3
decrease in y, rise = 0
slope $= \frac{\text{rise}}{\text{run}} = \frac{3-3}{4-1} = \frac{0}{3} = 0$

Note that the slope of the horizontal line is zero.

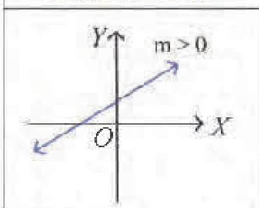
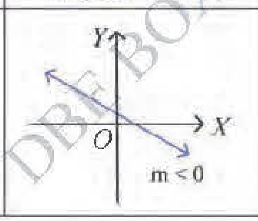
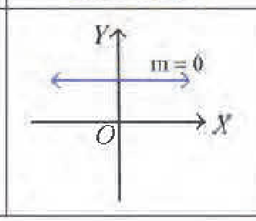
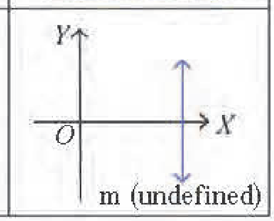
(iv) Undefined Slope



increase in x, run $= 0$
decrease in y, rise $= 3$
slope $= \frac{\text{rise}}{\text{run}} = \frac{4-1}{3-3} = \frac{3}{0}$ (undefined)
Since the denominator is zero,
so the ratio is undefined.

Note that the vertical line has undefined slope.

As shown in the examples, slope can be positive, negative, zero or undefined. By looking at the graph of a line, you may know these cases without calculation. The following table will help you.

Positive Slope	Negative Slope	Zero Slope	Undefined Slope
			

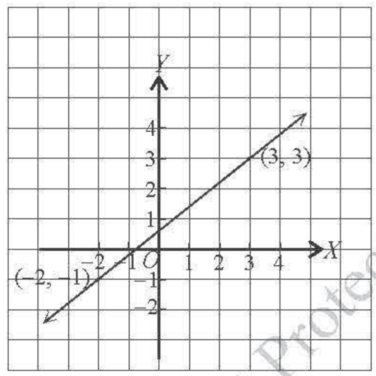
Example 7.
Find the slope of the line AB that passes through the points $A(4, -2)$ and $B(-1, 2)$.

Solution
Let (x_1, y_1) be $(4, -2)$ and (x_2, y_2) be $(-1, 2)$.

$$\text{Slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(-2)}{-1-4} = -\frac{4}{5}.$$

The slope of the line AB is $-\frac{4}{5}$.

Example 8.
Find the slope of the given line.



Solution
Let (3, 3) be (x_1, y_1) and (−2, −1) be (x_2, y_2).

$$\text{Slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-3}{-2-3} = \frac{-4}{-5} = \frac{4}{5}.$$

Example 9.
Find the slope of the line that passes through the following points.

x	0	1	2	3
y	5	5	5	5

Solution
Let us choose any two points on the line from the table.
Let (0, 5) be (x_1, y_1) and (2, 5) be (x_2, y_2).

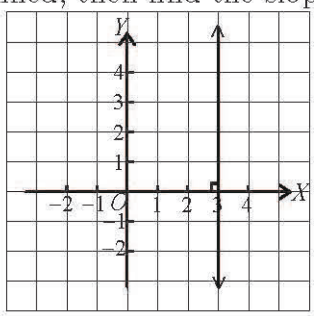
$$\text{Slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-5}{2-0} = \frac{0}{2} = 0.$$

Exercise 1.2

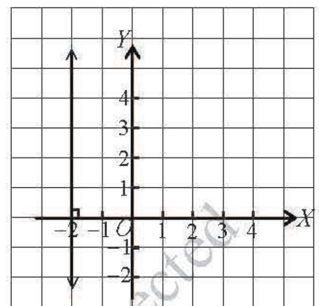
1. Complete each sentence.
   (a) The slope of the line passing through two points $(-6, 0)$ and $(2, 3)$ is ———.
   (b) The slope of the line joining the point (1, 2) and the origin is ———.
   (c) A vertical line has ——— slope.
   (d) A horizontal line has ——— slope.

2. For each graph state whether the slope is positive, negative, zero or undefined, then find the slope if possible.

(a)



(b)



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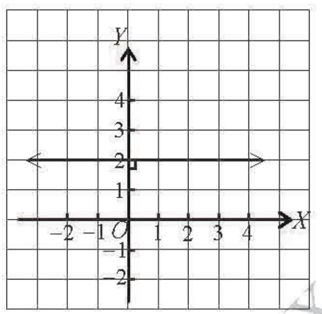
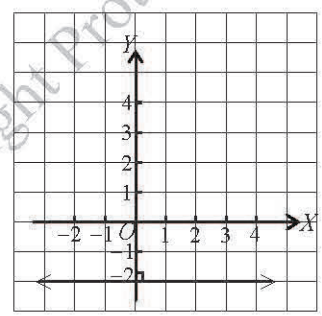


(d)



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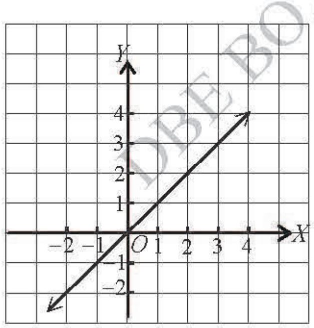

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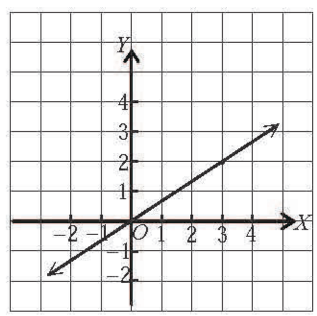

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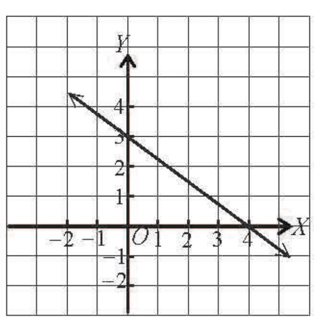

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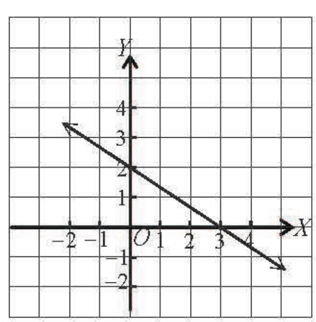


3. Which pairs of points given below will determine horizontal lines? Which ones vertical lines? Determine the slope of each line without calculation.
(a) $(5, 2)$ and $(-3, 2)$ (b) $(0, 5)$ and $(-1, 5)$ (c) $(2, 3)$ and $(2, 6)$
(d) $(0, 0)$ and $(0, -2)$ (e) $(1, -2)$ and $(-3, -2)$ (f) (a, b) and (a, c)

4. Find the slope of each line which contains each pair of points listed below.
(a) $A(0, 0)$ and $B(8, 4)$ (b) $C(10, 5)$ and $D(6, 8)$
(c) $E(-5, 7)$ and $F(-2, -4)$ (d) $G(23, 15)$ and $H(18, 5)$
(e) $I(-2, 0)$ and $J(0, 6)$ (f) $K(15, 6)$ and $L(-2, 23)$

5. Find the slope of each line which contains each pair of points listed below.
(a) $E(\frac{3}{4}, \frac{4}{5})$ and $F(-\frac{1}{2}, \frac{7}{5})$ (b) $G(-a, b)$ and $H(3a, 2b)$
(c) $L(\sqrt{12}, \sqrt{18})$ and $M(\sqrt{27}, \sqrt{8})$ (d) $P(0, a)$ and $Q(a, 0)$

6. Find p, q, r in the followings:
(a) The slope joining the points $(0, 3)$ and $(1, p)$ is 5.
(b) The slope joining the points $(-2, q)$ and $(0, 1)$ is -1.
(c) The slope joining the points $(-4, -2)$ and $(r, -6)$ is -6.

7. Find the slope corresponding to the following events.
(a) A man climbs 10 m for every 200 meters horizontally.
(b) A motorbike rises 3 m for every 10 meters horizontally.
(c) A plane takes off 1 km for every 5 kilometers horizontally.
(d) A submarine descends 120 m for every 15 meters horizontally.

8. A train climbs a hill with slope 0.05. How far horizontally has the train travelled after rising 15 meters?

9. The vertices of a triangle are the points $A(-2, 3), B(5, -4)$ and $C(1, 8)$. Find the slope of each side.

10. The vertices of a parallelogram are the points $P(1, 4), Q(3, 2), R(4, 6)$ and $S(2, 8)$. Find the slope of each side.

11. A line having a slope of -1 contains the point $(-2, 5)$. What is the y-coordinate of the point on that line whose x-coordinate is 8?

1.3 Lines in the Coordinate Plane

Graph of a Linear Equation

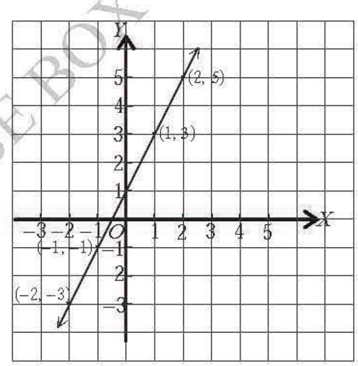
A linear equation is an algebraic equation, in which each term has an exponent of one. To draw the graph of a linear equation $y = mx + c$, you need to plot the coordinates of the points on the line. Construct a table of values of x and y. Choose any two or more values of x within the given interval. If the interval is not given you may choose any values of x. To find the value of y, substitute each value of x in the given linear equation. x and y are the variables in the equation, which means you may take any values. Then plot the points. You will get a straight line graph.

For example, we consider the graph of $y = 2x + 1$.

To draw the given graph, you may choose any the real values of x. Work out the y values and put them in a table.

x	-2	-1	0	1	2
y	-3	-1	1	3	5

Plot the points in the coordinate plane.



To find the slope of given linear equation $y = 2x + 1$, take any points on the line.

If you choose (x_1, y_1) be $(1,3)$ and (x_2, y_2) be $(2,5)$, then

$$\text{slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{2-1} = \frac{2}{1} = 2.$$

If you choose (x_1, y_1) be $(-2, -3)$ and (x_2, y_2) be $(1, 3)$, then

$$\begin{aligned} \text{slope } m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3 - (-3)}{1 - (-2)} = \frac{6}{3} = 2. \end{aligned}$$

Now you can see how the slope of this straight line is related to the equation $y = 2x + 1$. Notice that the slope of the line is equal to the coefficient of x in the given linear equation $y = mx + c$. The point where the line cuts the Y-axis (y-intercept) has a y coordinate that is equal to the constant term in the equation.

Slope-Intercept Form

The equation of the form $y = mx + c$ is the equation of a straight line with slope m and y-intercept c, which is called the slope-intercept form.

Point-Slope Form

We can also find the equation of a line if we are given a point on the line and its slope.

Consider a line with slope m, and passes through the known point $A(x_1, y_1)$. Let $P(x, y)$ be any point on the line. Then

$$m = \frac{y - y_1}{x - x_1}.$$

$$y - y_1 = m(x - x_1).$$

This is the equation of a straight line, with slope m, and passes through the point (x_1, y_1) which is called the point-slope form.

Example 10.

Find the y-intercept and the slope of each line.

(a) $y - 3x + 4 = 0$ (b) $y + 5x = 1$ (c) $x + y = 8$

Solution

equation	$y = mx + c$	slope	y-intercept
$y - 3x + 4 = 0$	$y = 3x - 4$	3	-4
$y + 5x = 1$	$y = -5x + 1$	-5	1
$x + y = 8$	$y = -x + 8$	-1	8

Example 11.

Find the equation of the line with slope -5 and passes through the point $(2, 0)$ and draw the graph.

Solution

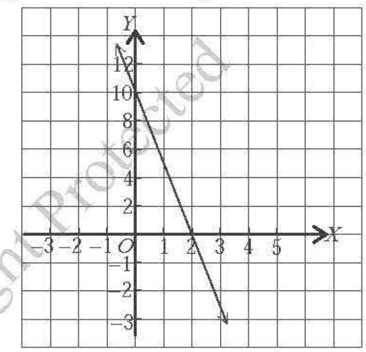
The equation of the line is $y = -5x + c$, where c is a constant.

Since the point (2, 0) is on the line, substituting $x = 2$ and $y = 0$ in the equation,

$0 = -5(2) + c$

$c = 10.$

Thus the equation of the line is $y = -5x + 10$.



Alternative Method:

Since the point (2, 0) is on the line, substituting $x_1 = 2$ and $y_1 = 0$, and slope $m = -5$ in point-slope form,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= -5(x - 2) \\ y &= -5x + 10. \end{aligned}$$

Thus the equation of the line is $y = -5x + 10$.

Example 12.

Find the equation of the line passing through the points $A(1, 3)$ and $B(4, 9)$. Find also the y-intercept.

Solution

Let (x_1, y_1) be $(1, 3)$ and (x_2, y_2) be $(4, 9)$.

$$\text{slope } m = \frac{9-3}{4-1} = \frac{6}{3} = 2.$$

$\therefore$ the equation of the line is $y = 2x + c$, where c is y-intercept.

Since the line passing through (1, 3), substituting $x = 1$ and $y = 3$ in the equation,

$y = 2x + c$

$3 = 2(1) + c$

$c = 1.$

Thus the equation of the line is $y = 2x + 1$ and the y-intercept is 1.

Alternative Method:

Let (x_1, y_1) be $(1, 3)$ and (x_2, y_2) be $(4, 9)$.

$$\text{slope } m = \frac{9-3}{4-1} = \frac{6}{3} = 2.$$

Since the point (1, 3) is on the line, substituting $x = 1$ and $y = 3$, and slope $m = 2$ in point-slope form,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= 2(x - 1) \\ y &= 2x + 1. \end{aligned}$$

Thus the equation of the line is $y = 2x + 1$, and the y-intercept is 1.

Example 13.

The coordinates of the points A and B are $(2, 5)$ and $(-1, 5)$ respectively. Find the equation of the line AB.

Solution

Let (x_1, y_1) be $(2, 5)$ and (x_2, y_2) be $(-1, 5)$.

$$\text{slope } m = \frac{5-5}{-1-2} = \frac{0}{-3} = 0.$$

Equation of AB is of the form $y = 0x + c$. Since the point (2, 5) is on the line, substituting $x = 2$ and $y = 5$ into the equation,

$5 = 0 \times 2 + c$

$c = 5$.

Thus the equation of the line is $y = 5$.

Alternative Method:

The point (2, 5) is on the line. Substituting $x = 2$ and $y = 5$, and slope $m = 0$ in point-slope form,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 5 &= 0(x - 2) \\ y &= 5. \end{aligned}$$

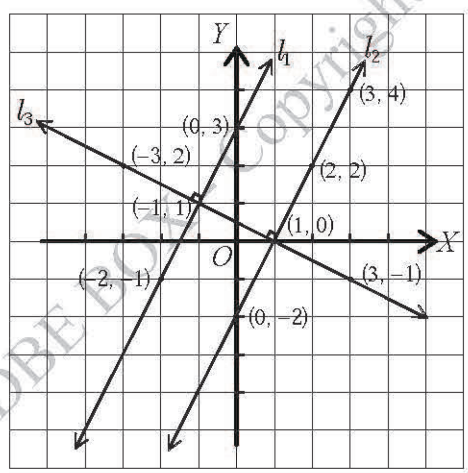
Thus the equation of the line is $y = 5$.

Note that the equation of the horizontal line is $y = a$, $a =$ constant and the equation of the vertical line is $x = b$, $b =$ constant. On a line, all segments have the same slope. Points on the same straight line are said to be collinear.

Parallel and Perpendicular Lines

Any two horizontal lines are parallel. Any two vertical lines are parallel. Vertical line and horizontal line are perpendicular. *Two non-vertical lines are parallel if and only if they have the same slope. Two non-vertical lines are perpendicular if and only if the product of their slopes is* -1 (*i.e., one is the negative reciprocal of the other*).

For example, in the following figure, the lines l_1 and l_2 are parallel, and the lines l_1 and l_2 are perpendicular to the line l_3.



$$\text{slope of line } l_1 = m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{0-(-1)} = \frac{2}{1} = 2$$

$$\text{slope of line } l_2 = m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{3-2} = \frac{2}{1} = 2$$

$$\text{slope of line } l_3 = m_3 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-2}{3-(-3)} = \frac{-3}{6} = -\frac{1}{2}$$

$\therefore m_1 = m_2,\ m_1 m_3 = m_2 m_3 = -1.$

Exercise 1.3

1. Sketch the following lines.
   (a) $y = 3$ (b) $x = -2$ (c) $y = 5x$ (d) $y = -3x$ (e) $y = \frac{1}{2}x$

2. Graph each line in the same coordinate plane.
   (a) $y = x + 2$ (b) $y = x + 3$ (c) $y = x + 5$
   (d) $y = x - 1$ (e) $y = x - 2$ (f) $y = x - 4$

3. Graph each line.
   (a) $y = 2x + 2$ (b) $y = \frac{1}{2}x + 2$ (c) $y = -\frac{1}{2}x + 2$

4. Find the slope and y-intercept for the following equations and sketch their graphs.
   (a) $y = x - 2$ (b) $y = -3x - 3$ (c) $y = \frac{1}{2}x + 1$
   (d) $y = -\frac{1}{2}x + 1$ (e) $y + 3x = 3$ (f) $x - y = 3$

5. Find the equation of the straight line with the given slope and y-intercept.
   (a) slope 3, y-intercept 4 (b) slope 2, y-intercept 0
   (c) slope 0, y-intercept 2 (d) slope 0, y-intercept 0

6. Find the equation of the line which has a slope m of $-\frac{2}{3}$ and passes through the point (9, 4).

7. A line has slope -2 and y-intercept 6, find its x-intercept.

8. Find the equation of the line which:
   (a) has a slope of 5 and passes through the point (2, 9)
   (b) has a slope of 1 and passes through the point $(1, -2)$
   (c) has a slope of -3 and passes through the point $(-1, 6)$
   (d) has a slope of -2 and passes through the point $(-1, 4)$.

9. Find the slope and equation of the line joining the following pairs of points.
   (a) (2, 4) and (6, 8) (b) $(-3, 5)$ and $(6, -1)$ (c) $(-2, 1)$ and $(-4, -2)$

10. Determine which of the pairs of lines in each case with given equations are parallel or perpendicular or neither.
(a) $y = 3x - 2$ and $y = 3x + 9$ (b) $y = \frac{2}{3}x - 5$ and $y = \frac{3}{2}x - 5$
(c) $y = 3x - 2$ and $y = -\frac{1}{3}x + 9$ (d) $y = \frac{2}{3}x - 5$ and $y = -\frac{3}{2}x - 5$

11. Find the equation of the line which is parallel to the line:
(a) with equation $y = 4x + 2$ and passes through (0, 8)
(b) with equation $y = -x + 3$ and passes through (0, 5)
(c) with equation $y = -2x - 3$ and passes through (0, −7)
(d) with equation $y = -\frac{4}{5}x - 3$ and passes through $(0, \frac{1}{2})$

12. Find the equation of the line which is perpendicular to the line:
(a) with equation $y = 5x - 4$ and passes through (0, 7)
(b) with equation $y = -x + 7$ and passes through (0, 4)
(c) with equation $y = -2x + 3$ and passes through (0, −4)
(d) with equation $y = x - \frac{3}{2}$ and passes through $(0, \frac{5}{4})$

13. Show that the line through (3n, 0) and (0, 7n) is parallel to the line through (0, 21n) and (9n, 0).

14. Prove that the triangle whose vertices are $H(-12, 1), K(9, 3)$ and $M(11, -18)$ is a right triangle.

15. Given the points $P(1, 2), Q(5, -6)$ and $R(b, b)$, determine the value of b so that angle PQR is a right angle.

16. A right-angled isosceles triangle has vertices at (0, 5), (5, 0) and (−5, 0). Find the equation of each of the three sides.

17. Determine the slope of each side of the quadrilateral whose vertices are $A(5, 6), B(13, 6), C(11, 2)$ and $D(1, 2)$. Can you tell what kind of a quadrilateral it is?

18. Given the points $D(-4, 6), E(1, 1)$, and $F(4, 6)$, find the slopes of DE and EF. Are the points D, E and F collinear, explain why?

19. Prove that the quadrilateral with vertices $A(-2, 2), B(2, -2), C(4, 2)$ and $D(2, 4)$ is a trapezoid with perpendicular diagonals.

20. Find the slopes of the six lines determined by the points $A(-5, 4)$, $B(3, 5), C(7, -2), D(-1, -3)$. Prove that $ABCD$ is a rhombus.

Chapter 2

Exponents and Radicals

Exponents are mathematical shorthand that tells us to multiply the same number by itself for a specific number of times. In this chapter, you will learn about positive integral exponents, zero exponent, negative integral exponents, rational exponents, radicals and exponential equations.

2.1 Exponents

Simply stated exponents are shorthand for repeated multiplication of the same element by itself. The exponent corresponds to the number of times the base is used as a factor. For instance, the shorthand for multiplying two copies of three is such as 3^2. Such products of repeated factors are called powers. Powers can be expressed as expanded form or factor form.

In the example $3 \times 3 = 3^2$, the exponent is 2 and base is 3. 3^2 can be also read as **3 to the second power** or **3 squared**.

2.1.1 Positive Integral Exponents

Integral here means "**integer**". So the exponent (or power) is an integer. In a given product, factors may occur more than once.

In writing a product, a raised dot ($\cdot$) is often used instead of cross ($\times$) to denote multiplication.

Thus,

$3 \times 3 = 3 \cdot 3 = 3^2$
$5 \times 5 \times 5 = 5 \cdot 5 \cdot 5 = 5^3$
$a \times a \times a \times a = a \cdot a \cdot a \cdot a = a^4$, and so on.

In general, if a is any real number and n is a positive integer, then the n^{th} power of a is

$$\underbrace{a \times a \times a \times a \times \cdots \times a}_{n\ factors} = a^n$$

where the number a is called the base and n is called the exponent or index.

2.1.2 Zero and Negative Integral Exponents

Definition. For any real number x, if $x \neq 0$, then $x^0 = 1$.

For example,

$$\begin{aligned} 3^0 &= 1 \\ (-4)^0 &= 1 \end{aligned}$$

Note that 0^0 is indeterminate.

Definition. For any real number x, if $x \neq 0$, then $x^{-n} = \frac{1}{x^n}$.

For example,

$$\begin{aligned} 5^{-2} &= \frac{1}{5^2} = \frac{1}{25} \\ \left(\frac{1}{3}\right)^{-4} &= \frac{1}{\left(\frac{1}{3}\right)^4} = \frac{1}{\frac{1}{3^4}} = 3^4 \end{aligned}$$

Consequently $\frac{1}{x^{-n}} = x^n$.

Example 1.

Evaluate the following:

(i) $\dfrac{3^{-4}}{4^{-3}}$ (ii) $5^0 - 5x^0 - (5x)^{-1} - (5x)^0$.

Solution

(i) $\dfrac{3^{-4}}{4^{-3}} = \dfrac{4^3}{3^4} = \dfrac{64}{81}$

(ii) $5^0 - 5x^0 - (5x)^{-1} - (5x)^0 = 1 - 5(1) - \dfrac{1}{5x} - 1 = 1 - 5 - \dfrac{1}{5x} - 1 = -5 - \dfrac{1}{5x}$

2.1.3 Rules for Integral Exponents

We assume that x and y be any real numbers and m and n are positive integers.

Rule 1 (Multiplication)

$$\boxed{x^m \cdot x^n = x^{m+n}}$$

For example,

$$\begin{aligned} 3^3 \cdot 3^5 &= 3^{3+5} = 3^8 \\ x^2 \cdot x^6 &= x^{2+6} = x^8 \end{aligned}$$

Rule 2 (Division)

$$\boxed{x^m \div x^n = \tfrac{x^m}{x^n} = \begin{cases} x^{m-n}, & \text{if } m > n \\ 1, & \text{if } m = n \\ \dfrac{1}{x^{n-m}}, & \text{if } m < n, x \neq 0 \end{cases}}$$

For example,

$$\begin{aligned} 3^5 \div 3^2 &= \frac{3^5}{3^2} = 3^{5-2} = 3^3 \\ a^3 \div a^8 &= \frac{a^3}{a^8} = \frac{1}{a^{8-3}} = \frac{1}{a^5} \\ a^4 \div a^4 &= \frac{a^4}{a^4} = 1 \end{aligned}$$

Rule 3 (Power of a Power)

$$\boxed{(x^m)^n = x^{mn}}$$

For example,

$$(2^3)^4 = 2^{3\cdot 4} = 2^{12}$$
$$(x^4)^2 = x^{4\cdot 2} = x^8$$

Rule 4 (Power of a Product)

$$\boxed{(xy)^n = x^n \cdot y^n}$$

For example,

$$(2\cdot 3)^4 = 2^4 \cdot 3^4$$
$$(xy)^5 = x^5 \cdot y^5$$

Rule 5 (Power of a Quotient)

$$\boxed{\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}, \quad y \neq 0}$$

For example,

$$\left(\frac{3}{5}\right)^2 = \frac{3^2}{5^2}$$
$$\left(\frac{x}{y}\right)^3 = \frac{x^3}{y^3}, \quad y \neq 0$$

The following examples are computed by the use of Rule 1 through Rule 5.

Example 2.

Simplify and name the rules used.

(i) $x^7 \cdot x^4$ (ii) $a^{15} \div a^7$ (iii) $(b^2)^3$ (iv) $a^5 \cdot b^5$ (v) $\dfrac{y^7}{4^7}$

Solution

(i) $x^7 \cdot x^4 = x^{7+4} = x^{11}$ (Multiplication Rule)

(ii) $a^{15} \div a^7 = a^{15-7} = a^8$ (Division Rule)

(iii) $(b^2)^3 = b^6$ (Power of a Power Rule)

(iv) $a^5 \cdot b^5 = (ab)^5$ (Power of a Product Rule)

(v) $\frac{y^7}{4^7} = (\frac{y}{4})^7$ (Power of a Quotient Rule)

Example 3.

Simplify and name the rules used.

(i) $\left(\frac{-81x^3y^4}{27xy^3}\right)^3$ (ii) $(a^{-1}b^{-3})^{-2}$

Solution

(i)

$$\begin{aligned} \left(\frac{-81x^3y^4}{27xy^3}\right)^3 &= (-3x^2y)^3 && \text{(Division Rule)} \\ &= (-3)^3(x^2)^3y^3 && \text{(Power of a Product Rule)} \\ &= -27x^6y^3 && \text{(Power of a Power Rule)} \end{aligned}$$

(ii)

$$\begin{aligned} (a^{-1}b^{-3})^{-2} &= (a^{-1})^{-2}(b^{-3})^{-2} && \text{(Power of a Product Rule)} \\ &= a^2b^6 && \text{(Power of a Power Rule)} \end{aligned}$$

Example 4.

Evaluate $\left(\frac{16 \cdot 27}{25}\right)^2 \left(\frac{50}{36}\right)^3$.

Solution

$$\begin{aligned} \left(\frac{16 \cdot 27}{25}\right)^2 \left(\frac{50}{36}\right)^3 &= \left(\frac{2^4 \cdot 3^3}{5^2}\right)^2 \left(\frac{2 \cdot 5^2}{2^2 \cdot 3^2}\right)^3 \\ &= \frac{(2^4 \cdot 3^3)^2}{(5^2)^2} \frac{(2 \cdot 5^2)^3}{(2^2 \cdot 3^2)^3} \\ &= \frac{2^8 \cdot 3^6}{5^4} \cdot \frac{2^3 \cdot 5^6}{2^6 \cdot 3^6} \\ &= 2^5 \cdot 5^2 \\ &= 32 \cdot 25 \\ &= 800 \end{aligned}$$

Example 5.

Evaluate $\dfrac{(x^2-y^2)^3}{(x+y)^3}$.

Solution

$$\begin{aligned}\frac{(x^2-y^2)^3}{(x+y)^3} &= \frac{((x+y)(x-y))^3}{(x+y)^3}\\ &= \frac{(x+y)^3(x-y)^3}{(x+y)^3}\\ &= (x-y)^3\end{aligned}$$

Exercise 2.1

1. Simplify by using the rules of exponents and name the rules used.

   (a) $\dfrac{36a^4b^5}{100a^7b^3}$ (b) $\dfrac{27a^2b^5}{(9a^2b)^2}$ (c) $\left(\dfrac{-135a^4b^5c^6}{315a^6b^7c^8}\right)^2$

   (d) $\left(\dfrac{x^4}{y^5}\right)^3\left(\dfrac{y^3}{x^2}\right)^2$ (e) $\dfrac{2^{3^2}}{(2^2)^3}$

2. Evaluate the following.

   (a) $\dfrac{54^2\times 12^3\times 64^2(3^2\times 4^3\times 5^2)^3}{(3^2\times 15\times 20^3)^4}$ (b) $\left(\dfrac{343}{36}\right)^3\left(\dfrac{540}{56}\right)^4$

   (c) $\left(\dfrac{33}{1056}\right)^3\left(\dfrac{768}{270}\right)^4\left(\dfrac{450}{48}\right)^3$

3. Simplify.

   (a) $\left(\dfrac{3^m}{15^n}\right)^3\left(\dfrac{45^n}{255^m}\right)^2$ (b) $\left(\dfrac{20^x}{400^y}\right)^2\left(\dfrac{150^{y^2}}{180^x}\right)^3$

   (c) $\dfrac{(x^3-y^3)(x+y)}{(x^2-y^2)^3}$ (d) $\dfrac{(x^{a-b}x^{b-c})^a\left(\dfrac{x^a}{x^c}\right)^c}{(x^bx^c)^a\div(x^{a+c})^c}$

4. Evaluate the following.

   (a) $(-3)^{-2}$ (b) -3^{-3} (c) $-2^0 + 5^{-1}$

   (d) $(-2)^{-3} + 2^{-2} - 2^{-4}$ (e) $5^0 - (-3)^0$ (f) $\dfrac{27^{-6}}{125^{-3}} \div \dfrac{9^{-2}}{25^{-4}}$

   (g) $(-5)^0 - (-5)^{-1} - (-5)^{-2} - (-5)^{-3}$ (h) $(-1)^{(-1)^{-1}}$

   (i) $\dfrac{(180^2)^{-3}(6 \cdot 90^{-2})^3}{(40^{-3})^2 \cdot 25^{-2}}$ (j) $\dfrac{(2^{-3} - 3^{-2})^{-1}}{(2^{-3} + 3^{-2})^{-1}}$

5. Simplify the following.

   (a) $(-3a^4)(4a^{-7})$ (b) $\left(\dfrac{2x^{-4}}{5y^2z^3}\right)^{-2}$

   (c) $\left(\dfrac{x^{2m+n}x^{3(m-n)}}{x^{m-2n}x^{2m-n}}\right)^{-3}$ (d) $\left(\dfrac{2x^{-3}y^2}{3^{-1}y^3}\right)^2 \left(\dfrac{4x^{-2}y^3}{3x^5}\right)^3 \div \left(\dfrac{81x^{-2}}{y^{-3}}\right)^{-2}$

   (e) $\dfrac{2x + y}{x^{-1} + 2y^{-1}}$ (f) $(x^{-2} - y^{-1})^{-3}$

   (g) $\dfrac{x^{-2} - y^{-2}}{x^{-1} + y^{-1}}$ (h) $\dfrac{(x + y^{-1})^2}{1 + x^{-1}y^{-1}}$

2.1.4 Rational Exponents

In this section, you will examine the root of a number and the rational exponents.

Definition. If n is a positive integer and x and y are real numbers, such that $x^n = y$, then x is called the n^{th} root of y.

For example,

(i) $2^3 = 8$

2 is the cube root of 8.

(ii) $(-2)^3 = -8$
-2 is the cube root of -8.
(iii) $(-3)^4 = 81$
-3 is the fourth root of 81.
(iv) $3^4 = 81$
3 is the fourth root of 81.

In general, for $x^n = y$, if n is odd, there is only one real n^{th} root of y, no matter whether y is negative or zero or positive. In this case the real n^{th} root of y is denoted by $\sqrt[n]{y}$ and is called **the principal n^{th} root of y**.

For example,
$\sqrt[3]{8} = 2$ by (i)
$\sqrt[3]{-8} = -2$ by (ii)

If n is even and y is positive, there are two real n^{th} roots of y, one positive and the other negative. In that case the positive n^{th} root of y is denoted by $\sqrt[n]{y}$ and is called **the principal n^{th} root of y**.
For example,
By (iii) and (iv), 3 and -3 are fourth roots of 81 and the principal fourth root of 81, $\sqrt[4]{81} = 3$.

Notice the following:
(i) $x^2 = -4$ has no real number x because the square of any nonzero real number is positive.
(ii) It is important to understand the difference between $\sqrt[n]{-b}$ and $-\sqrt[n]{b}$. $\sqrt[n]{-b}$ is the principal n^{th} root of of $-b$ and $-\sqrt[n]{b}$ is the negative of the principal n^{th} root of b.
(iii) $\sqrt{x^2} = \begin{cases} x & \text{if } x \text{ is positive or zero,} \\ -x & \text{if } x \text{ is negative.} \end{cases}$
For example,
(i) $\sqrt{9} = \sqrt{3^2} = 3$
(ii)$\sqrt{9} = \sqrt{(-3)^2} = -(-3) = 3$

Now we will extend the exponential concept to fractional exponents. First, we consider the exponents of the form $\frac{1}{n}$, where n is a positive integer. Here $b^{\frac{1}{n}}$ is the n^{th} root of b (provided such a root exists). Then we define as follows:

Definition. For a real number x and an integer n $(n \geq 2)$, $x^{\frac{1}{n}} = \sqrt[n]{x}$, when n is even, x must be positive or zero.

Example 6.

Simplify (a)$16^{\frac{1}{2}}$ (b) $8^{\frac{1}{3}}$ (c) $(-27)^{\frac{1}{3}}$.

Solution

(a) $16^{\frac{1}{2}} = \sqrt{16} = 4$

(b) $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

(c) $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$

Next, we now consider the more general fractional exponent expression $x^{\frac{m}{n}}$. If m and n are positive integers, to preserve the rules for exponents we want to write

$$(x^{\frac{1}{n}})^m = x^{\frac{m}{n}}.$$

This observation leads to the following definition.

Definition. If m and n are positive integers and $\frac{m}{n}$ is a rational number in lowest terms, then for any real number x,

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m,$$

when n is even, x must be positive or zero.

The following calculation for $(-8)^{\frac{2}{6}}$ is false because the exponent $\frac{2}{6}$ is not in lowest terms.

$(-8)^{\frac{2}{6}} = \sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2$ $\times$

Calculate $(-8)^{\frac{2}{6}}$ as follows:

$(-8)^{\frac{2}{6}} = (-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2$ ✓

Definition. If m and n are any positive integers, then for any real number $x \neq 0$, $x^{-\frac{m}{n}} = \frac{1}{x^{\frac{m}{n}}}$.

For example,

(i) $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4}$

(ii) $32^{-\frac{7}{5}} = \frac{1}{32^{\frac{7}{5}}} = \frac{1}{(\sqrt[5]{32})^7} = \frac{1}{2^7} = \frac{1}{128}$.

Example 7.
Simplify and express the answer with positive exponents $\left(\frac{2^m\sqrt{2^{-m}}}{\sqrt{2^{m+1}}}\right)^{-2m}$.

Solution

$$\begin{aligned}
\left(\frac{2^m\sqrt{2^{-m}}}{\sqrt{2^{m+1}}}\right)^{-2m} &= \left(\frac{2^m 2^{-\frac{m}{2}}}{2^{\frac{m+1}{2}}}\right)^{-2m} \\
&= \left(2^{m-\frac{m}{2}-\frac{m+1}{2}}\right)^{-2m} \\
&= \left(2^{-\frac{1}{2}}\right)^{-2m} \\
&= 2^m
\end{aligned}$$

Note that the rules for integral exponents hold for all rational exponents. The following are some useful expressions:

$$\begin{aligned}
x^2 - y^2 &= (x+y)(x-y) \\
x - y &= (x^{\frac{1}{2}})^2 - (y^{\frac{1}{2}})^2 = (x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}}) \\
x^{\frac{1}{2}} - y^{\frac{1}{2}} &= (x^{\frac{1}{4}})^2 - (y^{\frac{1}{4}})^2 = (x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} - y^{\frac{1}{4}}) \\
x^3 - y^3 &= (x-y)(x^2+xy+y^2) \\
x - y &= (x^{\frac{1}{3}})^3 - (y^{\frac{1}{3}})^3 = (x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}) \\
x^3 + y^3 &= (x+y)(x^2-xy+y^2) \\
x + y &= (x^{\frac{1}{3}})^3 + (y^{\frac{1}{3}})^3 = (x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})
\end{aligned}$$

Example 8.

Simplify $\dfrac{x - y^{-1}}{(x^{\frac{1}{2}} + y^{-\frac{1}{2}})(x^{\frac{1}{4}} - y^{-\frac{1}{4}})}$.

Solution

$$\begin{aligned}
\frac{x - y^{-1}}{(x^{\frac{1}{2}} + y^{-\frac{1}{2}})(x^{\frac{1}{4}} - y^{-\frac{1}{4}})} &= \frac{(x^{\frac{1}{2}} + y^{-\frac{1}{2}})(x^{\frac{1}{2}} - y^{-\frac{1}{2}})}{(x^{\frac{1}{2}} + y^{-\frac{1}{2}})(x^{\frac{1}{4}} - y^{-\frac{1}{4}})} \\
&= \frac{(x^{\frac{1}{2}} - y^{-\frac{1}{2}})}{(x^{\frac{1}{4}} - y^{-\frac{1}{4}})} \\
&= \frac{(x^{\frac{1}{4}} + y^{-\frac{1}{4}})(x^{\frac{1}{4}} - y^{-\frac{1}{4}})}{(x^{\frac{1}{4}} - y^{-\frac{1}{4}})} \\
&= x^{\frac{1}{4}} + y^{-\frac{1}{4}} \\
&= x^{\frac{1}{4}} + \frac{1}{y^{\frac{1}{4}}} \\
&= \frac{x^{\frac{1}{4}}y^{\frac{1}{4}} + 1}{y^{\frac{1}{4}}}
\end{aligned}$$

Exercise 2.2

1. Evaluate the following.

(a) $(125)^{\frac{2}{3}}$ (b) $(81)^{-\frac{3}{2}}$ (c) $(-27)^{\frac{2}{3}}$

(d) $\left(\dfrac{16}{81}\right)^{-\frac{3}{4}}$ (e) $\left(\dfrac{-125}{8} \div \dfrac{1}{64}\right)^{\frac{1}{3}}$ (f) $(0.125)^{-\frac{2}{3}}$

(g) $\left(\dfrac{64}{27}\right)^{-\frac{2}{3}}$ (h) $(-4)^{-1} + (-1)^{-4}$

2. Simplify the following.

   (a) $\sqrt[3]{4^2} \cdot 4^{\frac{2}{3}} \cdot \left(\frac{1}{4}\right)^{-\frac{2}{3}}$ (b) $\sqrt{\frac{512 \times 27^{-3} \times 81 \times 3^8}{3^4}}$

   (c) $\left(\left(\frac{3}{4}\right)^{-4}\right)^{-0.5} \cdot \sqrt{\left(\frac{4}{3}\right)^{-1}} \div 16^{-0.5}$

   (d) $(27)^{\frac{1}{4}} + \frac{24}{(8)^{-\frac{2}{3}}} + \frac{\sqrt[5]{2}}{(4)^{-\frac{2}{5}}}$ (e) $\frac{(243)^{\frac{4}{5}} + (64)^{\frac{2}{3}} - (216)^{\frac{1}{3}}}{(225)^{\frac{1}{2}} - (16)^{\frac{3}{4}}}$

3. Simplify the following.

   (a) $\frac{x - 5\sqrt{x}}{x - 2\sqrt{x} - 15} \div \left(1 + \frac{3}{\sqrt{x}}\right)^{-1}$ (b) $\sqrt[a]{\frac{\sqrt[b]{x}}{\sqrt[c]{x}}} \cdot \sqrt[b]{\frac{\sqrt[c]{x}}{\sqrt[a]{x}}} \cdot \sqrt[c]{\frac{\sqrt[a]{x}}{\sqrt[b]{x}}}$

   (c) $\left[\frac{x^m - y^m}{x^{\frac{m}{2}} - y^{\frac{m}{2}}} - \frac{x^m - y^m}{x^{\frac{m}{2}} + y^{\frac{m}{2}}}\right]^{-2}$

2.2 Radicals

In above section, the principal n^{th} root of b is denoted by $\sqrt[n]{b}$ and also written as $b^{\frac{1}{n}}$. There is another name for roots of a number.

Definition. The symbol $\sqrt[n]{b}$ is called a **radical**, $\sqrt[n]{\ }$ is the **radical sign**, n is the **order or index**, and b is called the **radicand**.

2.2.1 Rules for Radicals

For any positive integers m, n and k,

Rule 1 $\boxed{\left(\sqrt[n]{x}\right)^n = \sqrt[n]{x^n} = x}$

For example,

$$\left(\sqrt[3]{5}\right)^3 = 5$$
$$\sqrt[4]{b^4} = b$$

Rule 2 $\boxed{\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{xy}}$

For example,

$$\begin{aligned} \sqrt[3]{5} \cdot \sqrt[3]{20} &= \sqrt[3]{5 \cdot 20} = \sqrt[3]{100} \\ \sqrt[4]{x^2} \cdot \sqrt[4]{y} &= \sqrt[4]{x^2 y} \end{aligned}$$

Rule 3 $\boxed{\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}}$

For example,

$$\begin{aligned} \sqrt{\sqrt[3]{5}} &= \sqrt[6]{5} \\ \sqrt[3]{\sqrt[4]{x^5}} &= \sqrt[12]{x^5} \end{aligned}$$

Rule 4 $\boxed{\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}, \qquad y \neq 0}$

For example,

$$\begin{aligned} \sqrt[3]{\frac{5}{9}} &= \frac{\sqrt[3]{5}}{\sqrt[3]{9}} \\ \sqrt[4]{\frac{x^5}{y^3}} &= \frac{\sqrt[4]{x^5}}{\sqrt[4]{y^3}} \end{aligned}$$

Rule 5 $\boxed{\begin{array}{l} \text{(a) } \sqrt[n]{x^m} = \sqrt[kn]{x^{km}} \\ \text{(b) } \sqrt[n]{x^m} = \sqrt[\frac{n}{k}]{x^{\frac{m}{k}}}, \quad k \neq 0 \end{array}}$

For example,

$$\begin{aligned} \sqrt[3]{5} &= \sqrt[3\times 2]{5^2} = \sqrt[6]{5^2} \\ \sqrt[5]{b^3} &= \sqrt[5\times 3]{b^{3\times 3}} = \sqrt[15]{b^9} \\ \sqrt[6]{25} &= \sqrt[6]{5^2} = \sqrt[3]{5} \\ \sqrt[12]{b^3} &= \sqrt[4]{b} \end{aligned}$$

Example 9.

Simplify the following.

(a) $\sqrt{27}$ (b) $\sqrt[3]{16}$ (c) $\sqrt{\sqrt[3]{128}}$ (d) $\sqrt[5]{-32}$ (e) $\sqrt{72}$

Solution

(a) $\sqrt{27} = \sqrt{9 \times 3} = \sqrt{3^2 \times 3} = 3\sqrt{3}$

(b) $\sqrt[3]{16} = \sqrt[3]{8 \times 2} = \sqrt[3]{2^3 \times 2} = 2\sqrt[3]{2}$

(c) $\sqrt{\sqrt[3]{128}} = \sqrt[6]{128} = \sqrt[6]{2^6 \times 2} = 2\sqrt[6]{2}$

(d) $\sqrt[5]{-32} = \sqrt[5]{(-2)^5} = -2$

(e) $\sqrt{72} = \sqrt{36 \times 2} = \sqrt{6^2 \times 2} = 6\sqrt{2}$

Example 10.

Change the expression with the same radical and simplify the radicands.

(a) $5a\sqrt[3]{3}$ (b) $\sqrt[3]{5}\sqrt{3}$ (c) $\sqrt[4]{2}\sqrt[3]{3}$ (d) $\sqrt{2x}\sqrt[3]{3y}$

Solution

(a) $5a\sqrt[3]{3} = \sqrt[3]{(5a)^3}\sqrt[3]{3} = \sqrt[3]{125a^3}\sqrt[3]{3} = \sqrt[3]{375a^3}$

(b) $\sqrt[3]{5}\sqrt{3} = \sqrt[6]{5^2}\sqrt[6]{3^3} = \sqrt[6]{25 \times 27} = \sqrt[6]{675}$

(c) $\sqrt[4]{2}\sqrt[3]{3} = \sqrt[12]{2^3}\sqrt[12]{3^4} = \sqrt[12]{8 \times 81} = \sqrt[12]{648}$

(d) $\sqrt{2x}\sqrt[3]{3y} = \sqrt[6]{(2x)^3}\sqrt[6]{(3y)^2} = \sqrt[6]{8x^3 \times 9y^2} = \sqrt[6]{72x^3y^2}$

Exercise 2.3

1. Write the following in radical form.

   (a) $(5)^{\frac{1}{2}}$ (b) $(-9)^{\frac{1}{3}}$ (c) $(2)^{-\frac{1}{2}}$ (d) $\left(-\frac{3}{4}\right)^{\frac{2}{5}}$ (e) $\left(\frac{2}{7}\right)^{\frac{5}{2}}$

2. Write the following in fractional exponent form.

   (a) $\sqrt[6]{c^5}$ (b) $\sqrt[3]{-2}$ (c) $\sqrt[5]{a^4\sqrt[3]{b^5}}$ (d) $\sqrt[4]{\left(\frac{3}{7}\right)^3}$

3. Change the expression with the same radical and simplify the radicands.

   (a) $6\sqrt{2}$ (b) $3a\sqrt[3]{x}$ (c) $2\sqrt[5]{2}$ (d) $3\sqrt[4]{\frac{1}{2}}$ (e) $3\sqrt{x^3}$

4. Simplify.

   (a) $\sqrt{32}$ (b) $\sqrt[5]{-32}$ (c) $\sqrt[4]{\frac{81x^{16}}{16y^4}}$

   (d) $\sqrt[3]{\frac{81x^2}{4y}}$ (e) $\frac{9^{\frac{1}{2}}}{\sqrt[3]{27}}$ (f) $\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{75}{98}}$

   (g) $\sqrt[3]{\frac{-216}{8 \times 10^3}}$ (h) $\sqrt[n]{\frac{32}{2^{5+n}}}$

5. Rationalize the denominators.

   (a) $\frac{4\sqrt{35}}{3\sqrt{7}}$ (b) $\frac{20}{\sqrt{5}}$ (c) $\frac{18}{\sqrt[3]{2}}$

   (d) $\frac{\sqrt[3]{32}}{\sqrt[4]{27}}$ (e) $\frac{\sqrt[3]{36a^2}}{\sqrt[3]{9a}}$ (f) $\frac{\sqrt[3]{2}}{\sqrt[6]{12}}$

   (g) $\frac{1}{\sqrt[3]{xy^2}}$ (h) $\sqrt[m]{\frac{2x^2y^{3m}}{9x^5y^{4m-1}}}$

6. Reduce the order as far as possible.

   (a) $\sqrt[4]{25}$ (b) $\sqrt[6]{4}$ (c) $\sqrt[6]{8}$

   (d) $\sqrt[9]{8y^3}$ (e) $\sqrt[6]{27^3}$ (f) $\sqrt[8]{a^2b^4}$

   (g) $\sqrt[12]{64a^2b^6}$ (h) $(72)^{\frac{3}{5}}$ (i) $\sqrt[3]{768}$

7. Find the simplified forms.

   (a) $\sqrt{\frac{9}{50}}$ (b) $\sqrt[3]{\frac{-192}{49}}$ (c) $\sqrt[4]{16}$ (d) $2\sqrt[3]{56}$

2.3 Operations with Radicals

The product of two radicals of the same order is found directly from **Rule 2** of radicals. The product of two radicals of different orders can be found using the rule after they have been changed to the equivalent radicals of the same order.

Example 11.
Multiply $\sqrt{6}$ by $\sqrt{15}$ and express in the simplest form.

Solution

$$\sqrt{6}\cdot\sqrt{15} = \sqrt{6\cdot 15} = \sqrt{2\cdot 3\cdot 3\cdot 5} = 3\sqrt{10}$$

Example 12.

Multiply $\sqrt[3]{12}$ by $\sqrt{10}$.

Solution

$$\begin{aligned}
\sqrt[3]{12}\cdot\sqrt{10} &= \sqrt[6]{12^2}\cdot\sqrt[6]{10^3}\\
&= \sqrt[6]{12^2\cdot 10^3}\\
&= \sqrt[6]{(3\cdot 2^2)^2\cdot(2\cdot 5)^3}\\
&= \sqrt[6]{3^2\cdot 2^4\cdot 2^3\cdot 5^3}\\
&= \sqrt[6]{3^2\cdot 2^6\cdot 2\cdot 5^3}\\
&= 2\sqrt[6]{2250}
\end{aligned}$$

Example 13.
Simplify $\sqrt{3}(\sqrt{3}+\sqrt{8})$.
Solution

$$\begin{aligned}
\sqrt{3}(\sqrt{3}+\sqrt{8}) &= \sqrt{3}\cdot\sqrt{3}+\sqrt{3}\cdot\sqrt{8}\\
&= 3+\sqrt{24}\\
&= 3+\sqrt{2^2\cdot 6}\\
&= 3+2\sqrt{6}
\end{aligned}$$

The radicals with the same index and same radicand are called similar radicals and they can be added or subtracted. The following example shows the addition and subtraction of the radicals.

Example 14.

Simplify $\sqrt{200}+\sqrt{50}-\sqrt{18}$.
Solution

$$\begin{aligned}\sqrt{200}+\sqrt{50}-\sqrt{18} &= \sqrt{2\cdot 100}+\sqrt{2\cdot 25}-\sqrt{2\cdot 9}\\ &= 10\sqrt{2}+5\sqrt{2}-3\sqrt{2}\\ &= (10+5-3)\sqrt{2}\\ &= 12\sqrt{2}\end{aligned}$$

The algebraic sum of dissimilar radicals can be simplified by rationalizing the denominators.

Example 15.
Simplify $\sqrt{24}+\sqrt{\frac{2}{3}}-\sqrt[3]{\frac{2}{9}}$.
Solution

$$\begin{aligned}\sqrt{24}+\sqrt{\frac{2}{3}}-\sqrt[3]{\frac{2}{9}} &= \sqrt{2^2\cdot 6}+\frac{\sqrt{2}}{\sqrt{3}}-\frac{\sqrt[3]{2}}{\sqrt[3]{3^2}}\\ &= 2\sqrt{6}+\frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}-\frac{\sqrt[3]{2}}{\sqrt[3]{3^2}}\cdot\frac{\sqrt[3]{3}}{\sqrt[3]{3}}\\ &= 2\sqrt{6}+\frac{\sqrt{6}}{3}-\frac{\sqrt[3]{6}}{\sqrt[3]{3^3}}\\ &= 2\sqrt{6}+\frac{\sqrt{6}}{3}-\frac{\sqrt[3]{6}}{3}\\ &= (2+\frac{1}{3})\sqrt{6}-\frac{\sqrt[3]{6}}{3}\\ &= \frac{7}{3}\sqrt{6}-\frac{1}{3}\sqrt[3]{6}\end{aligned}$$

Example 16.

Simplify $\dfrac{\sqrt{2}-3\sqrt{3}}{2\sqrt{2}-\sqrt{3}}$.

Solution

$$\begin{aligned}
\frac{\sqrt{2}-3\sqrt{3}}{2\sqrt{2}-\sqrt{3}} &= \frac{\sqrt{2}-3\sqrt{3}}{2\sqrt{2}-\sqrt{3}}\cdot\frac{2\sqrt{2}+\sqrt{3}}{2\sqrt{2}+\sqrt{3}}\\
&= \frac{4+\sqrt{6}-6\sqrt{6}-9}{4\times 2-3}\\
&= \frac{-5-5\sqrt{6}}{5}\\
&= \frac{-5(1+\sqrt{6})}{5}\\
&= -(1+\sqrt{6})
\end{aligned}$$

In rationalizing the denominator, we multiply it by its conjugate. This process is based on the fact that

$$(a+b)(a-b)=a^2-b^2.$$

Each of the two factors is called **the conjugate of the other**. Thus

(1) $a+\sqrt{b}$ and $a-\sqrt{b}$

(2) $a+b\sqrt{c}$ and $a-b\sqrt{c}$

(3) $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$

(4) $a\sqrt{b}+c\sqrt{d}$ and $a\sqrt{b}-c\sqrt{d}$

are conjugate radicals.

Exercise 2.4

1. Simplify the following.

   (a) $3\sqrt{5}+7\sqrt{5}$ (b) $\sqrt{75}-\sqrt{12}$ (c)$3\cdot 3\sqrt{3}\cdot 3\sqrt{27}$

   (d) $2\sqrt{5}\cdot 3\sqrt{2}$ (e)$(4-\sqrt{3})^2$ (f) $(\sqrt{3}+2\sqrt{2})(\sqrt{3}+\sqrt{2})$

   (g) $(\sqrt{7}-\sqrt{6})(\sqrt{7}+\sqrt{6})(\sqrt{x}+1)(\sqrt{x}-1)$ (h) $\sqrt{75}-\frac{3}{4}\sqrt{48}-5\sqrt{12}$

   (i) $\sqrt{2x^2}+5\sqrt{32x^2}-2\sqrt{98x^2}$ (j) $\sqrt{20a^3}+a\sqrt{5a}+\sqrt{80a^3}$

2. Rationalise the denominators and simplify.

   (a) $\frac{2}{\sqrt{5}}$ (b) $\frac{5}{2+\sqrt{3}}$ (c) $\frac{12}{\sqrt{5}-\sqrt{3}}$

   (d) $\frac{\sqrt{2}+1}{2\sqrt{2}-1}$ (e) $\frac{\sqrt{7}+3\sqrt{2}}{\sqrt{7}-\sqrt{2}}$ (f) $\frac{\sqrt{17}-\sqrt{11}}{\sqrt{17}+\sqrt{11}}$

   (g) $\frac{1}{2\sqrt{2}-\sqrt{3}}$ (h) $\frac{\sqrt{6}+1}{3-\sqrt{5}}$

3. Write as a single fraction.

   (a) $\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{3}-1}$ (b) $\frac{2}{\sqrt{7}+\sqrt{2}}+\frac{1}{\sqrt{7}-\sqrt{2}}$

   (c) $\frac{1}{3+\sqrt{3}}+\frac{1}{\sqrt{3}-3}+\frac{1}{\sqrt{3}}$ (d)$\frac{7+\sqrt{5}}{7-\sqrt{5}}+\frac{\sqrt{11}-3}{\sqrt{11}+3}$

   (e) $\frac{3+2\sqrt{2}}{(\sqrt{3}-1)^2}$ (f) $\sqrt{\frac{x+1}{x-1}}+\sqrt{\frac{x-1}{x+1}}-\sqrt{\frac{1}{x^2-1}}$

   (g) $\sqrt{\frac{\sqrt[5]{32}+\sqrt{4}}{2^{-2}-2^{-3}}}$

2.4 Exponential Equations

The exponential equation is an equation in which a variable occurs in the exponent.

$3^x = 81$ and $2^{5x-1} = 32$ are some examples of exponential equations.

Some exponential equations may be solved using the fact that, if $x^n = x^m$, then $n = m$ for $x \neq 0$ and $x \neq 1$.

Example 17.

Solve the equation $2^{3x-1} = 32$.

Solution

$$\begin{aligned} 2^{3x-1} &= 32 \\ 2^{3x-1} &= 2^5 \\ 3x - 1 &= 5 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

Example 18.

Solve the equation $3^{2x+1} \cdot 27^{x-1} = 81$.

Solution

$$\begin{aligned} 3^{2x+1} \cdot 27^{x-1} &= 81 \\ 3^{2x+1} \cdot (3^3)^{x-1} &= 3^4 \\ 3^{2x+1} \cdot 3^{3x-3} &= 3^4 \\ 3^{5x-2} &= 3^4 \\ 5x - 2 &= 4 \\ 5x &= 6 \\ x &= \frac{6}{5} \end{aligned}$$

Example 19.

Solve the equation $2 \cdot 2^{2x} - 7 \cdot 2^x - 4 = 0$.
Solution

$$\begin{aligned} 2 \cdot 2^{2x} - 7 \cdot 2^x - 4 &= 0 \\ \textit{Let } 2^x = a. & \\ 2a^2 - 7a - 4 &= 0 \\ (2a+1)(a-4) &= 0 \\ 2a+1 &= 0 \quad \text{or} \quad a-4=0 \\ a &= -\frac{1}{2} \quad \text{or} \quad a=4 \\ 2^x &= -\frac{1}{2} \quad \text{or} \quad 2^x = 4 \\ & \text{impossible} \quad \text{(or)} \quad 2^x = 2^2 \\ & \therefore x = 2 \end{aligned}$$

Exercise 2.5

Solve the following equations.

1. $3^{2x-3} = 27^{2x}$
2. $5^{x^2-9} = 1$
3. $5^{x+1} = \frac{1}{625}$
4. $\left(\frac{1}{2}\right)^x = 64$
5. $2^{3x} \cdot 4^{x+1} = 128$
6. $3^{x+1} \cdot 9^{2-x} = \frac{1}{27}$
7. $\frac{27^{2x}}{3^{5-x}} = \frac{3^{2x+1}}{9^{x+3}}$
8. $8^{x-1} = \left(\frac{1}{32}\right)^{x+1}$
9. $10^{-x} = 0.000001$
10. $4^x + 4^{x+1} = 20$
11. $4 \cdot 2^{2x} + 3 \cdot 2^x - 1 = 0$

Chapter 3

Logarithms

In the seventeenth century, a Scottish mathematician John Napier developed method for efficiently performing calculations with large numbers. He found a method for finding the *product* of two numbers by *adding* two corresponding numbers, which he called *logarithms*. The *Richter scale* is a logarithmic scale used to measure the magnitude of an earthquake. pH is another logarithmic scale used to describe how acidic or basic an aqueous solution is.

3.1 Scientific Notation

Scientists often deal with very large or very small numbers. For instance, the speed of the light in vacuum is 299,792,458 metres per second and the mass of an electron is 0.000000000000000000000000000000910938 kilograms. To write such numbers more convenient, scientists make use of a compact form called the scientific notation.

Definition. A positive number is in scientific notation when it is written in the form $a \times 10^n$, where n is an integer and a is a real number satisfying the inequality $1 \leq a < 10$.

In scientific notation, the speed of light mentioned above is approximately 3.0×10^8 metres per second and the mass of an electron is approximately 9.1×10^{-31} kilograms. These numbers are obviously easier to write and remember when given in scientific notation. A simple rule to a write a number in scientific notation is as follow.

Place the decimal point after the first nonzero digit, and produce a new number lying between 1 and 10. Then determine the power of 10 by counting the number of places we moved the original decimal point to the marked decimal point. If we moved the point to the left, then the power is positive; and if we moved it to the right, it is negative.

An approximate number written in scientific notation $a \times 10^n$ indicates an accuracy to the number of digits in "a". The figures or digits in "a"are called **significant figures**. For example, there are three significant figures in the number 3.05×10^{-2}, in which significant digits are 3, 0 and 5.

Example 1.

Write the following numbers in scientific notation.

(a) 14,753 (b) 0.00632 (c) 0.23 (d) 0.00000912 (e) 1,000,000

Solution

(a) 147,53=1.4753×10^4

[Decimal point is moved four places to the left.]

(b) 0.00632=6.32×10^{-3}

[Decimal point is moved three places to the right.]

(c) 0.23=2.3×10^{-1}

[Decimal point is moved one place to the right.]

(d) 0.00000912=9.12×10^{-6}

[Decimal point is moved six places to the right.]

(e) 1,000,000=1×10^6

[Decimal point is moved six places to the left.]

Example 2.

Express the following numbers in ordinary decimal form.

(a) 7.354×10^5 (b) 3.2×10^{-1}

Solution

(a) 7.354×10^5=735400

(b) 3.2×10^{-1}=0.32

In carrying out calculations involving measurements, the general rule is that the accuracy of a calculated result is limited by the least accurate measurement involved in the calculation.

1. In addition and subtraction, the result should be rounded so that it has *the same number of digits after the decimal* as the measurement with the least number of digits to the right of the decimal.

   For example, $100.1 + 54.52147 = 154.62147$, which should be rounded as 154.6. Here we can write $100.1 + 54.52147 = 154.62147 \approx 154.6$ or briefly $100.1 + 54.52147 \approx 154.6$. Notice that 100.1 has one digit to the right of the decimal and 54.52147 has five, so the sum must have only one digit after the decimal.

2. In multiplication and division, the result should have *the same number of significant figures* as the measurement with the least.

   For example, $5.01 \times 45.0536 = 225.718536$, which should be rounded as 226. In this case, we can also write $5.01 \times 45.0536 = 225.718536 \approx 226$ or briefly $5.01 \times 45.0536 \approx 226$. Notice that 5.01 ($=5.01 \times 10^0$) has three significant figures and 45.0536 ($=4.50536 \times 10^1$) has six significant figures. So the product must have three significant figures.

Example 3.

Evaluate each of the followings and express the results in scientific notation:

(a) $4.215 \times 10^{-2} + 3.2 \times 10^{-4}$ [Addition]

(b) $8.97 \times 10^4 - 2.62 \times 10^3$ [Subtraction]

(c) $\left(6.73 \times 10^{-5}\right)\left(2.91 \times 10^2\right)$ [Multiplication]

(d) $\frac{6.4\times10^6}{1.92\times10^2}$ [Division]

(e) $\left(6.5 \times 10^{-3}\right)^2$ [Power]

(f) $\sqrt{3.6 \times 10^5}$ [*Root*]

Solution

(a) $4.215 \times 10^{-2} + 3.2 \times 10^{-4}$

$= 4.215 \times 10^{-2} + 0.032 \times 10^{-2}$

$= (4.215 + 0.032) \times 10^{-2} \quad = 4.247 \times 10^{-2}$

(b) $8.97 \times 10^4 - 2.62 \times 10^3$

$= 8.97 \times 10^4 - 0.262 \times 10^4$

$= (8.97 - 0.262) \times 10^4 = 8.708 \times 10^4 \approx 8.71 \times 10^4$

(c) $\left(6.73 \times 10^{-5}\right)\left(2.91 \times 10^{2}\right) = 19.5843 \times 10^{-3} \approx 1.96 \times 10^{-2}$

(d) $\frac{6.4\times10^6}{1.92\times10^2} \approx 3.3 \times 10^4$

(e) $\left(6.5 \times 10^{-3}\right)^2 = 42.25 \times 10^{-6} \approx 4.2 \times 10^{-5}$

(f) $\sqrt{3.6 \times 10^5} = \sqrt{36 \times 10^4} = 6.0 \times 10^2$

Example 4.

Evaluate $\frac{2,750,000 \times 0.015}{750}$ by transforming each number to scientific notation.

Solution

$$\frac{2,750,000 \times 0.015}{750} = \frac{\left(2.75 \times 10^6\right)\left(1.5 \times 10^{-2}\right)}{7.5 \times 10^2} = 0.55 \times 10^2 = 55$$

Exercise 3.1

1. How many significant figures are there in each of the following numbers?
   (a) 2.175 (b) 0.2175 (c) 0.0075
   (d) 89400 (e) 0.00046

2. Write in scientific notation.
   (a) 24.86 (b) 2.486 (c) 0.2486 (d) 0.002486
   (e) 0.073 (f) 0.0086 (g) 0.934 (h) 7
   (i) 0.00056857 (j) 6.843250

3. Write each number in ordinary decimal form.
   (a) 7.84×10^4 (b) 7.89×10^{-4}
   (c) 2.25×10^5 (d) 4.01×10^{-3}

4. Simplify and give the answers in scientific notation.
   (a) $2.3 \times 10^2 + 1.7 \times 10^2$ (b) $4.6 \times 10^{-3} - 2.5 \times 10^{-3}$
   (c) $(4.5 \times 10^6) \times (1.5 \times 10^{-2})$ (d) $\frac{7.6\times10^5}{1.9\times10^{-2}}$

5. Compute using scientific notation.
   (a) $\frac{2.5\times10^2}{0.25\times0.002}$ (b) $\frac{33{,}000{,}000\times0.4}{1.1\times30}$
   (c) $\frac{50\times0.014\times0.30}{10500}$ (d) $\frac{7000\times80\times300}{400}$

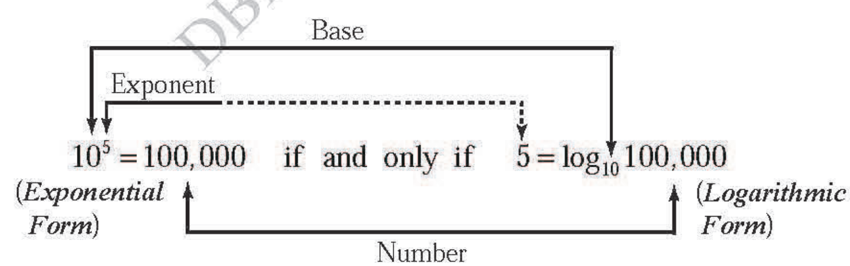
3.2 Definition of the Logarithm

We can easily see that $x = 3$ is the solution of the exponential equation $2^x = 8$. However, for some equations such as $2^x = \frac{1}{3}$ and $2.718^x = 5$, it is not easy to find the solutions. The following basic property of real numbers plays important role in defining logarithms.

"Given a positive number N, and a positive number b other than 1, the equation $N = b^x$ has exactly one solution for x."

Definition. Let N and b be positive real numbers, with $b \neq 1$. Then the **logarithm** of N (with respect) to the base b is the exponent by which b must be raised to yield N, and is denoted by $\log_b N$.

It means that $x = \log_b N$ is the solution of the equation $b^x = N$. In other words, $b^x = N$ if and only if $\log_b N = x$. We say that $b^x = N$ and $\log_b N = x$ are equivalent.



The followings are immediate consequences of the definition of logarithm:

L1.	$N = b^{\log_b N}$
L2.	$x = \log_b b^x$
L3.	$\log_b b = 1$
L4.	$\log_b 1 = 0$

Example 5.

Write each of the following in logarithmic form.

(a) $10^{-2} = 0.01$ (b) $4^{1/2} = 2$ (c) $7^2 = 49$

Solution

(a) $\log_{10} 0.01 = -2$ (b) $\log_4 2 = 1/2$ (c) $\log_7 49 = 2$

Example 6.

Express each of the following in exponential form.

(a) $\log_2 8 = 3$ (b) $\log_{10} 1 = 0$ (c) $\log_5 \left(\frac{1}{\sqrt{5}}\right) = -1/2$

Solution

(a) $2^3 = 8$ (b) $10^0 = 1$ (c) $5^{-1/2} = \frac{1}{\sqrt{5}}$

Example 7.

Find the value of each logarithm.

(a) $\log_5 25$ (b) $\log_2 16\sqrt{2}$ (c) $\log_{1/2} 8$

Solution

(a) Since $5^2 = 25$, we have $\log_5 25 = 2$. (using definition of logarithm)

or $\log_5 25 = \log_5 5^2 = 2$ (using L2)

(b) $\log_2 16\sqrt{2} = \log_2 \left(2^4 \times 2^{1/2}\right) = \log_2 2^{9/2} = \frac{9}{2}$

(c) $\log_{1/2} 8 = \log_{1/2} 2^3 = \log_{1/2} (1/2)^{-3} = -3$

Example 8.

Evaluate each expression.

(a) $3^{\log_3 7} + \log_5 125$ (b) $\log_7 7^9 - \log_3 \frac{1}{9}$

(c) $\log_5 (\log_2 (\log_3 9))$ (d) $10^{2+\log_{10} 5}$

Solution

(a) $3^{\log_3 7} + \log_5 125 = 7 + \log_5 5^3 = 7 + 3 = 10$

(b) $\log_7 7^9 - \log_3 \frac{1}{9} = 9 - \log_3 3^{-2} = 9 - (-2) = 9 + 2 = 11$

(c) $\log_5 (\log_2 (\log_3 9)) = \log_5 (\log_2 (\log_3 3^2)) = \log_5 (\log_2 2) = \log_5 1 = 0$

(d) $10^{2+\log_{10} 5} = 10^2 \times 10^{\log_{10} 5} = 100 \times 5 = 500$

Example 9.

(a) Given that $10^{0.3010} = 2$, find the value of $\log_{10} 16$.

(b) Solve $\log_3 (x^2 - 1) = 2$.

Solution

(a) Since $10^{0.3010} = 2$, we have $\log_{10} 2 = 0.3010$.

So $\log_{10} 16 = \log_{10} 2^4 = 4 \log_{10} 2 = 4\,(0.3010) = 1.204$.

(b) Since $\log_3 (x^2 - 1) = 2$, we have $x^2 - 1 = 3^2$.

$\therefore x^2 = 10$

$\therefore x = \pm\sqrt{10}$

Exercise 3.2

1. Write the following equations in logarithmic form.
   (a) $3^4 = 81$ (b) $9^{3/2} = 27$ (c) $10^{-3} = 0.001$
   (d) $3^{-1} = \frac{1}{3}$ (e) $\left(\frac{1}{4}\right)^{-3} = 64$

2. Write the following equations in exponential form.
   (a) $\log_{10}3 = 0.4771$ (b) $\log_6 0.001 = -3.855$ (c) $\log_{144}12 = \frac{1}{2}$
   (d) $-5 = \log_3 \frac{1}{243}$ (e) $\log_x 7 = y^2$, where , $0 < x < 1$

3. Solve the following equations
   (a) $\log_7 49 = x$ (b) $\log_x 10 = 1$ (c) $\log_{\sqrt{3}} x = 2$ (d) $x^{\log_x x} = 5$.

4. Evaluate.
   (a) $9^{\log_9 2} + 3^{\log_3 8}$ (b) $\log_4 4^5 \times \log_{10}10^2$ (c) $7^{\log_7 9} + \log_2\left(\frac{1}{2}\right)$
   (d) $\log_{\frac{1}{2}}\frac{1}{8} - 4\log_{10}10$ (e) $10^{1-\log_{10}3}$

5. Find the value of x in each of the following problems.
   (a) $\log_3(2x-5) = 2$, where $x > \frac{5}{2}$
   (b) $\log_{77}(\log_7 x) = 0$, where $x > 0$
   (c) $8 + 3^x = 10$, given that $\log_3 2 = 0.6309$

3.3 Properties of Logarithm

The following theorem concerns with some properties of logarithm.

Theorem 1. If M, N, b are positive real numbers, $b \neq 1$ and p is any real number, then

L5. $\log_b(MN) = \log_b M + \log_b N$

L6. $\log_b N^p = p\ \log_b N$

L7. $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$

Proof.

L5: Since $M = b^{\log_b M}$ and $N = b^{\log_b N}$ (using L1), we have
$MN = b^{\log_b M} b^{\log_b N} = b^{\log_b M + \log_b N}$
and hence $\log_b (MN) = \log_b M + \log_b N$.

L6: Since $N = b^{\log_b N}$, we have $N^p = \left(b^{\log_b N}\right)^p = b^{p \log_b N}$
and hence $\log_b N^p = p \ \log_b N$.

L7:
$$\begin{aligned}\log_b \left(\frac{M}{N}\right) &= \log_b (MN^{-1}) \\ &= \log_b M + \log_b N^{-1} \quad \text{(using L5)} \\ &= \log_b M - \log_b N \text{ (using L6)}\end{aligned}$$

Basic properties of exponents and logarithms can be summarized as follow.

Properties	**Exponents**	**Logarithms**
One-to-one Property	If $b^x = b^y$, then $x = y$.	If $\log_b M = \log_b N$, then $M = N$.
Product Property	$b^x \cdot b^y = b^{x+y}$	$\log_b (MN) = \log_b M + \log_b N$
Quotient Property	$\frac{b^x}{b^y} = b^{x-y}$	$\log_b \frac{M}{N} = \log_b M - \log_b N$
Power Property	$(b^x)^y = b^{x \cdot y}$	$\log_b N^p = p \ \log_b N$

Example 10.

If $p = \log_b 2$, $q = \log_b 3$ and $r = \log_b 5$, write $\log_b \frac{5\sqrt{3}}{2}$ in terms of p, q and r.

Solution

$\log_b \frac{5\sqrt{3}}{2} = \log_b 5 + \log_b \sqrt{3} - \log_b 2 = \log_b 5 + \frac{1}{2} \log_b 3 - \log_b 2 = r + \frac{1}{2} q - p$

Example 11.

Using $\log_2 3 = 1.5850$, find the values of
(a) $\log_2 24$ (b) $\log_2 0.75$.

Solution

(a) $\log_2 24 = \log_2 (2^3 \times 3) = \log_2 2^3 + \log_2 3 = 3 + 1.5850 = 4.5850$
(b) $\log_2 0.75 = \log_2 \left(\frac{3}{4}\right) = \log_2 \left(\frac{3}{2^2}\right) = \log_2 3 - \log_2 2^2 = 1.5850 - 2 = -0.4150$

Example 12.

Write each expression as a single logarithm.

(a) $2 + 3\log_5 x^2$ (b) $\log_3 2 + \log_9 81$

(c) $\log_b(3x) + \log_b(4y) - \log_b(2z)$

(d) $-\log_3(2s) + \frac{1}{2}\log_3(4t^2v^4) - 2\log_3(5u)$

Solution

(a) $2 + 3\ \log_5 x^2 = \log_5 5^2 + \log_5 (x^2)^3 = \log_5 25 + \log_5 x^6 = \log_5(25x^6)$

(b) $\log_3 2 + \log_9 81$

$= \log_3 2 + \log_9 9^2 = \log_3 2 + 2 = \log_3 2 + \log_3 3^2 = \log_3(2 \times 3^2) = \log_3 18$

(c) $\log_b(3x) + \log_b(4y) - \log_b(2z) = \log_b\left(\frac{3x \cdot 4y}{2z}\right) = \log_b\left(\frac{6xy}{z}\right)$

(d) $-\log_3(2s) + \frac{1}{2}\log_3(4t^2v^4) - 2\log_3(5u)$

$= -\log_3(2s) + \log_3(4t^2v^4)^{\frac{1}{2}} - \log_3(5u)^2$

$= \log_3 \frac{\sqrt{4t^2v^4}}{2s(5u)^2}$

$= \log_3 \frac{2\,t\,v^2}{2\,s \cdot 25u^2} = \log_3 \frac{tv^2}{25\,s\,u^2}$

Example 13.

Solve the following equation for x.

$\log_2(3x^2 - 1) - \log_2(2x) = 0$

Solution

$$\begin{aligned}
\log_2\left(3x^2 - 1\right) - \log_2(2x) &= 0 \\
\log_2\left(3x^2 - 1\right) &= \log_2(2x) \\
3x^2 - 1 &= 2x \\
3x^2 - 2x - 1 &= 0 \\
(3x + 1)(x - 1) &= 0 \\
3x + 1 = 0 \text{ or } x - 1 &= 0
\end{aligned}$$

$x = -\frac{1}{3}$ or $x = 1$

But $x = -\dfrac{1}{3}$ is impossible because $\log_2(-\frac{2}{3})$ is not defined.

$\therefore x = 1$

Example 14.

Suppose that $\log_b(xy^2) = 4$ and $\log_b\left(\frac{x^3}{y}\right) = 5$.

(a) Write the equation connecting $\log_b x$ and $\log_b y$.

(b) Find the values of $\log_b x$ and $\log_b y$.

(c) Find $\log_b(y^5\sqrt{x})$.

(d) Write x and y in terms of b.

Solution

(a) Since $\log_b(xy^2) = 4$,

$\log_b x + \log_b y^2 = 4$

$\log_b x + 2\ \log_b y = 4 \qquad (1)$

(b) Since $\log_b\left(\frac{x^3}{y}\right) = 5$,

$\log_b x^3 - \log_b y = 5$

$3\ \log_b x - \log_b y = 5$

$6\ \log_b x - 2\log_b y = 10 \qquad (2)$

Adding equations (1) and (2),

$7\ \log_b x = 14$

$\log_b x = 2$

Substituting $\log_b x = 2$ in (1),

$2 + 2\ \log_b y = 4$

$\log_b y = 1$

(c) $\log_b(y^5\sqrt{x}) = \log_b y^5 + \log_b\sqrt{x} = 5\log_b y + \frac{1}{2}\log_b x = 5\,(1) + \frac{1}{2}\,(2) = 6$

(d) Since $\log_b y = 1$, we have $y = b$.

Since $\log_b x = 2$, we have $x = b^2$.

Exercise 3.3

1. Replace $\square$ with the appropriate number.
   (a) $\log_3 24 = \log_3 6 + \log_3 \square$ (b) $\log_5 24 = \log_5 60 + \log_5 \square$
   (c) $\log_2 \square = 3\log_2 3$ (d) $\log_{10} 9 = \square \log_{10} 3$
   (e) $\log_8 5 = \log_8 \square - \log_8 11$
2. Write each expression as a single logarithm.
   (a) $\log_b 20 + \log_b 57 - \log_b 114$ (b) $3\log_b 8 - \log_b 12$
   (c) $\log_b x - 2\log_b y - \log_b a$ (d) $\log_2 3 + \log_4 15$
3. Write each expression in terms of $\log_b 2$, $\log_b 3$ and $\log_b 5$.
   (a) $\log_b 8$ (b) $\log_b 15$ (c) $\log_b 270$
   (d) $\log_b \frac{27\sqrt[3]{5}}{16}$ (e) $\log_b \frac{216}{\sqrt[3]{32}}$ (f) $\log_b \left(648\sqrt{125}\right)$
4. Evaluate each expression.
   (a) $\log_2 128$ (b) $\log_3 81^4$ (c) $\log_{1/2} 8$
   (d) $\log_8 2$ (e) $\log_3 \frac{\sqrt{3}}{81}$ (f) $\frac{\log_3 \sqrt{3}}{\log_3 81}$
   (g) $\frac{\log_2 25}{\log_2 5}$ (h) $\log_4 8$
5. Use $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$ to evaluate each of the following expressions.
   (a) $\log_{10} 6$ (b) $\log_{10} 1.5$ (c) $\log_{10} \sqrt{3}$
   (d) $\log_{10} 4$ (e) $\log_{10} 4.5$ (f) $\log_{10} 8$
   (g) $\log_{10} 18$ (h) $\log_{10} 5$
6. Solve the following equations for x.
   (a) $\log_a \frac{18}{5} + \log_a \frac{10}{3} - \log_a \frac{6}{7} = \log_a x$
   (b) $\log_b x = 2 - a + \log_b \left(\frac{a^2 b^a}{b^2}\right)$
   (c) $\log_b x^3 - \log_b x^2 = \log_b 5x - \log_b 4x$
   (d) $\log_{10} x + \log_{10} 3 = \log_{10} 6$
   (e) $8\log_b x = \log_b a^{\frac{3}{2}} + \log_b 2 - \frac{1}{2}\log_b a^3 - \log_b \frac{2}{a^4}$

7. Given that $\log_{10}5 = 0.6990$ and $\log_{10}x = 0.2330$. What is the value of x ?

8. Show that if $\log_e I = -\frac{R}{L}t + \log_e I_0$ then $I = I_0 e^{-\frac{Rt}{L}}$.

9. Show that if $\log_b y = \frac{1}{2}\log_b x + c$ then $y = b^c\sqrt{x}$.

10. Show that
    (a) $\frac{1}{4}\log_{10}8 + \frac{1}{4}\log_{10}2 = \log_{10}2$
    (b) $4\log_{10}3 - 2\log_{10}3 + 1 = \log_{10}90$.

11. Show that
    (a) $a^{2\log_a 3} + b^{3\log_b 2} = 17$
    (b) $3\ \log_6 1296 = 2\ \log_4 4096$.

12. Given that $\log_{10}12 = 1.0792$ and $\log_{10}24 = 1.3802$, deduce the values of $\log_{10}2$ and $\log_{10}6$.

13. If $\log_x a = 5$ and $\log_x 3a = 9$, find the values of a and x.

14. (a) If $\log_{10}2 = a$, find $\log_{10}8 + \log_{10}25$ in terms of a.
    (b) If $a = 10^x$ and $b = 10^y$, express $\log_{10}(a^4b^3)$ in terms of x and y .

15. (a) If $\log_2(4x-4) = 2$, find the value of $\log_4 x$.
    (b) Prove that if $\frac{1}{2}\log_3 M + 3\log_3 N = 1$ then $MN^6 = 9$.

3.4 Change of Base

Logarithms to any base can be computed by using logarithms to some other bases.

Theorem 2. Suppose a and b are any two positive real numbers other than 1. If N is any positive real number, then

L8. $\log_a N = \frac{\log_b N}{\log_b a}$

Proof.
Since $N = a^{\log_a N}$, we have
$\log_b N = \log_b a^{\log_a N} = (\log_a N)(\log_b a)$.
So $\log_a N = \frac{\log_b N}{\log_b a}$

Corollary 1. If a and N are any two positive real numbers other than 1, then

L9. $\log_a N = \frac{1}{\log_N a}$

Proof.
Using *L8* and *L3*, $\log_a N = \frac{\log_N N}{\log_N a} = \frac{1}{\log_N a}$.

Corollary 2. If a, p, N are any positive real numbers such that $a \neq 1$, then

L10. $\log_{a^p} N = \frac{1}{p} \log_a N$

Proof.
L10 follows from *L9* and *L6*.

Corollary 3. If a, b, k are any positive real numbers other than 1, then

L11. $a^{\log_k b} = b^{\log_k a}$

Proof.
Using L2 and L6, $a^{\log_k b} = a^{\log_k \left(a^{\log_a b}\right)} = a^{(\log_a b)(\log_k a)} = \left(a^{\log_a b}\right)^{\log_k a} = b^{\log_k a}$

Example 15.

Given that $\log_p x = 20$ and $\log_p y = 5$, find $\log_y x$ and $\log_x y$.

Solution

$\log_y x = \frac{\log_p x}{\log_p y} = \frac{20}{5} = 4$

$\log_x y = \frac{1}{\log_y x} = \frac{1}{4}$

Example 16.

Find the value of

(a) $2^{\frac{\log_5 3}{\log_5 2}}$ (b) $5^{\frac{1}{\log_7 5}}$ (c) $\log_3 5 \times \log_{25} 27$.

Solution

(a) $2^{\frac{\log_5 3}{\log_5 2}} = 2^{\log_2 3} = 3$

(b) $5^{\frac{1}{\log_7 5}} = 5^{\log_5 7} = 7$

(c) $\log_3 5 \times \log_{25} 27 = \log_3 5 \times \log_{5^2} 3^3$

$= \log_3 5 \times \frac{3}{2} \log_5 3$

$= \log_3 5 \times \frac{3}{2} \times \frac{1}{\log_3 5} = \frac{3}{2}$

Example 17.

Solve the equation $\log_3 x = 3 - 2\log_x 3$, where $x > 0$ and $x \neq 1$.

Solution

$$\begin{aligned}
\log_3 x &= 3 - 2\log_x 3 \\
\log_3 x &= 3 - \frac{2}{\log_3 x} \\
(\log_3 x)^2 &= 3\log_3 x - 2 \\
(\log_3 x)^2 - 3\log_3 x + 2 &= 0 \\
(\log_3 x - 2)(\log_3 x - 1) &= 0 \\
\log_3 x - 2 = 0 &\text{ or } \log_3 x - 1 = 0 \\
\log_3 x = 2 &\text{ or } \log_3 x = 1 \\
x = 3^2 &\text{ or } x = 3 \\
x = 9 &\text{ or } x = 3
\end{aligned}$$

Example 18.

Solve the logarithmic equation $\log_3 x = \log_9 (x+6)$.
Solution

$$\begin{aligned}
\log_3 x &= \log_9 (x+6) \\
\log_3 x &= \log_{3^2} (x+6) \\
\log_3 x &= \frac{1}{2}\log_3 (x+6) \\
2\log_3 x &= \log_3 (x+6) \\
\log_3 x^2 &= \log_3 (x+6) \\
x^2 &= x+6 \\
x^2 - x - 6 &= 0 \\
(x-3)(x+2) &= 0 \\
x = 3 \ &\text{or} \ x = -2
\end{aligned}$$

But $x = -2$ is impossible since $\log_3(-2)$ is undefined.
So $x = 3$.

Exercise 3.4

1. If $\log_a b + \log_b a^2 = 3$, find b in terms of a.

2. Show that
   (a) $\log_4 x = 2\log_{16} x$ (b) $\log_b x = 3\log_{b^3} x$ (c) $\log_2 x = (1 + \log_2 3)\log_6 x$.

3. If $a = \log_b c$, $b = \log_c a$ and $c = \log_a b$, prove that $abc = 1$.

4. Show that
   (a) $(\log_{10} 4 - \log_{10} 2)\log_2 10 = 1$ (b) $2\log_2 3(\log_9 2 + \log_9 4) = 3$.

5. Compute
   (a) $3^{\log_2 5} - 5^{\log_2 3}$ (b) $4^{\log_2 3}$ (c) $2^{\log_{2\sqrt{2}} 27}$.

3.5 Common Logarithm and Natural Logarithm

The logarithm of N to the base 10 is said to be a **common logarithm**, and is usually written as $\log N$. If $N = a \times 10^n$, then

$$\log N = \log (a \times 10^n) = \log 10^n + \log a = n + \log a;$$

in this case n and $\log a$ are respectively said to be the **characteristic** and the **mantissa** of $\log N$.

As a positive integer n become very large, the value of $(1 + 1/n)^n$ approaches an irrational number, which is denoted by e. The number e is called **Euler's number**, in honour of the Swiss mathematician Leonard Euler, and is approximately equal to 2.71828. The logarithm of N to the base e is called a *natural logarithm*, and is denoted by $\ln N$.

Example 19.

Given that $\log_{10}7 = 0.8451$; what are the characteristics and the mantissas of $\log_{10}0.007$ and $\log_{10}700$?

Solution

$\log_{10}7 = 0.8451$

$\log_{10}0.007 = \log_{10}\left(7 \times 10^{-3}\right) = \log_{10}10^{-3} + \log_{10}7 = -3 + 0.8451$

The characteristic and the mantissa of $\log_{10}0.007$ are -3 and 0.8451 respectively.

$\log_{10}700 = \log_{10}\left(7 \times 10^{2}\right) = \log_{10}10^{2} + \log_{10}7 = 2 + 0.8451$

The characteristic and the mantissa of $\log_{10}700$ are 2 and 0.8451 respectively.

Notice that the mantissas of $\log_{10}7$, $\log_{10}0.007$ *and* $\log_{10}700$ *are the same. Characteristics can be seen directly from the scientific notation of the numbers.*

Example 20.

Two nonnegative real numbers A and P are related by the formula $A = Pe^{0.085t}$. Given that $\ln 2 = 0.6931$, find the value of t for which A becomes 200% of P.

Solution

We have $A = Pe^{0.085t}$. If $A = 200\%$ of P, then

$\frac{200}{100} \times P = Pe^{0.085t}$.

So $2 = e^{0.085t}$

$\log_e 2 = 0.085\, t$

$t = \dfrac{\ln\ 2}{0.085} = \dfrac{0.6931}{0.0850} = \dfrac{6931}{850} = 8\dfrac{131}{850}$

Example 21.

Using $\log_{10}2 = 0.3010$, $\log_{10}9.87 = 0.9943$ and $\log_{10}8.5 = 0.9294$; evaluate $\dfrac{200 \times 98.7 \times 85}{8.5^3}$.

Solution

Let $p = \dfrac{200 \times 98.7 \times 85}{8.5^3}$. Then

$$\begin{aligned}
\log_{10}p &= \log_{10}\frac{200 \times 98.7 \times 85}{8.5^3} \\
&= \log_{10}200 + \log_{10}98.7 + \log_{10}85 - \log_{10}8.5^3 \\
&= \log_{10}\left(2 \times 10^2\right) + \log_{10}\left(9.87 \times 10^1\right) + \log_{10}(8.5 \times 10) - 3\log_{10}8.5 \\
&= 2 + \log_{10}2 + 1 + \log_{10}9.87 + 1 + \log_{10}8.5 - 3\log_{10}8.5 \\
&= 4 + \log_{10}2 + \log_{10}9.87 - 2\log_{10}8.5 \\
&= 4 + 0.3010 + 0.9943 - 2 \times 0.9294 \\
&= 5.2953 - 1.8588 \\
&= 3.4365
\end{aligned}$$

So $\dfrac{200 \times 98.7 \times 85}{8.5^3} = p = 10^{3.4365}$

Exercise 3.5

1. Given that $\log 2.345 = 0.3701$. What are the characteristics and the mantissas of each of the followings?
   (a) $\log 234,500$ (b) $\log 0.0002345$

2. Using $\log_{10}2.74 = 0.4378$, $\log_{10}2.83 = 0.4518$, $\log_{10}5.97 = 0.7760$, $\log_{10}6.21 = 0.7931$, $\log_{10}8.18 = 0.9128$ and $\log_{10}9.27 = 0.9671$, compute
   (a) $\left(\dfrac{28.3}{597 \times 621}\right)^2$ (b) $\dfrac{274^{\frac{1}{3}}}{927 \times 818}$ (c) $\dfrac{28.3\sqrt{0.621}}{597}$.

Chapter 4

Functions

In this chapter, product sets, relations and functions are introduced. A relation is described as a set of ordered pairs, and a function is described as a special kind of relation. Then some basic functions are illustrated by graphs. You will identify the relation, function, domain and range, and describe each of these mathematical concept in a given context. You will define the equality of functions, one-to-one function, inverse function and composite functions and solve with these functions in algebraic form.

4.1 Product Sets

Let a and b be any two elements. When we indicate such two elements a and b as an ordered, we take a as the first element and b as the second element and enclose the elements in parentheses (a, b). A pair of such elements a and b, written as (a, b), is called **an ordered pair**.

> Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Definition. Let A and B be any two sets. The product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. In symbol

$$A \times B = \{(a, b) \mid a \in A \quad \text{and} \quad b \in B\}.$$

$A \times B$ is read as A **cross** B.

Example 1.

Let $A = \{2, 4\}$, $B = \{1, 5, 6\}$. Find $A \times B$ and $B \times A$.
Solution

$A \times B = \{(2,1), (2,5), (2,6), (4,1), (4,5), (4,6)\}$.

$\times$	1	5	6
2	(2, 1)	(2, 5)	(2, 6)
4	(4, 1)	(4, 5)	(4, 6)

$B \times A = \{(1,2), (1,4), (5,2), (5,4), (6,2), (6,4)\}$.

$\times$	2	4
1	(1, 2)	(1, 4)
5	(5, 2)	(5, 4)
6	(6, 2)	(6, 4)

Similarly, we can write

$$\begin{aligned} A \times A &= \{(2,2), (2,4), (4,2), (4,4)\}, \\ B \times B &= \{(1,1), (1,5), (1,6), (5,1), (5,5), (5,6), (6,1), (6,5), (6,6)\}. \end{aligned}$$

Here the order of elements in the ordered pair is important since $(2,1)$ and $(1,2)$ are two different ordered pairs.

We can count the number of elements of $A \times B$. In example 1, A has 2 elements, B has 3 elements, $A \times B$ has 6 elements and $A \times A$ has 4 elements. Therefore,

the number of elements of $A \times B = 6 = 2 \times 3$,
and the number of elements of $A \times A = 4 = 2 \times 2$.

In general, if A has m elements and B has n elements, then $A \times B$ has mn elements.
Therefore, the number of elements of $A \times B = mn$.

Exercise 4.1

1. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$ be a universal set, $A = \{x \mid x \text{ is a prime number}\}$ and $B = \{x \mid 1 < x < 5\}$. Find $A \times B$ and $B \times A$.

2. If $A = \{1, 2\}$, $B = \{2, 3, 4\}$, find $(B \setminus A) \times (A \cup B)$.

3. Compute $\{a\} \times \{a\}$.

4. Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$ and $C = \{a, b\}$. Show that $A \times C \subset B \times C$.

5. $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4\}$, $D = \{1, 2\}$. Prove the following:

   (a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

   (b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

   (c) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

4.2 Relations

Definition. A **relation** is a set of ordered pairs.

For example, the set $R = \{(2, 2), (2, 4), (6, 3)\}$ is a relation.

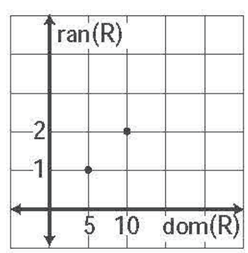
Definition. If R is a relation, the set of all first elements x of ordered pairs $(x, y) \in R$ is called the **domain** of R, denoted by $\text{dom}(R)$. The set of all second elements y is called the **range** of R, denoted by $\text{ran}(R)$.

For example, in the relation $R = \{(2, 2), (2, 4), (6, 3)\}$ has $\text{dom}(R) = \{2, 6\}$ and $\text{ran}(R) = \{2, 3, 4\}$.

Graph of a Relation

We can draw a graph which describes a relation as follows.

Consider the relation $R = \{(5, 1), (10, 2)\}$. We have that $\text{dom}(R) = \{5, 10\}$ and $\text{ran}(R) = \{1, 2\}$. Place the elements of $\text{dom}(R)$ on a horizontal line and the elements of $\text{ran}(R)$ on a vertical line. The dots represent the graph of R.



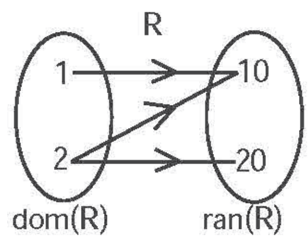
We can draw an arrow diagram of a relation as in the following example.

Example 2.

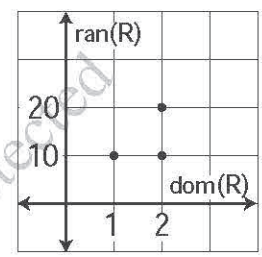
Let $R = \{(1, 10), (2, 10), (2, 20)\}$ be a relation. State the domain and range of R. Draw an arrow diagram and a graph of the relation.

Solution

$R = \{(1, 10), (2, 10), (2, 20)\}$. Domain of R is $\{1, 2\}$ and range of R is $\{10, 20\}$.



Arrow diagram of R

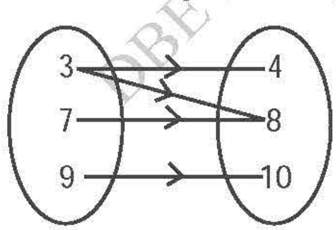


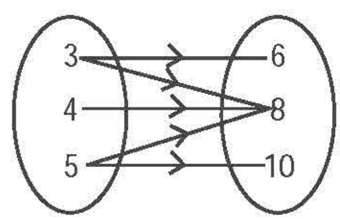
Graph of R

Exercise 4.2

1. Let $R = \{(1, 2), (2, 4), (2, 5), (3, 6), (4, 8)\}$ be a relation. State the domain and range of R. Draw a graph and an arrow diagram to describe R.

2. Write the sets of ordered pairs that represent the relations for each of the following arrow diagrams. Draw the graph of each relation.





4.3 Functions

Definition. A **function** f is a relation such that $(x, y) \in f$ and $(x, z) \in f$ implies $y = z$. The unique element y such that $(x, y) \in f$ is the **image** of x under f; we use the notation

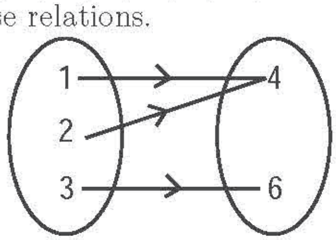
$$y = f(x) \quad \text{(read } f \text{ of } x\text{)} \quad \text{(or)} \quad f : x \mapsto y \ \text{ for } (x, y) \in f.$$

The domain and range of the function f are given by

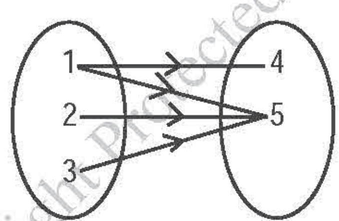
dom $(f) = \{x \mid (x, y) \in f\}$ and ran $(f) = \{y \mid (x, y) \in f\}$.

> Note that for every x in the domain of a function f there is exactly one y such that $(x, y) \in f$.

For example, the relation $\{(1,4),(2,4),(3,6)\}$ is a function. The relation $(1,4),(1,5),(2,5),(3,5)$ is not a function. See below the arrow diagrams of these relations.

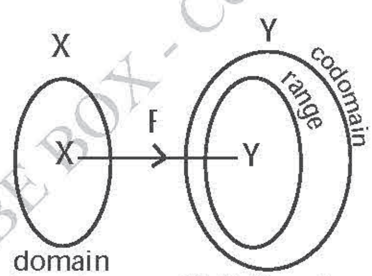


This relation is a function.



This relation is not a function.

Definition.



(i) We say that f is a function **on** X if $X = \text{dom}(f)$.

(ii) We call f is a function **from** X to Y (or a **mapping from** X **into** Y),

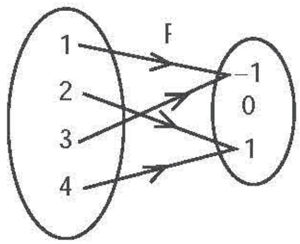
$$f : X \to Y,$$

if $\text{dom}(f) = X$ and $\text{ran}(f) \subset Y$. In this case Y is called the **codomain** of f.

(iii) If $Y = \text{ran}(f)$, then f is a function **onto** Y.

(iv) If f is a function from X to Y and $A \subset X$, then $\{f(a) \mid a \in A\}$ is denoted by $f(A)$ and is called the image of A.

(v) If $B \subset Y$ then $\{x \in X \mid f(x) \in B\}$ is denoted by $f^{-1}(B)$ and is called the inverse image of B.



For example, the given arrow diagram represents the function

$$f = \{(1,-1),(2,1),(3,-1),(4,1)\}$$

from the set $\{1,2,3,4\}$ to $\{-1,0,1\}$ with the domain $\{1,2,3,4\}$, the range $\{-1,1\}$ and the codomain $\{-1,0,1\}$.

Let $A = \{1,3\} \subset X$ and $B = \{1\} \subset Y$. Then $f(A) = \{-1\}$ and $f^{-1}(B) = \{2,4\}$.

Example 3.

Let $A = \{1,2,3,4\}$ and $B = \{4,5,6\}$. The relation from A to B is described by (a) $\{(1,4),(2,5),(3,6),(4,5)\}$, (b) $\{(1,4),(2,4),(3,4),(1,6)\}$.
Which relation is a function from A to B?

Solution

(a) The relation is a function from A to B because each element of A is related to exactly one element of B.

(b) The relation is not a function because the element 1 of A is related to two elements of B, namely 4 and 6.

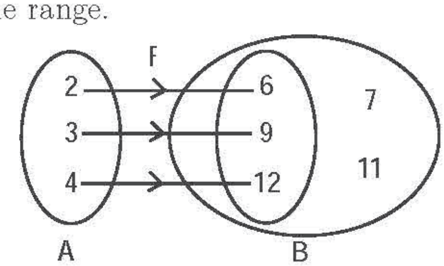
Example 4.

Let $A = \{2,3,4\}$ and $B = \{6,7,9,11,12\}$. Let f be a function from A to B such that $x \mapsto 3x$ where $x \in A$. Find the range of f.

Solution

The domain $A = \{2,3,4\}$. Let us find the range.

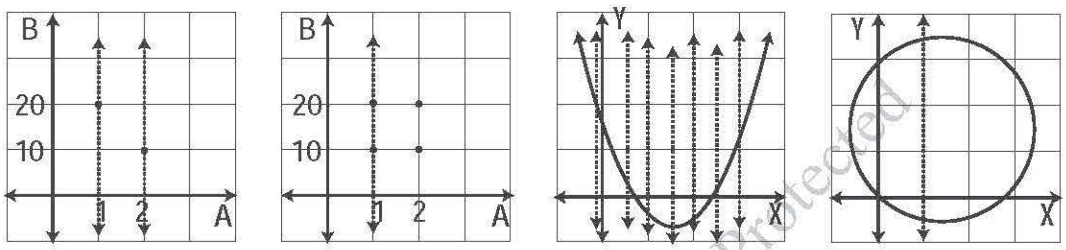
$2 \mapsto 6$
$3 \mapsto 9$
$4 \mapsto 12$



The range of f is $\{6,9,12\}$. It is clear that $\text{ran}(f) \subset B$. Thus f is a function from A into B.

> **Vertical Line Test:** A relation is a function if each vertical line intersects the graph at most one point.

We can use the vertical line test to determine whether the following graphs represent functions or not.



A function | Not a function | A function | Not a function

Example 5.

Let $\mathbb{R}$ be the set of all real numbers. Let the function f be defined by $f(x) = x^2 + x + 3$ for every $x \in \mathbb{R}$. Evaluate (a) $f(2)$ (b) $f(-3)$ (c) $f(-x)$.

Solution

$f(x) = x^2 + x + 3.$

(a) $f(2) = 2^2 + 2 + 3 = 9.$

(b) $f(-3) = (-3)^2 + (-3) + 3 = 9.$

(c) $f(-x) = (-x)^2 + (-x) + 3 = x^2 - x + 3.$

Example 6.

Let the function $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = px + q$, where p and q are real numbers. If $f(1) = 4$ and $f(-2) = 1$, find p and q.

Solution

$f(x) = px + q.$

Since $f(1) = 4$, $p(1) + q = 4$ gives

$$p + q = 4. \qquad (1)$$

Since $f(-2) = 1$, $p(-2) + q = 1$ gives

$$-2p + q = 1. \qquad (2)$$

Solving (1) and (2), we get $p = 1$ and $q = 3$.

The Domain of a Function

When the domain of a function is not specified, then assume that it is the set of all possible real numbers for which the function makes sense.

Example 7.

State the domain of
(a) $f(x) = \dfrac{1}{x-1}$ (b) $h(x) = \dfrac{x-3}{2}$ (c) $f(x) = \dfrac{1}{x^2-4}$.

Solution

(a) $f(x) = \dfrac{1}{x-1}$ makes sense when $x \neq 1$.
$\therefore$ domain of $f = \{x \mid x \neq 1, x \in \mathbb{R}\}$.

(b) $h(x) = \dfrac{x-3}{2}$ makes sense for all real numbers.
$\therefore$ domain of $h = \{x \mid x \in \mathbb{R}\}$.

(c) $f(x) = \dfrac{1}{x^2-4}$ make sense when $x^2 - 4 \neq 0$. i.e., $x \neq 2$ and $x \neq -2$.
$\therefore$ domain of $f = \{x \mid x \neq 2 \quad \text{and} \quad x \neq -2,\ x \in \mathbb{R}\}$.

Equality of Functions

Two functions f and g are equal (and we write $f = g$) if and only if

1. f and g have the same domain, and
2. $f(x) = g(x)$ for each element x in the domain.

Example 8.

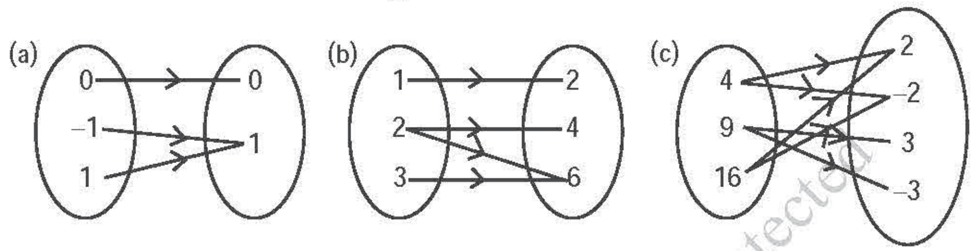
Determine whether $f(x) = x - 2$ and $g(x) = \dfrac{x^2-4}{x+2}$ are equal functions or not.

Solution

$\text{dom}(f) = \mathbb{R}$ and $\text{dom}(g) = \{x \mid x \neq -2,\ x \in \mathbb{R}\}$.
The domain of f and g are not the same. Therefore $f \neq g$.

Exercise 4.3

1. Determine whether each relation is a function or not. If it is a function, state the domain and range.



2. Consider the following relations. Determine whether each relation is a function or not. If it is a function, write down the domain and range.
   (a) $\{(1,3),(2,5),(3,7),(4,9)\}$ (b) $\{(-2,5),(-1,3),(0,1),(-1,1)\}$
   (c) $\{(1,3),(2,3),(3,2),(2,1)\}$ (d) $\{(2,4),(3,6),(4,6),(7,14)\}$
   (e) $\{(0,0),(1,1),(3,3),(4,4)\}$ (f) $\{(2,a),(4,c),(5,a),(4,e)\}$

3. Let f be a function from $\mathbb{R} \to \mathbb{R}$. Which of the following statements are true?
   (a) If $f(x) = 5 - x$, the image of -3 under f is 8.
   (b) If $f(x) = x^2 + 9$, the image of -3 under f is zero.
   (c) If $f(x) = 3x + 4$, then $f(a) = a$ implies that $a = -2$.
   (d) If $f(x) = x + 3$, there is only one value $a \in \mathbb{R}$ such that $f(a) = 0$.
   (e) If $f(x) = x^2 - 1$, then there are exactly two values $a \in \mathbb{R}$ such that $f(a) = 0$.

4. Illustrate the function $f : x \mapsto x + 2$ with an arrow diagram for the domain $\{3, 5, 7, 9, 10\}$. Write down the range of f.

5. Let the domain of function $h : x \mapsto 0$ be $\{2, 4, 6, 7\}$. What is the range of h? Draw an arrow diagram for h.

6. Let the domain of function $f : x \mapsto 3x$ be the set of natural numbers less than 5. State the domain and range.

7. Let $g : \mathbb{R} \to \mathbb{R}$ be given by
   (a) $g(x) = 3 - 4x$. Find $g(1)$, $g(3)$, $g(-2)$, $g(x+3)$, $g(\frac{1}{2})$.

(b) $g(x) = 2x - 5$. Find $g(3)$, $g(\frac{1}{2})$, $g(0)$, $g(-4)$, $g(4)$.
If $g(a) = 99$, find a.
(c) $g(x) = \dfrac{x+5}{2}$. Find the images of 3, 0, -3. Find x if $g(x) = 0$.
(d) $g(x) = 3x - 1$. Find x such that $g(x) = 20$.
(e) $g(x) = 3x + 1$. Find x such that $g(x) = 22$.

8. A function f from A to A, where A is the set of positive integers, is given by $f(x) =$ the sum of all possible divisors of x.
For example $f(6) = 1 + 2 + 3 + 6 = 12$.
(a) Find the values of $f(2)$, $f(5)$, $f(13)$, $f(18)$.
(b) Show that $f(14) = f(15)$ and $f(3) \cdot f(5) = f(15)$.

9. Let $A =$ the set of positive integers greater than 3 and $B =$ the set of all positive integers. Let $d : A \to B$ be a function given by $d(n) = \frac{1}{2}n(n-3)$, the number of diagonals of a polygon of n sides.
(a) Find $d(6)$, $d(8)$, $d(10)$, $d(12)$.
(b) How many diagonals will a polygon of 20 sides have?

10. Determine whether f and g are equal functions or not. Give reason:
(a) $f(x) = x^2 + 2$, $g(x) = (x+2)^2$ (b) $f(x) = \dfrac{x^2-1}{x+1}$, $g(x) = x - 1$
(c) $f(x) = x^2$, $g(x) = |x|^2$ (d) $f(x) = \dfrac{x+2}{x^2-4}$, $g(x) = \dfrac{1}{x-2}$

11. State the domain of the following functions.
(a) $f(x) = \sqrt{x-2}$ (b) $f(x) = \dfrac{1}{2x-1}$
(c) $f(x) = \dfrac{4}{x-3}$ (d) $f(x) = \dfrac{2}{x^2-1}$

4.3.1 The Graph of a Function

Let f be any function. The graph of f is the set of ordered pairs given by the function f as

$$\{(x, y) \mid x \in \text{dom}(f) \text{ and } y = f(x)\}.$$

The graph of function f is illustrated by plotting the set of ordered pairs in xy-plane.

We introduce the following curves of basic functions f given by $y = f(x)$.

Linear function: $y = mx + c$ where m and c are constants.

Quadratic function: $y = ax^2$ where a is a constant.

Absolute value function: $y = |x|$.

Square root function: $y = \sqrt{x}$.

Rational function: $y = \dfrac{ax+b}{cx+d}$ where a, b, c and d are constants.

To draw the graph of a function

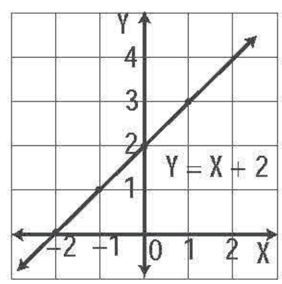
1. Make a suitable xy-table for the given function.
2. In the xy- plane, plot the points obtained from the xy- table.
3. Draw a curve through the points.

The Graph of Linear Function $y = mx + c$

Example 9.

Graph of $y = x + 2$ $(x \in \mathbb{R})$.

x	...	-2	-1	0	1	...
y	...	0	1	2	3	...



When $y = 0$, $x = -2$.
The graph cuts the x-axis at $(-2, 0)$
When $x = 0$, $y = 2$.
The graph cuts the y-axis at $(0, 2)$.

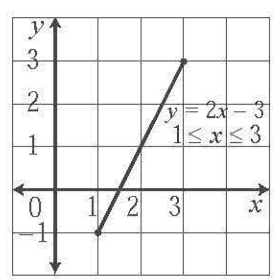
From the figure,
Domain $= \mathbb{R}$.
Range $= \mathbb{R}$.

Example 10.

Graph of $y = 2x - 3$ $(1 \le x \le 3)$.

x	1	...	2	...	3
y	-1	...	1	...	3

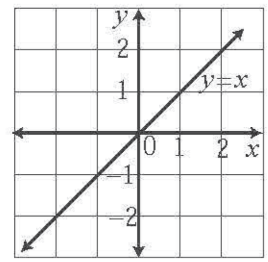
Domain $= \{x \mid 1 \le x \le 3\}$.
Range $= \{y \mid -1 \le y \le 3\}$.



Identity Function. The identity function I on $\mathbb{R}$ is the function $I : \mathbb{R} \to \mathbb{R}$ defined by

$$y = I(x) = x$$

i.e., the image of every $x \in \mathbb{R}$ is just itself.



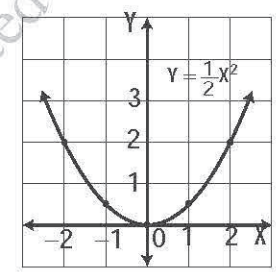
The Graph of Quadratic Function $y = ax^2$ $(a \neq 0)$

Example 11.

Graph of $y = \frac{1}{2}x^2$.

x	...	-2	-1	0	1	2	...
y	...	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	...

Domain $= \mathbb{R}$.
Range $=\{y \mid y \geq 0\}$.

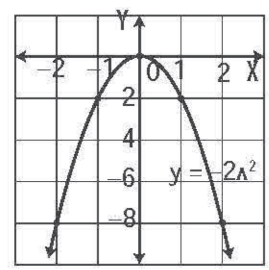


Example 12.

Graph of $y = -2x^2$.

x	...	-2	-1	0	1	2	...
y	...	-8	-2	0	-2	-8	...

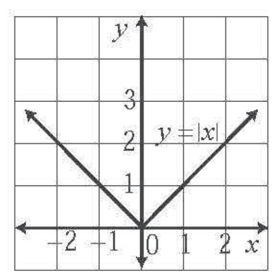
Domain $= \mathbb{R}$.
Range $= \{y \mid y \leq 0\}$.



The Graph of Modulus Function $y = |x|$
If x is a real number, we define

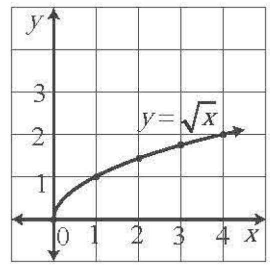
$$|x| = \begin{cases} -x & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ x & \text{if } x > 0. \end{cases}$$

$|x|$ is called the **modulus**, or the **absolute value** of x.



The function $f : \mathbb{R} \to [0, \infty)$ defined by $f(x) = |x|$ is known as the modulus function.

The Graph of Square Root Function $y = \sqrt{x}$



The function f, defined by $f(x) = \sqrt{x}$, is known as the square root function.
Domain $= \{x \mid x \geq 0\}$.
Range $= \{y \mid y \geq 0\}$.

Exercise 4.4

1. Sketch the graphs of:
   (a) $y = x - 1$ (b) $y = -x - 2$ (c) $y = -x + 2$
   (d) $y = 2x + 1$ (e) $y = 3x^2$ (f) $y = -3x^2$
   (g) $y = \frac{1}{3}x^2$ (h) $y = \sqrt{2x}$ (i) $y = |2x|$

2. Sketch the graphs of $y = 2x$ and $y = \frac{1}{2}x$ in the same plane. What do you notice from the graphs? Explain.

3. Sketch the graphs of $y = \frac{1}{2}x^2$ and $y = 2x^2$ in the same plane. What do you notice from the graphs? Explain.

(i) The Graph of $y = \dfrac{k}{x}$, $(k \neq 0)$

To draw a graph of

$$y = \frac{k}{x} \qquad \text{for} \quad k \neq 0,$$

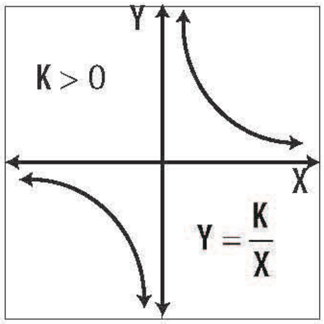
we consider two cases, $k > 0$ and $k < 0$.

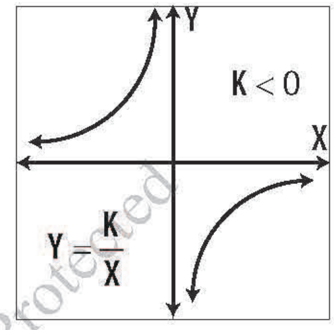
If $k > 0$, both x and y are positive, or negative. Therefore the point (x, y) lies in the quadrant I and quadrant III.

The values of y approach to 0 (zero) while the absolute values of x get larger, and the absolute values of y get larger while the values of x approach 0 (zero). However the curves never intersect both the lines $x = 0$ and $y = 0$. These lines are called the asymptotes. The line $x = 0$ is the **vertical asymptote** and the line $y = 0$ is the **horizontal asymptote**.

If $k < 0$, then $x < 0$ gives $y > 0$ and $x > 0$ gives $y < 0$. Therefore the point (x, y) lies in the quadrant II and quadrant IV.

Therefore we can conclude that if $k > 0$, the curves are in quadrant I and III and if $k < 0$, the curves are in quadrant II and IV.



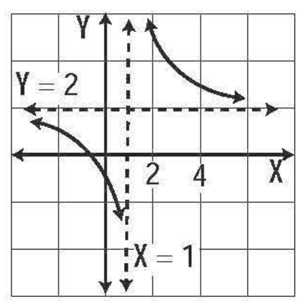


Domain $= \{x \mid x \neq 0,\ x \in \mathbb{R}\}$.
Range $= \{y \mid y \neq 0,\ y \in \mathbb{R}\}$.

(ii) The Graph of $y = \dfrac{k}{x-p} + q$

This equation is defined for $x \neq p$. The line $x = p$ is a vertical asymptote. When the absolute value of x gets larger and larger, y approaches to q. But the curves never intersect the line $y = q$. This line is a horizontal asymptote.

Example 13.



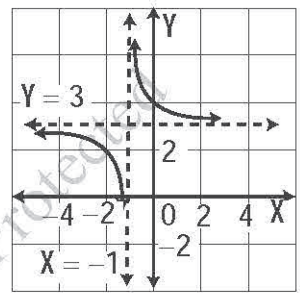
Let us draw the graph of $y = \dfrac{3}{x-1} + 2$.
This equation is defined for $x \neq 1$.
Hence vertical asymptote is $x = 1$.
Since $x = \dfrac{1+y}{y-2}$, horizontal asymptote is $y = 2$.
Domain $= \{x \mid x \neq 1,\ x \in \mathbb{R}\}$.
Range $= \{y \mid y \neq 2,\ y \in \mathbb{R}\}$.
If $x > 1$, then $y > 2$, since $\dfrac{3}{x-1} > 0$.
If $x < 1$, then $y < 2$, since $\dfrac{3}{x-1} < 0$.

(iii) The Graph of $y = \frac{ax+b}{cx+d}$

To draw the graph we need to transform the above function as that of case (ii).

Example 14.

Draw the graph of $y = \frac{3x+4}{x+1}$.



Solution

$$\frac{3x+4}{x+1} = \frac{3(x+1)+1}{x+1} = \frac{1}{x+1} + 3$$

We shall now draw the graph of $y = \frac{1}{x+1} + 3$.

As in case 2 we see that
vertical asymptote is $x = -1$,
horizontal asymptote is $y = 3$.
Domain $= \{x \mid x \neq -1,\ x \in \mathbb{R}\}$.
Range $= \{y \mid y \neq 3,\ y \in \mathbb{R}\}$.

Note: For the graph of $y = \frac{ax+b}{cx+d}$, the vertical asymptote is $x = -\frac{d}{c}$ and the horizontal asymptote is $y = \frac{a}{c}$.

Exercise 4.5

1. Sketch the graphs of the following functions.
   (a) $y = \frac{1}{x}$, (b) $y = \frac{3}{x}$, (c) $y = -\frac{2}{x}$, (d) $y = -\frac{1}{2x}$, (e) $y = \frac{1}{3x}$.
   State the domain and range of each function.

2. Sketch the graphs of:
   (a) $y = -\frac{2}{x}+1$, (b) $y = \frac{2}{x-3}$, (c) $y = -\frac{1}{x+1}-1$, (d) $y = \frac{2}{x+1}+2$.
   State the domain and range of each function.

3. Sketch the graphs of:
   (a) $y = \frac{x+1}{x-1}$, (b) $y = \frac{-3x+4}{x-2}$, (c) $y = \frac{2x-3}{3x+1}$.
   State the domain and range of each function.

4.3.2 One-to-One Functions

Definition. A function f is called one-to-one

$$\text{if } x, y \in \text{dom}(f), \text{ and } x \neq y \text{ implies } f(x) \neq f(y).$$

In other words

$$\text{if } x, y \in \text{dom}(f), \text{ and } f(x) = f(y), \text{ then } x = y.$$

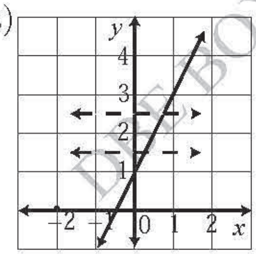
As an example, let $D = \{1, 2, 3\}$ and $R = \{3, 6, 9\}$. Define the function $f : D \to R$ such that $f(x) = 3x$. Then $f(1) = 3$, $f(2) = 6$, $f(3) = 9$. Thus we see that f is one-to-one.

It is easy to check whether the given function is one-to-one or not by using the horizontal line test.

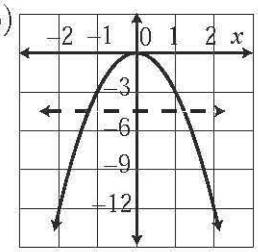
Horizontal Line Test: A real valued function is **one-to-one** if every horizontal line intersects the graph of the function at most one point.

Exercise 4.6

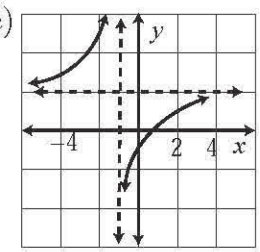
1. Determine whether each of the following function is a one-to-one function or not. If it is not one-to-one, explain why not.

(a)


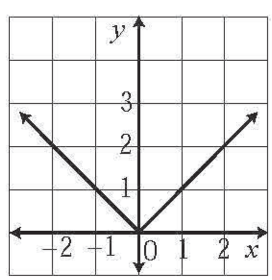
$f(x) = 2x + 1$

(b)


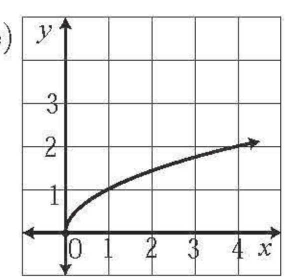
$f(x) = -3x^2$

(c)


$f(x) = \dfrac{2x-1}{x+1}$

(d)


$f(x) = |x|$

(e)


$f(x) = \sqrt{x}$

2. Draw the graph of the each given function and determine whether each is a one-to-one function or not.

   (a) $f(x) = 3x + 2$ (b) $f(x) = x - 3$ (c) $f(x) = 4x^2$

   (d) $f(x) = 2|x|$ (e) $f(x) = \dfrac{2x+3}{x+2}$ (f) $f(x) = 4x^2 (0 \leq x \leq 4)$

   (g) $f(x) = \sqrt{x} (x \geq 0)$.

4.3.3 Inverse Functions

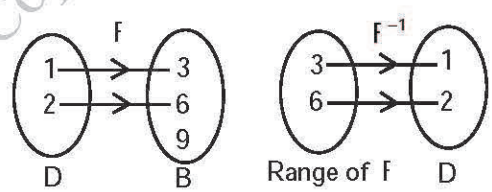
Definition. Let D be the domain and R be the range of function f given by $y = f(x)$. Suppose that f is a one-to-one function. The **inverse** function f^{-1} is defined by

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y.$$

The domain of f^{-1} is R and the range of f^{-1} is D. The symbol f^{-1} is for the inverse of f and is read f **inverse**.

Example 15.

Let $D = \text{dom}(f) = \{1, 2\}$. The function f is defined by $f(1) = 3$, $f(2) = 6$. Then its inverse $f^{-1} : \text{ran}(f) \to D$ is such that $f^{-1}(3) = 1$, $f^{-1}(6) = 2$.



Example 16.

Let the function f be given by $f(x) = 2x + 3$ and let y be the image of x under f. Find the formula for f^{-1}.

Solution

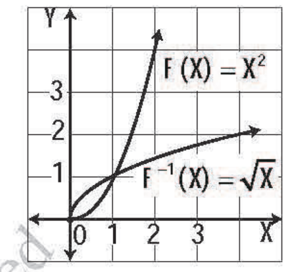
Let $y = f(x)$. Then $y = 2x + 3$. This gives

$$\begin{aligned} 2x &= y - 3 \\ x &= \frac{y-3}{2}. \\ \text{Hence,} \quad f^{-1}(y) &= \frac{y-3}{2}. \end{aligned}$$

Therefore $f^{-1}(x) = \dfrac{x-3}{2}$.

Example 17.

Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = x^2$ and let y be the image of x under f. Find the formula for f^{-1}.



Solution
Let $y = f(x)$. Thus $y = x^2$ gives $x = \pm\sqrt{y}$.
Since $\text{dom}(f) = \{x \mid x \geq 0\}$, $\text{ran}(f) = \{y \mid y \geq 0\}$, pick $f^{-1}(y) = \sqrt{y}$.
Therefore $f^{-1}(x) = \sqrt{x}$.

Example 18.

The function f is given by $f(x) = \dfrac{2}{3 - 4x}$. State the domain of f. Find the formula for f^{-1} and state the domain and range of f^{-1}.

Solution
$f(x) = \dfrac{2}{3 - 4x}$. Therefore $\text{dom}(f) = \{x \mid \ x \neq \dfrac{3}{4}, x \in \mathbb{R}\}$.

Let $f(x) = y$, then $\dfrac{2}{3 - 4x} = y$. This gives

$$\begin{aligned} 4xy &= 3y - 2 \\ x &= \frac{3y - 2}{4y} \\ f^{-1}(y) &= \frac{3y - 2}{4y}. \end{aligned}$$

Thus $f^{-1}(x) = \dfrac{3x - 2}{4x}$.
Domain of $f^{-1} = \{x \mid x \neq 0, x \in \mathbb{R}\}$, Range of $f^{-1} = \{y \mid y \neq \frac{3}{4}, y \in \mathbb{R}\}$.

Example 19.

The function f is given by $f(x) = \dfrac{2x + 3}{x - 5}, x \neq 5$. Find $f^{-1}(3)$.

Solution
Let $\ a = f^{-1}(3)$. Then $f(a) = 3$.

$$\begin{aligned} \frac{2a + 3}{a - 5} &= 3 \\ 2a + 3 &= 3a - 15 \\ a &= 18 \\ f^{-1}(3) &= 18 \end{aligned}$$

Exercise 4.7

1. Find the formula for f^{-1} and state the domain of f^{-1} when the function f is given by
   (a) $f(x) = 2x - 3$, (b) $f(x) = 1 + 3x$, (c) $f(x) = 1 - x$,
   (d) $f(x) = \dfrac{x+9}{2}$, (e) $f(x) = \frac{1}{3}(4x - 5)$, (f) $f(x) = \dfrac{2x+5}{x-7}$,
   (g) $f(x) = \dfrac{3}{x-2}$, (h) $f(x) = \dfrac{13}{2x}$.

2. $A = \{x \mid x \geq 0,\ x \in \mathbb{R}\}$ and g, h are functions from A to A defined by $g(x) = 2x,\ h(x) = x^2$.
   (a) Find the formula for the inverse functions g^{-1}, h^{-1}.
   (b) Evaluate $g^{-1}(7),\ h^{-1}(5)$.

3. Function f is given by $f(x) = \dfrac{2x-5}{x-3}$.
   (a) State the value of x for which f is not defined.
   (b) Find the value of x for which $f(x) = 0$.
   (c) Find the inverse function f^{-1} and state the domain of f^{-1}.

4. Function f is given by $f(x) = \dfrac{x+a}{x-2}$ and that $f(7) = 2$, find
   (a) the value of a, and (b) $f^{-1}(-4)$.

4.3.4 Composition of Functions

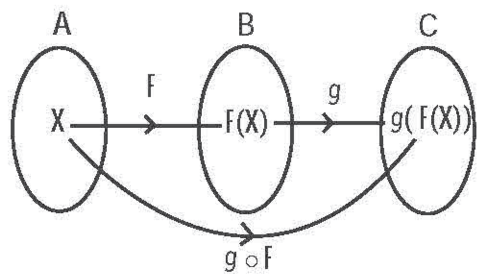
Definition. If f and g are functions such that $\text{ran}(f) \subset \text{dom}(g)$, then the **composite function** of f and g is the new function $g \circ f$ (read as g **circle** f) with $\text{dom}(g \circ f) = \text{dom}(f)$ such that

$$(g \circ f)(x) = g(f(x))$$

for all $x \in \text{dom}(f)$.

If $f : A \to B$ and $g : B \to C$ are given functions and x is any element in A, then we may illustrate the function $g \circ f$ as follows.

$$x \longmapsto f(x) \longmapsto g(f(x)).$$



Example 20.

Functions f and g are defined by $f(x) = x^2$ and $g(x) = 3x + 1$. Find
(a) $(g \circ f)(2)$ (b) $(f \circ g)(2)$ (c) $(g \circ f)(x)$ (d) $(f \circ g)(x)$.

Solution

(a) $(g \circ f)(2) = g(f(2)) = g(4) = 12 + 1 = 13.$

(b) $(f \circ g)(2) = f(g(2)) = f(7) = 7^2 = 49.$

(c) $(g \circ f)(x) = g(f(x)) = g(x^2) = 3x^2 + 1.$

(d) $(f \circ g)(x) = f(g(x)) = f(3x + 1) = (3x + 1)^2.$

Example 21.

Let the functions f and g be given by $f(x) = \dfrac{1}{2x - 1}$ and $g(x) = \dfrac{1}{x + 1}$.
(a) If dom $(f) = \{x \mid x \neq 0 \text{ and } x \neq \frac{1}{2}\}$, find $(g \circ f)(x)$.
(b) If dom $(g) = \{x \mid x \neq -1 \text{ and } x \neq 1\}$, find $(f \circ g)(x)$.
Solution

(a) $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{1}{2x - 1}\right) = \dfrac{1}{\frac{1}{2x-1} + 1} = \dfrac{2x - 1}{2x}.$

$\therefore (g \circ f)(x) = \dfrac{2x - 1}{2x}.$

(b) $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{1}{x + 1}\right) = \dfrac{1}{\frac{2}{x+1} - 1} = \dfrac{x + 1}{1 - x}.$

$\therefore (f \circ g)(x) = \dfrac{x + 1}{1 - x}.$

Example 22.

Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{x+1}{x-1}$.

(a) What is the domain of f for which the function $g \circ f$ exists? Find $(g \circ f)(x)$.

(b) What is the domain of g for which the function $f \circ g$ exists? Find $(f \circ g)(x)$.

Solution

(a) Since the function $g \circ f$ exists,

$$\begin{aligned} \mathrm{dom}(f) &= \mathrm{dom}(g \circ f) \\ &= \{x \in \mathbb{R} \mid x \neq 0 \quad \text{and} \quad f(x) \neq 1\} \\ &= \{x \in \mathbb{R} \mid x \neq 0 \quad \text{and} \quad \frac{1}{x} \neq 1\} \\ &= \{x \in \mathbb{R} \mid x \neq 0 \quad \text{and} \quad x \neq 1\} \\ &= \mathbb{R} \setminus \{0, 1\}. \end{aligned}$$

By definition, $(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{\frac{1}{x}+1}{\frac{1}{x}-1}$.

$$\therefore (g \circ f)(x) = \frac{1+x}{1-x}.$$

(b) Since the function $f \circ g$ exists,

$$\begin{aligned} \mathrm{dom}(g) &= \mathrm{dom}(f \circ g) \\ &= \{x \in \mathbb{R} \mid x \neq 1 \quad \text{and} \quad g(x) \neq 0\} \\ &= \{x \in \mathbb{R} \mid x \neq 1 \quad \text{and} \quad \frac{x+1}{x-1} \neq 0\} \\ &= \{x \in \mathbb{R} \mid x \neq 1 \quad \text{and} \quad x+1 \neq 0\} \\ &= \{x \in \mathbb{R} \mid x \neq 1 \quad \text{and} \quad x \neq -1\} \\ &= \mathbb{R} \setminus \{-1, 1\}. \end{aligned}$$

By definition, $(f \circ g)(x) = f(g(x)) = f\left(\frac{x+1}{x-1}\right) = \frac{1}{\frac{x+1}{x-1}}$.

$$\therefore (f \circ g)(x) = \frac{x-1}{x+1}.$$

Properties of Composite Functions

1. The composition of linear functions is a linear function.
2. The composition of one-to-one functions is a one-to-one function.
3. Let $f : A \to B$, $\quad g : B \to C$ and $h : C \to D$ be given functions. Then
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
4. Let $f : A \to B$ be functions and $I_A : A \to A$ be an identity function on A. Then
$$f \circ I_A = f.$$

Moreover if $I_B : B \to B$ an identity function on B, then $I_B \circ f = f$.

Example 23.

Let f and g be defined by $f(x) = 2x + 3$ and $g(x) = 5x - 4$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(5x - 4) \\ &= 2(5x - 4) + 3 \\ &= 10x - 5\end{aligned} \qquad \begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x + 3) \\ &= 5(2x + 3) - 4 \\ &= 10x + 11\end{aligned}$$

We see that both $f \circ g$ and $g \circ f$ are linear when f and g are linear functions.

Example 24.

Let the functions f, g and h be defined by $f(x) = 3x$, $g(x) = x - 1$ and $h(x) = x^2$.

$$\begin{aligned}(h \circ (g \circ f))(x) &= h((g \circ f)(x)) \\ &= h(g(f(x))) \\ &= h(g(3x)) \\ &= h(3x - 1) \\ &= (3x - 1)^2\end{aligned} \qquad \begin{aligned}((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) \\ &= (h \circ g)(3x) \\ &= h(g(3x)) \\ &= h(3x - 1) \\ &= (3x - 1)^2\end{aligned}$$

Therefore $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$ for all x.
Hence $h \circ (g \circ f) = (h \circ g) \circ f$.

Note. The composition of functions does not, in general, obey the commutative law.

For example, let us consider the functions f and g of Example 23.

We have $(f \circ g)(x) = 10x - 5.$

$(f \circ g)(1) = 5.$

But $(g \circ f)(x) = 10x + 11.$

$(g \circ f)(1) = 21.$

Hence $f \circ g \neq g \circ f$.

Exercise 4.8

1. Functions f and g are given by $f(x) = 2x + 1$ and $g(x) = 3x$.
   (a) Calculate $(g \circ f)(1)$ and $(g \circ f)(3)$.
   (b) Find the formula of $g \circ f$ and check the above images. State the domain of $g \circ f$.

2. The functions f and g are given by $f(x) = x + 2$ and $g(x) = x^2$.
   (a) Find the formulae for $g \circ f$, $g \circ g$, $f \circ g$, $f \circ f$ and their domains.
   (b) Find $(g \circ f)(-1)$ and $(g \circ f)(2)$.
   (c) Find $(f \circ g)(-1)$ and $(f \circ g)(2)$.

3. Find the formulae for composite functions $f \circ g$, $g \circ f$ and their domains in each case.
   (a) $f(x) = x + 1$, $g(x) = 2x^2 - x + 3$.
   (b) $f(x) = x^2 - 1$, $g(x) = 3x + 1$.
   (c) $f(x) = -x$, $g(x) = x$.
   (d) $f(x) = x^2$, $g(x) = \sqrt{x}$.

4. A function f is given by $f(x) = x + 1$. Find the function $g : \mathbb{R} \to \mathbb{R}$ in each of the following:
   (a) $(g \circ f)(x) = x^2 + 5x + 5$ (b) $(f \circ g)(x) = x^2 + 5x + 5$.

5. If $g : \mathbb{R} \to \mathbb{R}$ is given by $g(x) = x^2 + 3$, find the function f such that
   (a) $(f \circ g)(x) = 4x^2 + 3$ (b) $(g \circ f)(x) = 4x^2 + 3$.

6. Functions f and g are given by $f(x) = px - 2$ where p is a constant and $g(x) = 4x + 3$. Find the value of p for which $(f \circ g)(x) = (g \circ f)(x)$.

7. Let f and g be functions given by $f(x) = 3x - 1$ and $g(x) = x + 7$. Find the formulae of $f^{-1} \circ g$, $g^{-1} \circ f$ and state their domains. What are the values of $(f^{-1} \circ g)(3)$ and $(g \circ f^{-1})(2)$?

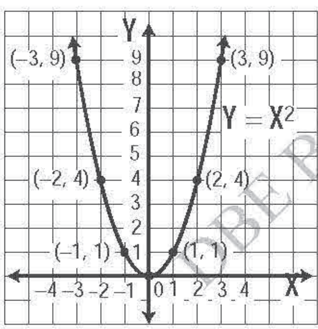
8. Let the functions f and g be given by $f(x) = 2x - 1$ and $g(x) = \dfrac{2x+3}{x-1}$.
   What is the domain of f for which the function $g \circ f$ exists? Find $(g \circ f)(x)$.
   Find the inverse function f^{-1} and g^{-1}.
   Evaluate $(f \circ g^{-1})(1)$ and $(f^{-1} \circ g^{-1})(1)$.

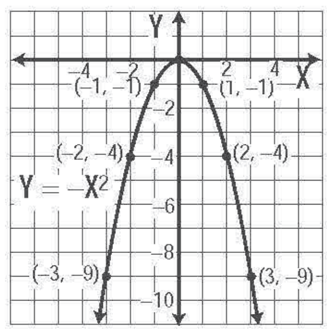
9. Let the functions f, g and h be $f(x) = x - 2$, $g(x) = x^3$ and $h(x) = 4x$. Show that $(h \circ g) \circ f = h \circ (g \circ f)$.

Chapter 5

Quadratic Functions

A function $y = ax^2 + bx + c$ where $a \neq 0$ is a **quadratic function** in the **standard form**. The graph of a quadratic function is called a **parabola**.





- If a is positive that is $a > 0$, then the parabola opens up as the graph of $y = x^2$.

- If a is negative that is $a < 0$, then the parabola opens down as the graph of $y = -x^2$.

Note that the graph of $y = -x^2$ is the **reflection on the x-axis** of the graph of $y = x^2$.

5.1 Graph of the Function $y = x^2 + bx + c$

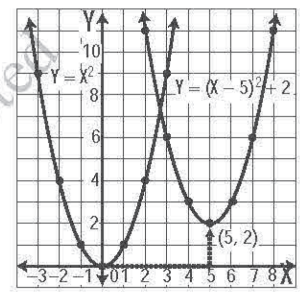
If $a = 1$ in the standard form, then the quadratic function is $y = x^2 + bx + c$. The graph of $y = x^2 + bx + c$ can be seen as the **translation** of the graph of $y = x^2$ as shown in the example below.

Example 1.

Consider the function $y = x^2 - 10x + 27$. Since

$$y = x^2 - 10x + 27 = (x - 5)^2 + 2;$$

the graph of $y = x^2 - 10x + 27$ is the translation of **positive** 5 units **horizontally** and **positive** 2 units **vertically** of the graph $y = x^2$.



How to change $y = x^2 + bx + c$ to the form $y = (x - h)^2 + k$

$$\begin{aligned} y &= x^2 + bx + c \\ &= x^2 + bx + (\frac{b}{2})^2 - (\frac{b}{2})^2 + c \\ &= (x + \frac{b}{2})^2 - (\frac{b}{2})^2 + c \\ &= (x + \frac{b}{2})^2 - \frac{b^2 - 4c}{4} \\ &= (x - h)^2 + k \quad \text{where } h = -\frac{b}{2} \text{ and } k = -\frac{b^2 - 4c}{4}. \end{aligned}$$

5.2 Graph of the Function $y = -x^2 + bx + c$

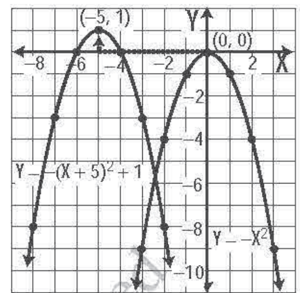
If $a = -1$ in the standard form, then the quadratic function is $y = -x^2 + bx + c$. The graph of $y = -x^2 + bx + c$ can be seen as the **translation** of the graph of $y = -x^2$ as shown in the following example.

Example 2.

Consider the function $y = -x^2 - 10x - 24$. Since

$$y = -x^2 - 10x - 24 = -(x+5)^2 + 1$$

the graph of $y = -x^2 - 10x - 24$ is the translation of **negative** 5 units **horizontally** and **positive** 1 unit **vertically** of the graph $y = -x^2$.



How to change $y = -x^2 + bx + c$ to the form $y = -(x-h)^2 + k$

$$\begin{aligned}
y &= -x^2 + bx + c \\
&= -(x^2 - bx) + c \\
&= -(x^2 - bx + (\frac{b}{2})^2 - (\frac{b}{2})^2) + c \\
&= -(x^2 - bx + (\frac{b}{2})^2) + (\frac{b}{2})^2 + c \\
&= -(x - \frac{b}{2})^2 + \frac{b^2 + 4c}{4} \\
&= -(x-h)^2 + k \quad \text{where } h = \frac{b}{2} \text{ and } k = \frac{b^2+4c}{4}.
\end{aligned}$$

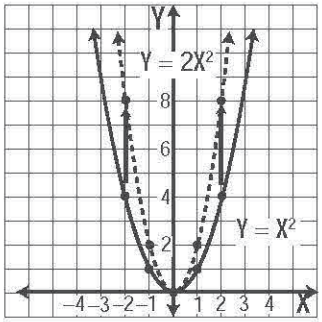
Exercise 5.1

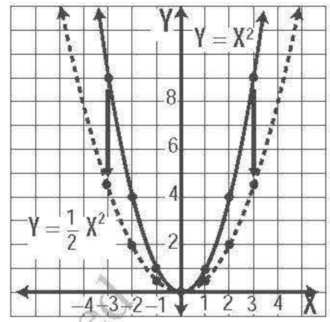
1. Compare the graphs of the following functions to the graph of $y = x^2$.
   (a) $y = x^2 + 2x + 3$ (b) $y = x^2 - 4x + 2$ (c) $y = x^2 + 4x + 4$

2. Compare the graphs of the following functions to the graph of $y = -x^2$.
   (a) $y = -x^2 + 2x + 3$ (b) $y = -x^2 - 4x - 7$ (c) $y = -x^2 + 4x - 4$

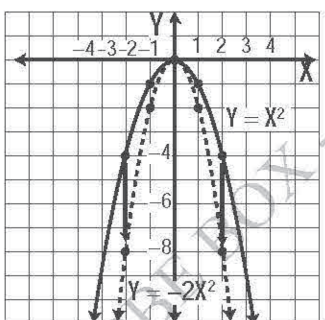
5.3 Graph of the Function $y = ax^2$

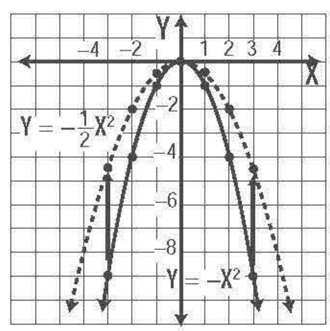
When a is **positive**, the graph of the function $y = ax^2$ is the vertical stretch of the scale factor a of the function $y = x^2$ as in the following figures.





When a is **negative**, the graph of the function $y = ax^2$ is the vertical stretch of the scale factor $|a|$ of the function $y = -x^2$ as in the following figures.





One can see that if the point (p, q) is on the graph $y = x^2$, then the point (p, aq) is on the graph $y = ax^2$. For example the point $(2, 4)$ is on the graph $y = x^2$, then

- the point $(2, 8)$ is on the graph $y = 2x^2$
- the point $(2, 2)$ is on the graph $y = \frac{1}{2}x^2$
- the point $(2, -8)$ is on the graph $y = -2x^2$
- the point $(2, -2)$ is on the graph $y = -\frac{1}{2}x^2$

5.4 Graph of the Function $y = ax^2 + bx + c$

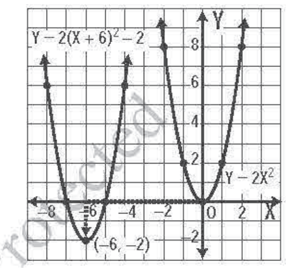
The graph of $y = ax^2 + bx + c$ can be seen as the **translation** of the graph of $y = ax^2$ as shown in the examples below.

Example 3.

Consider the function $y = 2x^2 + 24x + 70$. Since

$$y = 2x^2 + 24x + 70 = 2(x + 6)^2 - 2$$

the graph of $y = 2x^2 + 24x + 70$ is the translation of **negative** 6 units **horizontally** and **negative** 2 units **vertically** of the graph $y = 2x^2$.

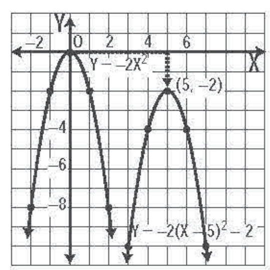


Example 4.

Consider the function $y = -2x^2 + 20x - 52$. Since

$$y = -2x^2 + 20x - 52 = -2(x - 5)^2 - 2$$

the graph of $y = -2x^2 + 20x - 52$ is the translation of **positive** 5 units **horizontally** and **negative** 2 units **vertically** of the graph $y = -2x^2$.



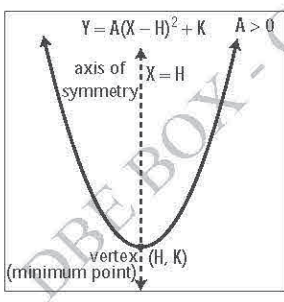
How to change $y = ax^2 + bx + c$ to the form $y = a(x - h)^2 + k$

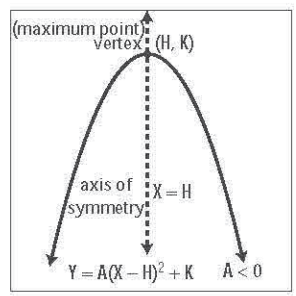
$$\begin{aligned}
y &= ax^2 + bx + c \\
&= a(x^2 + \frac{b}{a}x) + c \\
&= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2) + c \\
&= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) - a(\frac{b}{2a})^2 + c \\
&= a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c \\
&= a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} \\
&= a(x - h)^2 + k \qquad \text{where } h = -\frac{b}{2a} \text{ and } k = -\frac{b^2 - 4ac}{4a}.
\end{aligned}$$

Features of the graph $y = ax^2 + bx + c = a(x-h)^2 + k$

- vertex: $(h, k) = (-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a})$
- axis of symmetry: $x = -\frac{b}{2a}$
- y-intercept: $(0, c)$
- domain = the set $\mathbb{R}$ of all real numbers
- range $= \begin{cases} \{y \mid y \geq k\} & \text{when } a > 0; k \text{ is minimum} \\ \{y \mid y \leq k\} & \text{when } a < 0; k \text{ is maximum.} \end{cases}$

The form $y = a(x-h)^2 + k$ is called the **vertex form** of the quadratic function.



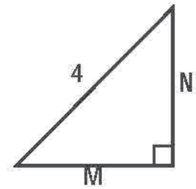


Example 5.

What is the largest area possible for a right triangle whose hypotenuse is 4 cm long?

Solution

Let m cm and n cm be two legs of the right triangle with hypotenuse 4 cm.



$$m^2 + n^2 = 4^2 = 16$$

$$n = \sqrt{16 - m^2}$$

Then area of the right triangle is $A = \frac{1}{2}mn$.

$$\begin{aligned} A &= \frac{1}{2}mn \\ &= \frac{1}{2}m\sqrt{16-m^2} \\ \therefore A^2 &= \frac{1}{4}m^2(16-m^2) \\ A^2 &= -\frac{1}{4}m^4 + 4m^2 \end{aligned}$$

Let $x = m^2$. Then

$$A^2 = -\frac{1}{4}x^2 + 4x.$$

The maximum value of A^2 occurs at $x = -\frac{4}{2(-\frac{1}{4})} = 8$.

So the maximum value of A occurs at $m = \sqrt{x} = \sqrt{8} = 2\sqrt{2}$.
When $m = 2\sqrt{2}$, $n = \sqrt{16-m^2} = 2\sqrt{2}$.
Since $\frac{1}{2}mn = \frac{1}{2}(2\sqrt{2})(2\sqrt{2}) = 4$, the largest area is 4 cm^2.

Exercise 5.2

1. Find the vertex form of each of the following quadratic functions. Find also y-intercept, axis of symmetry, vertex, and range of each of the functions.

   (a) $y = 2x^2 + 4x + 3$ (b) $y = 3x^2 - 6x + 2$ (c) $y = \frac{1}{2}x^2 + x - 4$
   (d) $y = -2x^2 + 2x + 3$ (e) $y = -3x^2 - 12x - 7$ (f) $y = -\frac{1}{2}x^2 - 3x - 4$

2. Find two positive numbers whose sum is 20 and whose product is as large as possible.

3. What is the largest area possible for a rectangle whose perimeter is 16 cm?

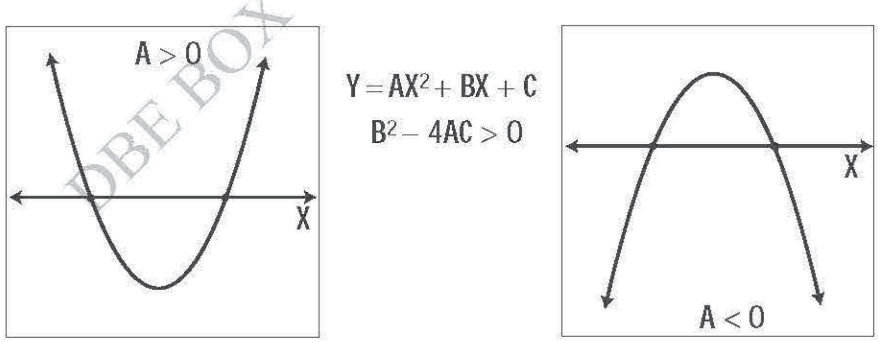
5.5 Discriminant of a Quadratic Function

The sign of the **discriminant** $b^2 - 4ac$ indicates whether or not the graph of the quadratic function passes through the x-axis.

First we consider the case $b^2 - 4ac > 0$.

- If $a > 0$, then the parabola opens up and $-\frac{b^2 - 4ac}{4a} < 0$. Thus the vertex $(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a})$ is below the x-axis so the graph cuts the x-axis at two points.
- If $a < 0$, then the parabola opens down and $-\frac{b^2 - 4ac}{4a} > 0$. Thus the vertex $(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a})$ is above the x-axis so the graph cuts the x-axis at two points.

When $b^2 - 4ac > 0$ the graph passes through the x-axis at two points.

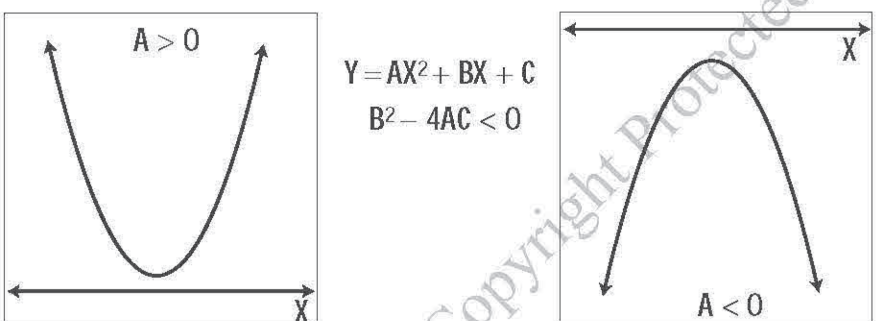


Next we consider the case $b^2 - 4ac < 0$.

- If $a > 0$, then the parabola opens up and $-\frac{b^2 - 4ac}{4a} > 0$. Thus the vertex $(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a})$ is above the x-axis so that the graph does not cut the x-axis.

- If $a < 0$, then the parabola opens down and $-\dfrac{b^2-4ac}{4a} < 0$. Thus the vertex $(-\dfrac{b}{2a}, -\dfrac{b^2-4ac}{4a})$ is below the x-axis so that the graph does not cut the x-axis.

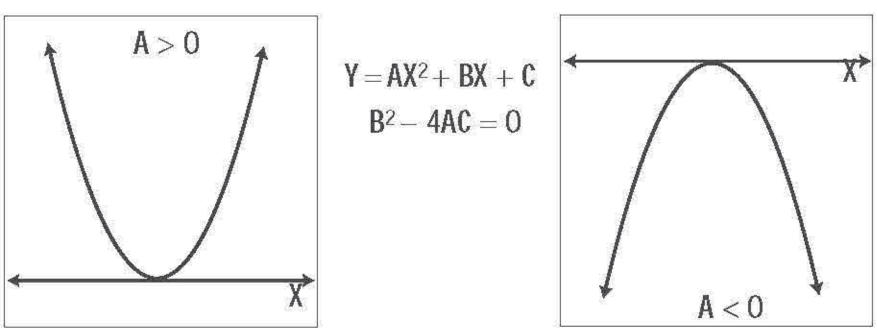
When $b^2 - 4ac < 0$ the graph does not pass through the x-axis.



Finally we consider the case $b^2 - 4ac = 0$.

When $b^2 - 4ac = 0$, the vertex $(-\dfrac{b}{2a}, -\dfrac{b^2-4ac}{4a}) = (-\dfrac{b}{2a}, 0)$ is on the x-axis so the graph meets the x-axis at exactly one point.

When $b^2 - 4ac = 0$ the graph meets the x-axis at exactly one point.



If $b^2 - 4ac \geq 0$, the quadratic function $y = ax^2 + bx + c$ can be written as

$$y = a(x - p)(x - q).$$

This is called the **intercept form** or **factor form** of the quadratic function. In the intercept form, p and q are x-intercepts and the x-coordinate of the vertex of the quadratic function can be found as $\frac{p+q}{2}$.

Exercise 5.3

1. Find the discriminant of each of the following quadratic functions. Also find the number of x-intercepts of each of the functions.

   (a) $y = 3x^2 - 4x + 3$ (b) $y = 2x^2 - 4x - 3$ (c) $y = \frac{1}{2}x^2 + x - 4$
   (d) $y = -x^2 + 6x - 9$ (e) $y = -3x^2 - 12x - 7$ (f) $y = -\frac{1}{2}x^2 - 3x - 4$

2. Find the intercept form of each of the quadratic functions. Also find the y-intercept, axis of symmetry, vertex, and range of each of the functions.

   (a) $y = 2x^2 - 2x - 12$ (b) $y = 3x^2 - 6x + 3$ (c) $y = \frac{1}{2}x^2 + x - 4$
   (d) $y = 2x^2 - 5x - 3$ (e) $y = -6x^2 - 7x + 5$ (f) $y = -\frac{1}{2}x^2 - 3x - 4$

5.6 Quadratic Formula of $ax^2 + bx + c = 0$

Finding the x-intercepts of the quadratic function $y = ax^2 + bx + c$ is the same as finding the solutions of the quadratic equation $ax^2 + bx + c = 0$. We will derive the quadratic formula for the solutions of $ax^2 + bx + c = 0$ as in the following.

$$
\begin{aligned}
ax^2 + bx + c &= 0\\
a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} &= 0\\
a(x + \frac{b}{2a})^2 &= \frac{b^2 - 4ac}{4a}\\
(x + \frac{b}{2a})^2 &= \frac{b^2 - 4ac}{4a^2}\\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}\\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a}\\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}\\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

In this formula, we again find the discriminant $b^2 - 4ac$.

As stated in Section 5.5

- if $b^2 - 4ac > 0$, then the quadratic equation has two real solutions as

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a};$$

- if $b^2 - 4ac = 0$, then the quadratic equation has only one real solution (repeated solution) as

$$x = -\frac{b}{2a};$$

- if $b^2 - 4ac < 0$, then the quadratic equation has no real solution.

Example 6.

Find the solutions for $2x^2 + 3x - 4 = 0$ by using quadratic formula.

Solution

Comparing the given equation $2x^2 + 3x - 4 = 0$ to the standard form $ax^2 + bx + c = 0$, we get $a = 2, b = 3, c = -4$. Then

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{3^2 - 4(2)(-4)}}{2(2)} \\ &= \frac{-3 \pm \sqrt{9 + 32}}{4} \\ &= \frac{-3 \pm \sqrt{41}}{4} \\ x &= \frac{-3 + \sqrt{41}}{4} \quad \text{or} \quad x = \frac{-3 - \sqrt{41}}{4} \end{aligned}$$

Example 7.

Solve $2x^2 - 3x + 4 = 0$.

Solution

Comparing the given equation $2x^2 - 3x + 4 = 0$ to the standard form $ax^2 + bx + c = 0$, we get $a = 2, b = -3, c = 4$. Then

$$b^2 - 4ac = (-3)^2 - 4(2)(4) = 9 - 32 = -23 < 0$$

.

Therefore the equation has no real solution.

Exercise 5.4

Solve the following equations by using the quadratic formula.

1. $x^2 + 2x - 1 = 0$
2. $x^2 + 4x + 4 = 0$
3. $p^2 - 6p + 3 = 0$
4. $t^2 - 4t - 8 = 0$
5. $3q^2 - 12q + 11 = 0$
6. $5z^2 + 3z - 4 = 0$

5.7 Miscellaneous Exercises

There are problems that can be solved by creating quadratic equation by using the given information. In this section we will solve some examples.

Example 8.

Find the solution set of the system of equations:

$$\begin{aligned} x^2 + y^2 &= 25 \\ y &= x - 1 \end{aligned}$$

Solution

$$\begin{aligned} x^2 + y^2 &= 25 \qquad (1) \\ y &= x - 1 \qquad (2) \end{aligned}$$

Substituting $y = x - 1$ in (1), we get

$$\begin{aligned} x^2 + (x-1)^2 &= 25 \\ x^2 + x^2 - 2x + 1 &= 25 \\ 2x^2 - 2x - 24 &= 0 \\ x^2 - x - 12 &= 0 \\ (x+3)(x-4) &= 0 \\ x = -3 \quad &\text{or} \quad x = 4 \end{aligned}$$

Substituting $x = -3$ in (2), $y = -3 - 1 = -4$.

Substituting $x = 4$ in (2), $y = 4 - 1 = 3$.

Therefore the solution set is $\{(-3, -4), (4, 3)\}$.

Example 9.

Find the solution set of the system of equations:

$$\begin{aligned} x^2 - xy + 2y^2 &= 8 \\ 3x - 2y &= 2 \end{aligned}$$

Solution

$$\begin{aligned} x^2 - xy + 2y^2 &= 8 \qquad (1) \\ 3x - 2y &= 2 \qquad (2) \end{aligned}$$